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ON COVERING SINGLE POINTS BY RANDOMLY ORDERED INTERVALS[1]

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A strictly increasing, pure jump process with stationary, independent increments hits a single point $r > 0$ with probability 0. Adapting a method of proof, due to Carleson, we obtain a similar result for processes with exchangeable increments. This enables us to solve a regularity problem from game theory concerning probabilities of covering single points by randomly ordered intervals.

1. Introduction. Chung's problem for processes with stationary, independent increments (i.e., Lévy processes) was formulated by Meyer (1969) as a problem on hitting probabilities. It conjectures that for an ascending, pure jump Lévy process with infinitely many jumps in each interval, the probability that this process hits an arbitrary point $r > 0$ is zero. Kesten (1969) proved this result and investigated also more general Lévy processes. Bretagnolle (1971) indicated important simplifications in the proofs. Carleson gave an analytic solution of Chung's problem (see Assouad (1971)).

The main theorem of this paper can be formulated also as a result on hitting probabilities of single points for a process X. This process has dependent increments. We will consider it locally at t as a Lévy process with a Lévy measure that is random and depends on t. It is then possible, despite this absence of time homogeneity, to use Carleson's method to obtain the required result. A solution of Chung's problem on Lévy processes can be obtained from our result by using a suitable conditioning on the Lévy process.

Let $(Y_i)_{i\geq 1}$ be a sequence of independent random variables, uniformly distributed on (0, 1). We may assume that $Y_i \neq Y_j$ for $i \neq j$. Write $i\,\alpha\, j$ if $Y_i < Y_j$. Then α is a linear order on the positive integers N, such that

$$P(k_1\alpha \cdots \alpha k_n) = \frac{1}{n!}$$

for any n-tuple $(k_1, \cdots, k_n)$ of distinct positive integers.

Assume $a_1 \geq a_2 \geq \cdots > 0$ are constants with total sum 1. Define open intervals

$$J_i := (\textstyle\sum_{j\alpha i} a_j,\ a_i + \sum_{j\alpha i} a_j), \qquad i \geq 1. \tag{1}$$

These random intervals are disjoint and contained in (0, 1). Their total length is 1. If $i\,\alpha\, k$ then $\sup J_i \leq \inf J_k$.

Theorem 1. $P(r \in \cup_{i\geq 1} J_i) = 1$ *for* $r \in (0, 1)$.

Call $i \in N$ the *pivot* for r if

$$\textstyle\sum_{j\alpha i} a_j < r \leq a_i + \sum_{j\alpha i} a_j. \tag{2}$$

In the theory of weighted majority games, pivots play a central role in the determination

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of the value of the game. Theorem 1 implies that there is a pivot with probability 1 for any $r \in (0, 1)$. This answers a question in Shapley (1962). Earlier Shapiro (1955) proved in an unpublished paper that this result was valid for λ—a.e. r, where λ is the Lebesgue measure on $(0, 1)$.

Call $i \in \{1, \cdots, n\}$ an *n-pivot* if

$$\sum_{j\alpha i, 1 \le j \le n} a_j < r \le a_i + \sum_{j\alpha i, 1 \le j \le n} a_j. \tag{3}$$

By Theorem 1 the inequalities in (2) are strict, apart from a P-null set. It follows that with probability 1

$$\lim_{n\to\infty} n\text{-}pivot = pivot$$

and the pivot is determined by the order α on a sufficiently large finite random subset of $\mathbb{N}$ (see also Artstein (1971)).

A slightly weaker result than Theorem 1 can be proved easily.

Proposition 2. *$P(r \in \cup_{i\ge1} J_i) = 1$ for λ—a.e. r.*

Proof. Let χ_{J_i} be the *indicator function* of J_i. The proposition follows from

$$\int_0^1 P(r \in \cup_{i\ge1} J_i)\, dr = E \int_0^1 \sum_{i\ge1} \chi_{J_i}(r)\, dr = E \sum_{i\ge1} \lambda(J_i) = 1.$$

Here we used that $\chi_{J_i}(r)$, with $J_i = J_i(\omega)$, is jointly measurable in (r, ω). □

Let us now reformulate Theorem 1 as a result on hitting probabilities. Define

$$X_t := \sum_{i\ge1} a_i\, \chi_{[Y_i,1]}(t), \qquad 0 \le t \le 1. \tag{4}$$

Note that $J_i = (X_{Y_i-}, X_{Y_i})$. We prove

Theorem 3. *$P(\exists\, t : X_t = r) = 0$ for $r \in (0, 1)$.*

By symmetry considerations this implies $P(\exists\, t : X_{t-} = r) = 0$ and hence Theorem 1 follows from Theorem 3.

The process X is right-continuous. We may assume that in each rational interval in $(0, 1)$ there lies some Y_i. Hence X is strictly increasing. Write $T_r := \inf\{t > 0 : X_t \geqq r\}$, $r \in (0, 1)$, and note that $\nexists\, t : X_t = r$ if and only if $X_{T_r} > r$. By proposition 2

$$P(X_{T_r} > r) = 1 \quad \text{for} \quad \lambda\text{—a.e. } r, \tag{5}$$

and we have to show that the equality in (5) holds for all $r \in (0, 1)$. Analogous to a result in Meyer (1969) we prove in Section 2, as a first step in the proof of Theorem 3:

Proposition 4. *For any $r \in (0, 1)$*

$$P(X_{T_r} > r) = E \int_0^{T_r} \sigma_s(r - X_s)\, ds. \tag{6}$$

The function σ_s will be specified later. Formula (6) describes the probability that the process X overshoots the level r. Intuitively the formula becomes clear by assuming that locally at t the process is a Lévy process with Lévy meassre λ_t. Then the intensity of a jump exceeding x is $\sigma_t(x) = \lambda_t(x, \infty)$. In Section 3 we investigate (6) by means of Carleson's method, to show that in (5) the equality holds for all $r \in (0, 1)$.

2. The overshot formula. To prove Proposition 4 we use a martingale as in Meyer (1969). Actually, in view of our approach in Section 3, we will derive a slightly stronger result in Corollary 6.

Define for $0 \le t < 1$

$$\sigma_t(x) := \textstyle\sum_{i \ge 1} \frac{\chi_{[0, Y_i)}(t)}{1 - t} \delta_{a_i}(x, \infty), \qquad x > 0, \tag{7}$$

where δ_a is the probability measure degenerate at a. Let $\mathscr{F}_t$ be the saturation of the σ-field $\sigma(Y_i \wedge t,\ i \ge 1)$, $0 \le t \le 1$. Then it is easily proved that X is adapted to the family $\mathscr{F}_t$, $0 \le t \le 1$, which is increasing and right-continuous.

LEMMA 5. *Let f be continuously differentiable with bounded continuous derivatives on* $[0, 1]$. *The process*

$$[f(X_t) - f(X_0)] - \int_0^t \int_0^\infty [f(X_s + u) - f(X_s)]\, d\sigma_s(u)\, ds \tag{8}$$

is a martingale with respect to $\mathscr{F}_t$, $0 \le t \le 1$.

PROOF. By (7) we have for $0 \le t < t + h \le 1$

$$\begin{aligned} &E\left(\int_t^{t+h} \int_0^\infty [f(X_s + u) - f(X_s)]\, d\sigma_s(u)\, ds \,|\, \mathscr{F}_t\right) \\ &\qquad = \textstyle\sum_{i \ge 1} E\left(\int_t^{t+h} [f(X_s + a_i) - f(X_s)] \frac{\chi_{[0, Y_i)}(s)}{1 - s}\, ds \,|\, \mathscr{F}_t\right). \end{aligned} \tag{9}$$

By our requirements on f these integrals are well defined. Write

$$X_t^i := \textstyle\sum_{j \ne i, j \ge i} a_j\, \chi_{[Y_j, 1]}(t), \qquad 0 \le t \le 1.$$

So X^i is obtained by removing the ith term of the sum in (4). The terms of the sum in (9) can be written as

$$E\left(\int_t^{t+h} [f(X_s^i + a_i) - f(X_s^i)] \frac{\chi_{[0, Y_i)}(s)}{1 - s}\, ds \,|\, \mathscr{F}_t\right). \tag{10}$$

Note that X^i and Y_i are independent and also conditionally independent, given $\mathscr{F}_t$. On $\{Y_i > t\}$ the conditional distribution of Y_i, given $\mathscr{F}_t$, is the uniform distribution on $(t, 1]$. Hence (10) equals on $\{Y_i > t\}$

$$\begin{aligned} &E\left(\int_t^{t+h} [f(X_s^i + a_i) - f(X_s^i)] \left[\int_t^1 \frac{\chi_{[0, y]}(s)}{1 - s} \frac{dy}{1 - t}\right] ds \,|\, \mathscr{F}_t\right) \\ &= E\left(\int_t^{t+h} [f(X_s^i + a_i) - f(X_s^i)]\, ds \cdot \frac{1}{1 - t} \,|\, \mathscr{F}_t\right) \\ &= E\left(\int_t^{t+h} \int_{X_s^i}^{X_s^i + a_i} f'(u)\, du\, ds \frac{1}{1 - t} \,|\, \mathscr{F}_t\right). \end{aligned}$$

Note that $X_{Y_i} = X_{Y_i-} + a_i$ a.s. Hence, using the properties of Y_i mentioned earlier, (10) equals on $\{Y_i > t\}$

$$E\left(\int_{X_{Y_i-}}^{X_{Y_i}} f'(u)\, du\, \chi_{(t, t+h]}(Y_i) \,|\, \mathscr{F}_t\right).$$

Therefore (9) equals

$$\sum_{i\geq 1:\, Y_i > t} E\left(\int_{X_{Y_i-}}^{X_{Y_i}} f'(u)\, du\, \chi_{(t,t+h]}(Y_i) \,|\, \mathscr{F}_t\right)$$

$$= E\left(\int_{X_t}^{X_{t+h}} f'(u)\, du \,|\, \mathscr{F}_t\right) = E\big(f(X_{t+h}) - f(X_t) \,|\, \mathscr{F}_t\big).$$

Hence (8) is a martingale. □

Proposition 4 is an obvious consequence of the following corollary.

COROLLARY 6. *If τ is an $(\mathscr{F}_t)$ stopping time and* $0 < r < 1$ *then*

$$P(X_{T_r\wedge\tau} > r) = E\int_0^{T_r\wedge\tau} \sigma_s(r - X_s)\, ds. \tag{11}$$

PROOF. Let $f_1 \leq f_2 \leq \cdots$ be a sequence of functions, satisfying the conditions of the last lemma, such that f_n is nondescending, vanishes on $[0, r]$ and equals 1 on $\left[r + \frac{1}{n}, 1\right]$. Because $\{T_r \leq t\} = \{X_t \geq r\} \in \mathscr{F}_t$, the random variable T_r is an $(\mathscr{F}_t)$-stopping time. Hence by Lemma 5

$$E\, f_n(X_{T_r\wedge\tau}) = E\int_0^{T_r\wedge\tau} \int_0^{\infty} f_n(X_s + u)\, d\sigma_s(u)\, ds$$

and so, by letting $n \to \infty$

$$P(X_{T_r\wedge\tau} \in (r, \infty)) = E\int_0^{T_r\wedge\tau} \sigma_s(r - X_s)\, ds. \qquad \square$$

Formula (11) can also be written another way. Let ν be the *occupation measure* of X, i.e.,

$$\nu(B) := \int_0^1 \chi_B(X_t)\, dt,\ B \subset [0, 1] \quad \text{measurable.}$$

Then $T_u = \nu[0, u]$. Note that T_u, $0 < u < 1$, is the right inverse of X. We will substitute $s = T_u$ in (11). The process X is strictly increasing. Hence ν is nonatomic and has no mass on any of the countably many intervals $[X_{t-}, X_t]$ for which t is a discontinuity point of X. On the complement of these intervals holds $X_{T_u} = u$ and so (11) becomes

$$P(X_{T_r\wedge\tau} > r) = E\int_0^{r\wedge X_\tau} \sigma_{T_u}(r - u)\nu\, (du). \tag{12}$$

3. Proof of Theorem 3. This section uses Carleson's method to solve Chung's problem to obtain Theorem 3. An account of this method of proof can be found in Assouad (1971). Some of the changes in the argument we follow were taken from an earlier lecture by Chung.

Define for $0 \leq t < 1$

$$\sigma_t^\theta(x) := \frac{1}{\theta}\int_0^\theta \sigma_t(x - y)\, dy,\ \theta > 0.$$

Denote $\sigma \equiv \sigma_0$ and $\sigma^\theta \equiv \sigma_0^\theta$. By (5) and (12) we have

$$1 = \frac{1}{\theta}\int_{r-\theta}^{r} P(X_{T_x} > x)\,dx = \frac{1}{\theta}\int_{r-\theta}^{r} E\int_0^x \sigma_{T_u}(x-u)\,\nu(du)\,dx.$$

By using the convention that $\sigma_s = 0$ on $[-\infty, 0]$ we may replace $\int_0^x$ by $\int_0^r$ to obtain

$$1 = E\int_0^r \sigma_{T_u}^\theta(r-u)\,\nu(du).$$

The discontinuity points of $\sigma_t(r-\cdot)$ are contained in the ν-null set $\{r - a_i : i \geq 1\} \cup \{r\}$. Hence, if the bounded convergence theorem could be applied, the theorem could be proved by letting $\theta \downarrow 0$.

It is however necessary to make several truncations. First we consider σ_s, $0 \leq s \leq t < 1$. By (7) we have with $\zeta_i := \chi_{[0,Y_i)}(s)$, $i \geq 1$,

$$(1-s)\sigma_s(x) = \textstyle\sum_{i=1}^{\sigma(x)} \zeta_i. \tag{13}$$

Because $\lim_{x\downarrow 0}\sigma(x) = \infty$, the strong law of large numbers implies

$$\lim_{x\downarrow 0}(1-s)\frac{\sigma_s(x)}{\sigma(x)} = 1 - s \qquad \text{a.s.}$$

Using that ζ_i is monotone in s, one easily sees that the convergence is a.s. uniform in $s \in [0, t]$. Hence

$$\lim_{x\downarrow 0}\frac{\sigma_s(x)}{\sigma(x)} = 1 \qquad \text{uniformly for} \quad s \in [0, t] \qquad \text{a.s.}$$

and so for $\rho \downarrow 0$

$$\tau_\rho := \sup\{t < 1 : \tfrac{1}{2} < \frac{\sigma_s(x)}{\sigma(x)} < 2,\ s \leq t,\ x \leq \rho\} \uparrow 1 \text{ a.s.}$$

Note that σ_s is $\mathscr{F}_s$ – measurable and that $\{\tau_\rho < t\} \in \mathscr{F}_t$. Hence τ_ρ is a stopping time.

Consider a stopping time τ of the form $\tau := \tau_\rho \wedge t_0$, $0 < \rho$, $t_0 < 1$. By (5) and (12) we have

$$P(\tau > T_r) \leq \frac{1}{\theta}\int_{r-\theta}^{r} P(\tau \geq T_x)\,dx$$

$$= \frac{1}{\theta}\int_{r-\theta}^{r} P(X_{T_x\wedge\tau} > x)\,dx = E\int_0^{r\wedge X_\tau} \sigma_{T_u}^\theta(r-u)\nu(du).$$

Writing $\nu_\tau(du) := \nu(du)\chi_{[0,X_\tau)}(u)$ we obtain

$$P(\tau \geq T_r) \leq E\int_0^{r-\rho} \sigma_{T_u}^\theta(r-u)\nu_\tau(du) + E\int_0^{\rho} \sigma_{T_{r-u}}^\theta(u)\mu(du), \tag{14}$$

where $\mu(du) := \nu_\tau(r - du)\chi_{[0,\rho)}(u)$. Because $\tau \leq t_0$ we obtain by using (7) that ν_τ—a.s. for $u \leq r - \rho$

$$\sigma_{T_u}^\theta(r-u) \leq \frac{1}{1-t_0}\sigma^\theta(\rho) \leq \frac{1}{1-t_0}\sigma\left(\frac{1}{2}\rho\right) \qquad \text{if} \quad \theta \leq \frac{1}{2}\rho.$$

Hence we can apply the bounded convergence theorem on the first integral in (14). Therefore by (12) and (14)

$$
\begin{aligned}
(15)\qquad P(X_{T_r} > r) &\geq E \int_0^{r-\rho} \sigma_{T_u}(r-u)\nu_\tau(du) \\
&\geq P(\tau \geq T_r) - \liminf_{\theta\downarrow 0} E \int_0^{\rho} \sigma^\theta_{T_{r-u}}(u)\mu(du).
\end{aligned}
$$

For $\rho \downarrow 0$ and $t_0 \uparrow 1$ we have $OP(\tau \geq T_r) \uparrow 1$. Hence it is sufficient to prove that for any t_0, the last term in (15) tends to 0 as $\rho \downarrow 0$. Because $\tau \leq \tau_\rho$ and μ is concentrated on $[0, \rho]$ we have μ—a.s.

$$
(16)\qquad \sigma^\theta_{T_{r-u}}(u) = \frac{1}{\theta}\int_{u-\theta}^{u} \sigma_{T_{r-u}}(x)dx \leq \frac{1}{\theta}\int_{u-\theta}^{u} 2\sigma(x)\,dx = 2\,\sigma^\theta(u).
$$

Hence it suffices to prove

$$
(17)\qquad \liminf_{\theta\downarrow 0} E \int \sigma^\theta(u)\mu(du) \to 0 \qquad \text{for} \quad \rho \downarrow 0.
$$

The proof will use that

$$
E \int_0^r \sigma_{T_u}(r-u)\nu\, du)
$$

is a probability because of (12) and hence

$$
(18)\qquad E\int \sigma(u)\mu(du) \leq E\int_0^\rho 2\sigma_{T_{r-u}}(u)\mu(du) \to 0 \qquad \text{for} \quad \rho \downarrow 0.
$$

Based on the behavior of

$$
J(x) := \frac{\frac{1}{x}\int_0^x \sigma(y)\,dy}{\sigma(x)}, \qquad x > 0,
$$

we distinguish three subcases in the proof of (17), that have to be dealt with by increasingly refined estimates.

Case 1. $\limsup_{x\downarrow 0} J(x) < \infty$. Take ρ so small that $\sup_{x\leq\rho} J(x) \leq B < \infty$. Because σ is nonincreasing on $(0, \infty)$, we have if $0 < u < b$

$$
\frac{1}{u}\int_0^u \sigma(x)dx \geq \frac{1}{b-a^+}\int_{a^+}^b \sigma(x)dx, \qquad \text{where} \quad a^+ := a \vee 0.
$$

Hence μ—a.s. $J(u) \leq B$ and therefore

$$
(19)\qquad \sigma^\theta(u) = \frac{1}{\theta}\int_{u-\theta}^u \sigma(x)dx \leq \frac{1}{\theta}\frac{u-(u-\theta)^+}{u}\int_0^u \sigma(x)dx \leq J(u)\sigma(u) \leq B\sigma(u).
$$

Therefore by (18)

$$
E\int \sigma^\theta(u)\mu(du) \leq B\,E\int \sigma(u)\mu(du) \to 0 \text{ for } \rho \downarrow 0.
$$

Case 2. $\liminf_{x\downarrow 0} J(x) < \infty$, $\limsup_{x\downarrow 0} J(x) = \infty$. Suppose $A > \liminf_{x\downarrow 0} J(x)$ and take a sequence $x_n \downarrow 0$ such that $J(x_n) \le A$. Define for any $B > A$

$$y_n := \inf\{x > 0 : J(y) \le B \quad \text{for} \quad y \in [x, x_n]\}.$$

Note that y_n is a continuity point of J for otherwise $J(y_n-) < J(y_n) \le B$, contradicting the definition of y_n. Hence $J(y_n) = B$. We also have $y_n \le (A/B)x_n$ because

$$y_n B = y_n J(y_n) = \frac{\int_0^{y_n} \sigma(x)\,dx}{\sigma(y_n)} \le \frac{\int_0^{x_n} \sigma(x)\,dx}{\sigma(x_n)} = x_n J(x_n) \le x_n A.$$

Let $\theta_n := y_n$. For $u \in [y_n, x_n]$ we have by (19) that $\sigma^{\theta_n}(u) \le B\,\sigma(u)$ and for $u \in [0, y_n)$ we have the same bound because $\sigma^{\theta_n}(u) \le \sigma^{\theta_n}(y_n) = B\,\sigma(y_n) \le B\,\sigma(u)$. Hence

$$E\int_0^{x_n} \sigma^{\theta_n}(u)\mu(du) \le B\,E\int_0^{x_n} \sigma(u)\mu(du) \to 0 \quad \text{for} \quad n \to \infty \tag{20}$$

because $E\int \sigma(u)\mu(du) < \infty$ by (18).

Note that if B is chosen such that $B \ge 2A$ then $x_n \ge 2\,\theta_n$. To obtain (17) we still need other estimates. Let

$$D_\theta := \{u \ge 2\theta : \sigma^\theta(u) \ge 2\,\sigma(u)\};\ C_\theta := [2\theta, \infty) - D_\theta.$$

Then

$$E\int_{C_\theta} \sigma^\theta(u)\mu(du) \le 2\,E\int \sigma(u)\mu(du) \tag{21}$$

becomes arbitrarily small if $\rho \downarrow 0$ by (18). Lemma 8 will permit us to conclude

$$E\int_{D_{\theta_n}} \sigma^{\theta_n}(u)\mu(du) \le \frac{8}{J(\theta_n)} = \frac{8}{B}. \tag{22}$$

Because B may be any number exceeding A, (22) can be made arbitrarily small. Hence (20), (21) and (22) imply (17).

To prove Lemma 8 we use the estimate:

LEMMA 7. $E\,\nu_\tau[x - \theta, x] \le \dfrac{4\theta}{\int_0^\theta \sigma(y)\,dy}$, $\theta \le \rho$.

PROOF. By (12)

$$1 \ge \frac{1}{2\theta}\int_{x-\theta}^{x+\theta} E\int_0^y \sigma_{T_u}(y - u)\nu(du)\,dy.$$

Replace $\int_0^y$ by $\int_0^{x+\theta}$, exchange integrals and use the definitions of $\tau \le \tau_\rho$ and ν_τ to get

$$2\theta \ge E\int_0^{x+\theta}\int_{x-\theta}^{x+\theta} \sigma_{T_u}(y - u)\,dy\,\nu_\tau(du)$$

$$\ge E\int_{x-\theta}^{x}\int_u^{u+\theta} \sigma_{T_u}(y - u)\,dy\,\nu_\tau(du) \ge E\int_{x-\theta}^{x}\int_0^\theta \tfrac{1}{2}\,\sigma(y)dy\,\nu_\tau(du). \qquad \square$$

LEMMA 8. $E \int_{D_\theta} \sigma^\theta(u)\mu(du) \le \frac{8}{J(\theta)}$.

PROOF. Define $\lambda_1 := \inf D_\theta$, $\lambda_k := \inf D_\theta \cap [\lambda_{k-1} + \theta, \infty)$, $k > 1$. Note that $\lambda_k \in D_\theta$ and that $D_\theta \subset \cup_{k\ge1}\omega_k$, where $\omega_k := [\lambda_k, \lambda_k + \theta)$. For $k \ge 1$

$$\sigma^\theta(\lambda_k) \le \sigma(\lambda_k - \theta) \le \sigma(\lambda_{k-1}) \le ½\, \sigma^\theta(\lambda_{k-1}) \le \cdots \le \frac{1}{2^{k-1}}\, \sigma^\theta(\lambda_1) \le \frac{1}{2^{k-1}}\, \sigma(\theta).$$

Hence

$$E \int_{D_\theta} \sigma^\theta(u)\mu(du) \le \textstyle\sum_{k\ge1} E \displaystyle\int_{\omega_k} \sigma^\theta(\lambda_k)\mu(du) \le \textstyle\sum_{k\ge1} \frac{1}{2^{k-1}}\, \sigma(\theta)\, \sup_{k\ge1} E\mu(\omega_k)$$

$$\le \frac{1}{1 - ½}\, \sigma(\theta) \frac{4\theta}{\int_0^\theta \sigma(x)\, dx} = \frac{8}{J(\theta)}.$$

Here the last inequality used Lemma 7. □

Case 3. $\lim_{x\downarrow0} J(x) = \infty$. In case 2 we used (20) to prove that

$$E \int_0^{2\theta} \sigma^\theta(u)\mu(du) \to 0 \tag{23}$$

along some subsequence $\theta \downarrow 0$. This part of the argument of case 2 has to be revised to deal with case 3. Define $\Sigma(y) := \int_0^y \sigma(x)dx$ and $T(y) := E \int_0^y \Sigma(u)\mu(du)$. Because $E \int_0^{2\theta} \sigma^\theta(u)\mu(du) \le \frac{1}{\theta} E \int_0^{2\theta} \Sigma(y)\mu(dy) \le \frac{T(2\theta)}{\theta}$ it is sufficient to show that $\liminf_{y\downarrow0} \frac{1}{y}\, T(y) = 0$. We have to disprove $\frac{T(y)}{y} \ge \alpha > 0$ for sufficiently small y. The function $g(y) := -\frac{\sigma(y)}{\Sigma(y)}$ is nondecreasing. Note that it suffices to prove $\int_0^r T(y)dg(y) < \infty$ and $\int_0^z y\, d\, g(y) = \infty$ for all $z > 0$. By exchanging integrals

$$\begin{aligned}\int_0^r T(y)dg(y) &= \int_0^r E \int_0^y \Sigma(u)\mu(du)\; d - \frac{\sigma(y)}{\Sigma(y)}\\ &= E \int_0^r \Sigma(u) \left(\frac{\sigma(u)}{\Sigma(u)} - \frac{\sigma(r)}{\Sigma(r)}\right)\mu(du)\\ &\le E \int_0^r \sigma(u)\mu(du) < \infty\end{aligned}$$

by (18). Furthermore by using partial integration

$$\int_0^z y\, d - \frac{\sigma(y)}{\Sigma(y)} = - \frac{\sigma(y)}{\Sigma(y)} \cdot y \,\Big|_0^z + \int_0^z \frac{\sigma(y)}{\Sigma(y)}\, dy.$$

Because $y\, \sigma(y) \le \Sigma(y)$ the first term on the right is finite. The second term equals $\log \Sigma(y)\,|_0^z = \infty$, $z > 0$. Hence (23) holds along some subsequence. Using (21) and Lemma 8 we obtain (17).

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