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ON THE LAW OF LARGE NUMBERS

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Suppose X_n is an i.i.d. sequence of random variables with mean μ and that t_n is a nondecreasing sequence of positive integers such that $t_n \le n$. Let $S_n = X_1 + \cdots + X_n$. We give conditions under which

$$\max_{t_n \le k \le n} \left| \frac{S_n - S_{n-k}}{k} - \mu \right| \to 0$$

almost surely and we discuss sharpness.

1. Introduction. The motivation for the work done here was a specific statistical problem. Suppose X_k is an independent sequence of random variables whose distributions approach (in some appropriate sense) a fixed distribution F with mean μ. Then under mild regularity conditions $S_n/n \to \mu$ almost surely where $S_n = X_1 + \cdots + X_n$. There will usually be bias (or nonrandom error) associated with the earlier X_k's. One might hope to reduce this bias by "throwing away" some of these earlier X_k's, by considering averages $(S_n - S_{n-t_n})/t_n$. It is still desirable that $(S_n - S_{n-t_n})/t_n \to \mu$ almost surely. How many X_k's must be kept (how fast must t_n grow) in order to get this almost sure convergence?

In this paper we give some answers to this question in the case when X_n is an i.i.d. sequence. Note that if lim sup $(n/t_n) < \infty$ so that one asymptotically keeps at least some fixed proportion of the X_k's, then

$$\frac{S_n - S_{n-t_n}}{t_n} = \frac{S_n}{n} \cdot \frac{n}{t_n} + \frac{S_{n-t_n}}{n - t_n}\left(1 - \frac{n}{t_n}\right).$$

Since n/t_n is bounded it easily follows that $(S_n - S_{n-t_n})/t_n \to EX_k$ almost surely whether $n - t_n \to \infty$ or not. On the other hand, if $t_n \equiv 1$ then $(S_n - S_{n-t_n})/t_n = X_n$ "oscillates" over the support of the distribution F.

Throughout this paper X_k will be an i.i.d. sequence of random variables having finite mean μ and distribution function F. $m(\theta) = Ee^{\theta X_k}$ and $S_n = X_1 + \cdots + X_n$. C will be used to denote various positive constants whose exact values are irrelevant.

In Section 2 the X_k's have a finite moment generating function in an open interval about μ. In Section 3 we require $E|X_k|^r < \infty$ for some $r \ge 1$. We consider random sequences t_n since one might wish to decide how many X_k's to keep (throw out) based on the data. It is clear that most of the results presented here are true (with appropriate modifications) if the tail probabilities $P\{|X_k - EX_k| \ge t\}$ have appropriate uniform bounds.

2. Results assuming moment generating functions.

THEOREM 2.1. *Suppose*

(2.1) *$m(\theta)$ is finite for θ in some open interval containing μ and*

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(2.2) *$\{t_n\}$ is a sequence of positive integers such that $t_n \le n$ and $t_n/(\log n) \to \infty$.*

Then

$$\max_{t_n \le k \le n} \left| \frac{S_n - S_{n-k}}{k} - \mu \right| \to 0 \quad \textit{almost surely.} \tag{2.3}$$

PROOF. Fix $\epsilon > 0$. According to a classical result due to Cramér [2] there is a ρ such that $0 \le \rho < 1$ and such that for every n and k

$$P\left\{ \left| \frac{S_n - S_{n-k}}{k} - \mu \right| \ge \epsilon \right\} \le 2\rho^k \text{ so that}$$

$$P\left\{ \max_{t_n \le k \le n} \left| \frac{S_n - S_{n-k}}{k} - \mu \right| \ge \epsilon \right\} \le \frac{2}{1-\rho} \exp\{t_n \log \rho\}.$$

Now

$$\sum_{n=1}^{\infty} \exp\{t_n \log \rho\} = \sum_{n=1}^{\infty} \exp\left\{ \frac{t_n}{\log n} (\log n)(\log \rho) \right\}.$$

If n is large enough, say $n \ge n_0$, then $t_n(\log \rho)/(\log n) < -2$ so that the expression above is bounded by $\sum_{n=1}^{n_0} \exp\{t_n \log \rho\} + \sum_{n>n_0} n^{-2}$ and is finite. Thus

$$\sum_{n=1}^{\infty} P\left\{ \max_{t_n \le k \le n} \left| \frac{S_n - S_{n-k}}{k} - \mu \right| \ge \epsilon \right\} < \infty$$

so by the Borel-Cantelli Lemma

$$P\left\{ \max_{t_n \le k \le n} \left| \frac{S_n - S_{n-k}}{k} - \mu \right| \ge \epsilon \text{ infinitely often} \right\} = 0.$$

Since $\epsilon > 0$ was arbitrary the theorem is proved.

THEOREM 2.2. *Suppose* (2.1) *holds and that*

(2.4) *$\{t_n\}$ is a sequence of positive integer valued random variables such that $t_n \le n$ and $t_n/(\log n) \to \infty$ almost surely.*

Then (2.3) *holds.*

PROOF. Fix $\epsilon > 0$. For each positive integer m there exists a positive integer n_m such that

$$P\left\{ \inf_{n \ge n_m} \frac{t_n}{\log n} \ge m \right\} \ge 1 - \epsilon 2^{-m}.$$

Assume that $\{n_m\}$ has been chosen to be strictly increasing. Now define

$$t_n^* = 1 \quad \text{for} \quad 1 \le n < n_1 \tag{2.5a}$$

and for $m \ge 1$ define

$$t_n^* = [m(\log n)] \quad \text{for} \quad n_m \le n < n_{m+1} \tag{2.5b}$$

where $[C]$ is the greatest integer in C. Then $\{t_n^*\}$ satisfies condition (2.2)—and the conclusion (2.3)—and if $A = \{t_n \ge t_n^* \text{ for all } n\}$, then $P(A) \ge 1 - \epsilon$. It follows that

$$P\left\{ \max_{t_n \le k \le n} \left| \frac{S_n - S_{n-k}}{k} - \mu \right| \to 0 \right\}$$

$$\ge P\left(\left\{ \max_{t_n^* \le k \le n} \left| \frac{S_n - S_{n-k}}{k} - \mu \right| \to 0 \right\} \cap A \right) \ge 1 - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the proof is complete.

SHARPNESS. The preceding proofs seem to have been fairly "sloppy" and one might conjecture that the results are not nearly sharp, but that is not the case.

Suppose $0 < p < 1$, $q = 1 - p$, $\{X_n\}$ is an i.i.d. sequence of random variables such that $P\{X_n = p\} = q$ and $P\{X_n = -q\} = p$, and that $t_n = \max\{1, [b(\log n)]\}$ where $b > 0$. Let $m_n = [2bn(\log n)]$.

For large enough n we have $m_{n+1} - m_n \geq [2b(\log n)] > t_{m_n}$ so that the "blocks" of random variables $\{X_{m_n - t_{m_n}}, X_{m_n - t_{m_n}+1}, \cdots, X_{m_n}\}$ are disjoint and hence independent. We will argue that there is an $\epsilon > 0$ such that

$$\sum_{n=1}^{\infty} P\left\{\frac{S_{m_n} - S_{m_n - t_{m_n}}}{t_{m_n}} > \epsilon\right\} = \infty \quad \text{so that} \quad P\left\{\frac{S_{m_n} - S_{m_n - t_{m_n}}}{t_{m_n}} > \epsilon \text{ i.o.}\right\} = 1$$

and hence $(S_n - S_{n-t_n})/t_n$ does not converge to 0 almost surely (i.e., (2.3) does not hold).

Let $\rho_\epsilon = \min_{\theta>0}(e^{-\epsilon\theta}m(\theta))$. This minimum certainly exists (at least for ϵ small enough) and it is easy to see that $\lim_{\epsilon\downarrow 0} \rho_\epsilon = 1$.

According to Theorem 1 of Bahadur and Rao [1] the series

$$\sum_{n=1}^{\infty} P\left\{\frac{S_{m_n} - S_{m_n - t_{m_n}}}{t_{m_n}} > \epsilon\right\} \quad \text{and} \quad \sum_{n=1}^{\infty} (\rho_\epsilon)^{t_{m_n}}/(t_{m_n})^{1/2}$$

both converge or both diverge. However

$$\sum_{n=1}^{\infty} (\rho_\epsilon)^{t_{m_n}}/(t_{m_n})^{1/2} \geq C + \sum_{n=1}^{\infty} (\rho_\epsilon)^{2b\log n}/(2b\log n)^{1/2} = C + \sum_{n=1}^{\infty} n^{2b(\log\rho_\epsilon)}/(2b\log n)^{1/2}.$$

This last series diverges if ϵ is close enough to zero that $2b(\log \rho_\epsilon) > -\frac{1}{2}$.

THEOREM 2.3 *Suppose $m(\theta) = \infty$ for all $\theta > \mu$ or all $\theta < \mu$. Then there exists a nondecreasing sequence $\{t_n\}$ satisfying* (2.2) *such that*

$$P\left\{\max_{t_n \leq k \leq n}\left|\frac{S_n - S_{n-k}}{k} - \mu\right| \to 0\right\} = 0. \tag{2.6}$$

PROOF. For notational convenience we assume $\mu = 0$. If $\theta > 0$ then $\sum_{n=1}^{\infty} P\{\theta|Y| \geq \log n\} < \infty$ if and only if $Ee^{\theta|Y|} < \infty$. It follows that $\sum_{n=1}^{\infty} P\{|X_n| \geq m \log n\} = \infty$ for every positive integer m. Choose n_m recursively so that $n_m/(\log n_m) \geq m + 1$ and so that $\sum_{n=n_{m-1}+1}^{n_m} P\{|X_n| \geq m \log n\} \geq 1$. Then if $t_n = \max\{1, [m(\log n)]\}$ for $n_{m-1} < n \leq n_m$ (with $n_0 = 0$), it follows that t_n is a nondecreasing sequence satisfying (2.2) and that $P\{|X_n| \geq t_n$ infinitely often$\} = 1$. Let $k_n = \max\{t_n, t_{n-1} + 1\}$. Then if $|X_n| \geq t_n$ and n is large enough

$$\left|\frac{S_{n-1} - S_{n-k_n}}{k_n - 1} - \frac{S_n - S_{n-k_n}}{k_n}\right| \geq \left|\frac{X_n}{k_n}\right| - \frac{1}{k_n}\left|\frac{S_{n-1} - S_{n-k_n}}{k_n - 1}\right| \geq \frac{4}{5} - \left|\frac{S_{n-1} - S_{n-k_n}}{k_n - 1}\right|$$

so that either

$$\left|\frac{S_{n-1} - S_{n-k_n}}{k_n - 1}\right| > \frac{1}{5} \quad \text{or else} \quad \left|\frac{S_n - S_{n-k_n}}{k_n}\right| > \frac{1}{5}.$$

It follows that

$$P\left\{\max_{t_n \leq k \leq n}\left|\frac{S_n - S_{n-k}}{k}\right| > \frac{1}{5} \text{ i.o.}\right\} = 1.$$

3. Results assuming $E|X_k|^r$ is finite. We first need two lemmas.

LEMMA A. *Suppose $t > 1$. Then $E|X_k|^t < \infty$ and $EX_k \equiv 0$ if and only if*

$$\sum_{n=1}^{\infty} n^{t-2} P\{\max_{1\leq k\leq n}|S_k| \geq n\epsilon\} < \infty \quad \textit{for every} \quad \epsilon > 0. \tag{3.1}$$

PROOF. This is just a mild strengthening—when $t > 1$—of Theorem 1 of Katz [3].

Katz' Theorem 1 is mainly a corollary to his Theorem 2 when $t > 1$. (Note that $r > 1$ in Theorem 2 so the case $t = r = 1$ is not covered.) Katz' proof of his Theorem 2 is modified by setting $A_n = \{\max_{1 \le k \le n} | S_k | > n^{r/t}\}$ and $A_n^{(3)} = \{\max_{1 \le k \le n} | \sum'^{k}_{\alpha=1} X_\alpha | > 2^{(i-2)r/t}\}$ and by using corollary 3.3.2 on page 144 of [5] to help in bounding the moment $E[\max_{1 \le k \le n} | \sum'^{k}_{\alpha=1} X_\alpha |]^{2M_j}$ obtained when Markov's inequality is applied to bound $P(A_n^{(3)})$.

Katz' Theorem 1 for $t = 1$ comes from Spitzer's Theorem 4.1 in [4]. It is not clear to the authors whether Lemma A is true for $t = 1$, but the modification indicated here will not work in that case.

LEMMA B. *Suppose* $r > 1$, $E | X_k |^r < \infty$, $EX_k \equiv 0$, $b > 0$, $\epsilon > 0$, *and* $\alpha = (r-1)^{-1}$. *Then*

$$\sum_{n=1}^{\infty} P\{\max_{1 \le k \le bn^\alpha} | S_k | \ge n^\alpha \epsilon\} < \infty. \tag{3.2}$$

PROOF. Suppose $r \ge 2$ so that $\alpha \le 1$. Let $p(a, b) = P\{\max_{1 \le k \le a} | S_k | \ge b\}$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} p(bn^\alpha, n^\alpha \epsilon) &= \sum_{m=1}^{\infty} \sum_{\{n | m-1 < bn^\alpha \le m\}} p(bn^\alpha, n^\alpha \epsilon) \\ &\le \sum_{m=1}^{\infty} \sum_{\{n | m-1 < bn^\alpha \le m\}} p(m, Cm\epsilon) \\ &= \sum_{m=1}^{\infty} \#\{n \mid m-1 < bn^\alpha \le m\} p(m, Cm\epsilon) \\ &\le \sum_{m=1}^{\infty} Cm^{\frac{1}{\alpha}-1} p(m, Cm\epsilon). \end{aligned} \tag{3.3}$$

Now $(1/\alpha) - 1 = r - 2$ so (3.3) is finite by Lemma A.

Now suppose $1 < r < 2$ so that $\alpha > 1$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} p(bn^\alpha, n^\alpha \epsilon) &\le \sum_{n=1}^{\infty} [b(n+1)^\alpha - bn^\alpha - 1]^{-1} \sum_{\{m | bn^\alpha \le m < b(n+1)^\alpha\}} p(m, Cm\epsilon) \\ &\le C \sum_{n=1}^{\infty} n^{-(\alpha-1)} \sum_{\{m | bn^\alpha \le m < b(n+1)^\alpha\}} p(m, Cm\epsilon) \\ &\le C \sum_{m=1}^{\infty} m^{-(\alpha-1)/\alpha} p(m, Cm\epsilon). \end{aligned}$$

Which, again, is finite from Lemma A.

THEOREM 3.1. *Suppose* $r \ge 1$, $EX_k \equiv \mu$ *and*

(3.4) $\{t_n\}$ *is a sequence of integers such that* $1 \le t_n \le n$ *and such that for* $n \ge n_0$ *we have* $t_n = [bn^{1/r}]$ *where* $0 < b$ (*if* $r = 1$ *then* $0 < b \le 1$).

Then (2.3) *holds if and only if* $E | X_k |^r < \infty$.

PROOF. For notational convenience we assume that $\mu = 0$ throughout this proof. We first prove that (2.3) holds if $E | X_k |^r < \infty$. The case $r = 1$ is a special (and easy) case. We consider it first.

Suppose $\epsilon > 0$. Fix ω such that $n^{-1} S_n(\omega) \to 0$ and then choose n' so that $| n^{-1} S_n(\omega) | < b\epsilon$ if $n \ge n'$. We get

$$\begin{aligned} \left| \frac{S_n - S_{n-k}}{k} \right| &\le \left| \frac{S_n}{n} \cdot \frac{n}{k} \right| + \left| \frac{S_{n-k}}{n-k} \cdot \frac{n-k}{k} \right| I_{\{n-k<n'\}}(k) \\ &\quad + \left| \frac{S_{n-k}}{n-k} \cdot \frac{n-k}{k} \right| I_{\{n-k \ge n'\}}(k), \end{aligned}$$

$$\max_{[bn] \le k \le n} \left| \frac{S_n - S_{n-k}}{k} \right| \le \left| \frac{S_n}{n} \right| \cdot \frac{n}{[bn]} + \max_{1 \le j < n'} \left| \frac{S_j}{j} \right| \cdot \frac{n'}{[bn]} + (b\epsilon) \cdot \frac{n}{[bn]},$$

and

$$\limsup \max_{[bn]\le k\le n} \left|\frac{S_n - S_{n-k}}{k}\right| \le 0\cdot\left(\frac{1}{b}\right) + \max_{1\le j<n'} \left|\frac{S_j}{j}\right| \cdot 0 + \epsilon = \epsilon.$$

Since $n^{-1}S_n(\omega) \to 0$ almost surely it follows that

$$P\left\{\limsup \max_{[bn]\le k\le n} \left|\frac{S_n - S_{n-k}}{k}\right| \le \epsilon\right\} = 1$$

for every $\epsilon > 0$ so that (2.3) holds.

Now suppose $r > 1$. It suffices to prove that

$$P\left\{\max_{t_n\le k\le n} \left|\frac{S_n - S_{n-k}}{k}\right| \ge \epsilon \text{ i.o.}\right\} = 0 \tag{3.5}$$

for every $\epsilon > 0$.

Fix $\epsilon > 0$. Define $\beta = r/(r-1)$ and $n_k = [ak^\beta]$ with $k > 0$. By the Mean Value Theorem there exists $x_k \in (k-1, k)$ so that $k^\beta - (k-1)^\beta = \beta(x_k)^{\beta-1}$. Note that

$$\frac{n_k - n_{k-1}}{t_{n_k}} \simeq \frac{a(k^\beta - (k-1)^\beta)}{ba^{1/r}k^{\beta-1}} = \left(\frac{a^{1/\beta}\beta}{b}\right)\left(\frac{x_k}{k}\right)^{\beta-1} \to a^{1/\beta}\beta b^{-1} \tag{3.6}$$

and that

$$\frac{t_{n_k}}{ba^{1/r}k^{\beta-1}} \simeq 1 \simeq \frac{t_{n_k}}{t_{n_{k-1}}}. \tag{3.7}$$

Let a be such that $a^{1/\beta}\beta b^{-1} = 5/8$. Fix b' and b'' so that $b' > ba^{1/r} > b'' > 0$. Then from Lemma B, $\sum_{k=1}^\infty P\{\max_{1\le j\le b'k^\alpha} |S_j| \ge b''k^\alpha\epsilon/7\} < \infty$ where $\alpha = 1/(r-1) = \beta - 1$ so that

$$\textstyle\sum_{k=1}^\infty P\{\max_{1\le j\le t_{n_k}} |S_j| \ge t_{n_k}\epsilon/7\} < \infty.$$

Let

$$T_k = \max_{1\le j\le t_{n_k}} |S_{n_k} - S_{n_k-j}|$$

and $A_k = \{T_k \ge t_{n_k}\epsilon/7\}$. Then $\sum_{k=1}^\infty P(A_k) < \infty$ and $P\{\omega \mid \omega \in A_k \text{ i.o.}\} = 0$. Fix ω so that $\omega \in A_k$ only finitely often and choose k_0 so that for all $k \ge k_0$

(1) $\omega \notin A_k$

(2) $\frac{1}{2} t_{n_k} < n_k - n_{k-1} < \frac{3}{4} t_{n_k}$,

and

(3) $t_{n_k} < 4\, t_{n_{k-1}}/3$.

Now choose $n > n_{k_0+1}$ and ν so that $t_n \le \nu \le n$. There exist k' and k'' so that $n_{k'} < n \le n_{k'+1}$ and either

(1) $k_0 < k''$ and $n_{k''-1} < n - \nu \le n_{k''}$

or

(2) $k_0 = k''$ and $1 \le n - \nu \le n_{k''}$.

Note that $t_n/(n_{k'+1} - n_{k'}) \ge (t_{n_{k'}}/t_{n_{k'+1}})(t_{n_{k'+1}}/(n_{k'+1} - n_{k'})) > 1$ and that $k'' \le k'$. Then if

$$D = \max_{1\le j\le n_{k_0}} |S_{n_{k_0}}(\omega) - S_{n_{k_0}-j}(\omega)|,$$

for our fixed ω we have

$$\begin{aligned}|S_n - S_{n-\nu}| &\le |S_{n_{k'+1}} - S_n| + |S_{n_{k'+1}} - S_{n_{k'}}| + \textstyle\sum_{k=k''+1}^{k'} |S_{n_k} - S_{n_{k-1}}| + |S_{n_{k''}} - S_{n-\nu}| \\ &\le 2T_{k'+1} + \textstyle\sum_{k=k''+1}^{k'} T_k + (T_{k''} \text{ or } D) \\ &\le 2t_{n_{k'+1}}\epsilon/7 + \textstyle\sum_{k=k''+1}^{k'} t_{n_k}\epsilon/7 + (t_{n_{k''}}\epsilon/7 + D)\end{aligned}$$

$$\leq \frac{\epsilon}{7}\left[\frac{8}{3}t_n + \sum_{k=k''+1}^{k'} t_{n_k} + t_n\right] + D$$

$$\leq \frac{\epsilon}{7}[8\nu/3 + \sum_{k=k''+1}^{k'} 2(n_k - n_{k-1}) + \nu] + D$$

$$\leq \frac{\epsilon\nu}{7}\left[\frac{8}{3} + 2 + 1\right] + D < 6\epsilon\nu/7 + D.$$

Thus

$$\max_{t_n \leq \nu \leq n}\left|\frac{S_n - S_{n-\nu}}{\nu}\right| < 6\epsilon/7 + D/t_n$$

which is less than ϵ if n is large enough. Thus if $\omega \in A_k$ only finitely often then

$$\max_{t_n \leq k \leq n}\left|\frac{S_n - S_{n-k}}{k}\right| \geq \epsilon$$

only finitely often so (3.5) holds and the proof that (2.3) holds if $E|X_k|^r < \infty$ is complete.

If $r = 1$ and $E|X_k|^r = \infty$ then $\mu = \pm\infty$ (or else is undefined) so (2.3) can't possibly hold. The following holds if $\mu = 0$ for $r > 1$ *and* if μ is replaced by 0 in (2.3) when $r = 1$.

It is well known that $E|Y|^r < \infty$ if and only if $\sum_{n=1}^{\infty} P\{|Y| \geq n^{1/r}\} < \infty$. Assume $E|X_k|^r = \infty$. Then $\sum_{n=1}^{\infty} P\{|X_n| \geq n^{1/r}\} = \infty$ so $P\{|X_n| \geq n^{1/r} \text{ i.o.}\} = 1$. Now use the argument of Theorem 2.3.

THEOREM 3.2. *Suppose* $r \geq 1$, $E|X_k|^r < \infty$, $EX_k \equiv \mu$, *and*

(3.8) $\{t_n\}$ *is a sequence of positive integer valued random variables such that* $t_n \leq n$ *and* $\liminf t_n/n^{1/r} > 0$ *almost surely.*

Then (2.3) *holds.*

PROOF. This is essentially an "immediate" corollary to Theorem 3.1.

ADDITIONAL REMARK ON SHARPNESS. If $E|X_k|^r < \infty$ but $E|X_k|^{r'} = \infty$ for all $r' > r$, then (2.3) does not hold if $t_n = [bn^{1/r'}]$. We will, however, construct a sequence $t_n = o(n^{1/r})$ for which (2.3) holds. For each positive integer m choose a positive integer n_m so that $n_m > n_{m-1}$, so that $[(m-1)^{-1}(n_m)^{1/r}] > [(m-2)^{-1}(n_{m-1})^{1/r}]$ for $m \geq 3$, and so that

$$P\left\{\max_{[m^{-1}n^{1/r}] \leq k \leq n}\left|\frac{S_n - S_{n-k}}{k}\right| \leq 2^{-m} \quad \text{for} \quad n \geq n_m\right\} \geq 1 - 2^{-m}.$$

Define $t_n = [n^{1/r}]$ for $1 \leq n \leq n_2$ and for $m \geq 2$ and $n_m < n \leq n_{m+1}$ define $t_n = \max\{[m^{-1} \cdot n^{1/r}], [(n_2)^{1/r}], [2^{-1}(n_3)^{1/r}], \cdots, [(m-1)^{-1}(n_m)^{1/r}]\} = \max\{[m^{-1}n^{1/r}], [(m-1)^{-1}(n_m)^{1/r}]\}$. Note that $\{t_n\}$ is a nondecreasing sequence of positive integers such that $t_n \leq [(m-1)^{-1}n^{1/r}] \leq n$ so that $t_n/n^{1/r} \to 0$. Now for each positive integer m_0 we get

$$P\{\max_{t_n \leq k \leq n}\left|\frac{S_n - S_{n-k}}{k}\right| \leq 2^{-m} \quad \textit{for all } n_m \leq n < n_{m+1} \textit{ and } m \geq m_0\} \geq 1 - 2^{-m_0+1}$$

and (2.3) follows from this.

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