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OPERATOR-STABLE DISTRIBUTION ON R^2 WITH MULTIPLE EXPONENTS

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Operator-stable distributions are the n-dimensional analogues of stable distributions when nonsingular matrices are used for scaling. Every full operator-stable distribution μ has an exponent, that is, a nonsingular linear transformation A such that for every $t > 0$ $\mu^t = \mu t^{-A} * \delta(a(t))$ for some function $a: (0, \infty) \to R^n$. Full operator-stable distributions on R^2 have multiple exponents if and only if they are elliptically symmetric; in this case the characteristic functions are of the form $\exp\{iy'Vw - c|Vy|^\gamma\}$ where V is positive-definite and self-adjoint, $0 < \gamma \le 2$, $c > 0$, and w is a point in R^2.

1. Introduction. Operator-stable distributions are the analogues of stable distributions in n dimensions. A nondegenerate distribution μ on R^n is said to be operator-stable if there exist independent identically distributed n-dimensional random vectors $\{X_n\}$, nonsingular $n \times n$ matrices $\{A_n\}$, and vectors $\{a_n\}$ in R^n such that the sequence $\{A_n \sum_1^n X_k - a_n\}$ converges in law to μ. In the present work attention is restricted to full measures, that is, measures which are not concentrated on a hyperplane in R^n. In [4], Sharpe showed that a full operator-stable distribution μ on R^n is infinitely divisible. If $\hat{\mu}(y)$ denotes the characteristic function of μ, and if $t > 0$, then $\hat{\mu}(y)^t$ is the characteristic function of the infinitely divisible distribution μ^t. Sharpe showed that there is a nonsingular $n \times n$ matrix A and there is a function $a: (0, \infty) \to R^n$ such that for all $t > 0$

$$\mu^t = t^A \mu * \delta(a(t))$$

where $t^A = \exp(A \ln t) = \sum_{k=0}^\infty (A \ln t)^k/k!$, and $t^A\mu = \mu t^{-A}$. We refer to such an A as an exponent for μ. In general, this exponent is not unique. In this paper we characterize those operator-stable distributions on R^2 which do not have unique exponents.

Let $\mathscr{S}(\mu)$, the symmetry group of μ, be the set of all nonsingular matrices B such that for some vector b in R^2, $\mu = B\mu * \delta(b)$, $(B\mu = \mu B^{-1})$. It follows from Theorem 1 of Billingsley [1], that if μ is full then $\mathscr{S}(\mu)$ is a compact subgroup of the general linear group $GL(n, R)$. A classical result (see for example Theorem 5 of Billingsley) says that there exists a closed subgroup $\mathscr{O}_0$ of the group $\mathscr{O}$ of $n \times n$ orthogonal matrices and there exists a positive-definite symmetric matrix V such that $\mathscr{S}(\mu) = V^{-1}\mathscr{O}_0 V$. (Any compact subgroup of $GL(n, R)$ is of this form.) Our first theorem is

THEOREM 1. *Let μ be full and operator-stable on R^2. Then μ has more than one exponent if and only if there exists a positive-definite symmetric matrix V such that $\mathscr{S}(\mu) = V^{-1}\mathscr{O}V$, where $\mathscr{O}$ denotes the group of* 2×2 *orthogonal matrices.*

The proof of Theorem 1 will also yield the following

COROLLARY. *Let μ be full and operator-stable on R^2. If $\mathscr{S}(\mu) = V^{-1}\mathscr{O}V$, where V is some positive-definite symmetric matrix, then the set $\mathscr{E}(\mu)$ of exponents for μ is*

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$$\mathscr{E}(\mu) = \left\{ V^{-1}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} V: -\infty < \beta < \infty \right\}$$

where $\alpha \geq \frac{1}{2}$ is some real number. It follows that if A is an exponent for μ, if A is not a multiple of the identity, and if A has real eigenvalues, then A is the unique exponent for μ.

REMARK. An exponent A for a full operator-stable distribution μ may have complex conjugate eigenvalues and yet be unique. For example, suppose that

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \quad \alpha > \tfrac{1}{2},$$

and suppose further that M is a Lévy measure concentrated on a single orbit of t^A and $t^A M = t \cdot M$. If $y = (y_1, y_2)'$, and if

$$\log \hat{\mu}(y) = \int_{R^2 \setminus \{0\}} \left(e^{iy'x} - 1 - \frac{iy'x}{1 + x'x} \right) M(dx),$$

then A is the unique exponent for μ. ($\mathscr{S}(\mu) = \{I\}$.)

Next we consider the Gaussian case. The following theorem is included for the sake of completeness; the easy proof is omitted.

THEOREM 2. *Let μ be a full Gaussian measure on R^2. Then μ is operator-stable with a positive-definite covariance matrix $\sum$. Let V denote the positive-definite symmetric square root of $\sum^{-1}$. Then $\mathscr{S}(\mu) = V^{-1}\mathscr{O}V$ and*

$$\mathscr{E}(\mu) = \left\{ V^{-1}\begin{pmatrix} \frac{1}{2} & -\beta \\ \beta & \frac{1}{2} \end{pmatrix} V: -\infty < \beta < \infty \right\}.$$

Our last theorem completes the characterization.

THEOREM 3. *Let μ be full and operator-stable on R^2 and suppose that μ is not purely Gaussian. If μ has more than one exponent, then there exists a positive-definite symmetric matrix V such that the characteristic function $\hat{\mu}(y)$ is given by*

$$\hat{\mu}(y) = \exp(iy'Vv - c\,|Vy|^{\gamma}),$$

where $v \in R^2$, $c > 0$ and $0 < \gamma < 2$. Conversely, if the characteristic function $\hat{\mu}(y)$ of μ is of the above form for some positive-definite symmetric matrix V, then

$$\mathscr{E}(\mu) = \left\{ V \begin{pmatrix} \frac{1}{\gamma} & -\beta \\ \beta & \frac{1}{\gamma} \end{pmatrix} V^{-1}: -\infty < \beta < \infty \right\}$$

$\mathscr{S}(\mu) = V\mathscr{O}V^{-1}$, and $d(VM) = kr^{-1-\gamma}\,dr\,d\theta$ in polar coordinates, where M is the Lévy measure of μ.

To prove these theorems we will need to use more of Sharpe's results and we state them below for the convenience of the reader. Their proofs may be found in Sharpe's fundamental paper [4].

(A). A nonsingular $n \times n$ matrix B is an exponent for some full operator-stable distribution μ on R^n if and only if

(i) the eigenvalues of B lie in the half-plane Re $z \geq \frac{1}{2}$ and

(ii) every eigenvalue of B having real part equal to $\frac{1}{2}$ is a simple root of the minimal polynomial of B.

(B). Let M be the Lévy measure of a full operator-stable distribution on R^n and let B be an exponent for μ. Then for any Borel subset D of $R^n\setminus\{0\}$, $Mt^{-B}(D) = t\cdot M(D)$. That is, $t^B M = t\cdot M$.

(C). Let M be a Lévy measure concentrated on a single orbit $\{t^B x_0 : t > 0\}$ satisfying $t^B M = t\cdot M$ and let $[x_0]$ denote the cyclic subspace generated by x_0 relative to B. Then every eigenvalue of the restriction of B to $[x_0]$ has real part greater than ½.

(D). Any full operator-stable measure μ on R^n can be decomposed into a product $\mu = \mu_1 * \mu_2$ of measures μ_i concentrated on subspaces V_i, $R^n = V_1 \oplus V_2$, where μ_1 is a full Gaussian measure on V_1 and μ_2 is a full operator-stable measure on V_2 having no Gaussian component.

(E). Any Lévy measure M for a full operator-stable distribution on R^n can be represented as a mixture of Lévy measures M_x, where M_x is concentrated on a single orbit $\{t^B x : t > 0\}$ and satisfies $t^B M_x = t\cdot M_x$ where $B \in \mathscr{E}(\mu)$. The measure M_x is characterized by the condition that $sM_x\{t^B x : t > s\}$ is constant for all s.

This paper is organized as follows. In Section 2 a description of the operator t^B is given. This is followed by some preliminary lemmas which are proved in Section 3. The proofs of Theorems 1 and 3 are given in Section 4.

2. A Description of the Operators t^B. The operators t^B on R^2 may be described using matrix representations with respect to a suitable basis.

First suppose that B is diagonalizable with two real eigenvalues, α_1 and α_2, which are not necessarily distinct. Then with respect to a suitable basis $\mathscr{C}$, B is represented by the matrix

$$[B]_{\mathscr{C}} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}.$$

A power series calculation shows that

$$[t^B]_{\mathscr{C}} = \begin{pmatrix} t^{\alpha_1} & 0 \\ 0 & t^{\alpha_2} \end{pmatrix}.$$

Next, suppose that there is a basis $\mathscr{C}$ such that $[B]_{\mathscr{C}}$ is, in Jordan cannonical form,

$$[B]_{\mathscr{C}} = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix},$$

where α is a real number. Putting

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and noting $N^2 = 0$, we see that $[B]_{\mathscr{C}} = \alpha I + N$ and $[t^B]_{\mathscr{C}} = t^{\alpha I + N} = t^{\alpha}(I + N \ln t)$. That is,

$$[t^B]_{\mathscr{C}} = \begin{pmatrix} t^{\alpha} & 0 \\ t^{\alpha} \ln t & t^{\alpha} \end{pmatrix}.$$

Finally, suppose that B has a pair of complex conjugate eigenvalues $\alpha \pm i\beta$. Then there exists a basis $\mathscr{C}$ such that

$$[B]_{\mathscr{C}} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

Another easy power series calculation shows that

$$[t^B]_{\mathscr{C}} = t^{\alpha} \begin{pmatrix} \cos(\beta \ln t) & -\sin(\beta \ln t) \\ \sin(\beta \ln t) & \cos(\beta \ln t) \end{pmatrix}.$$

We let $R(\theta)$ denote the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and note that $\{[t^{B-\alpha I}]_{\mathscr{C}} : t > 0\}$ is a compact group of nonsingular matrices. Then according to a classical theorem (mentioned above) there is a positive-definite symmetric matrix V and a closed subgroup $\mathcal{O}_0$ of the orthogonal matrices such that with respect to the usual basis the operators $\{t^{B-\alpha I}: t > 0\}$ are represented by the matrices $V^{-1}\mathcal{O}_0 V$. Since for every $t > 0$, the determinant $|t^{B-\alpha I}| = 1$, $\mathcal{O}_0$ is a subgroup of the group of rotations. From the matrix representation of $t^{B-\alpha I}$ obtained above, we have that $t^{B-\alpha I} = A^{-1}R(\beta \ln t)A$ with respect to the usual basis, where A is some nonsingular 2×2 matrix. But also, for some function $f: (0, \infty) \to R$, $t^{B-\alpha I} = V^{-1}R(f(t))V$, for all $t > 0$. Thus $R(f(t)) = D^{-1}R(\beta \ln t)D$, where $D = AV^{-1}$. It is clear from the entries of the matrices that $f(t)$ may be chosen to be continuously differentiable on $(0, \infty)$ and satisfy $f(1) = 0$. Then $|f'(t)|^2 = \beta^2/t^2$ follows by differentiating $R(f(t)) = D^{-1}R(\beta \ln t)D$ and then taking determinants. Hence, either $f(t) = \beta \ln t$ or $f(t) = -\beta \ln t$ and so either $t^{B-\alpha I} = V^{-1}R(\beta \ln t)V$ or $t^{B-\alpha I} = V^{-1}R(-\beta \ln t)V$.

3. Lemmas. In this section we establish some preliminary results. The first two lemmas together say that if μ has multiple exponents and if $\mathcal{S}(\mu)$ is a subgroup of the 2×2 orthogonal matrices, $\mathcal{O}$, then $\mathcal{S}(\mu) = \mathcal{O}$. Lemma 3 says that certain exponents are incompatible with a symmetry group that contains all the rotations. Finally, Lemma 4 establishes a form for the Lévy measure of μ when $\alpha I \in \mathcal{E}(\mu)$.

The assumption that $\mathcal{S}(\mu)$ is a subgroup of $\mathcal{O}$ is not very restrictive. Since $\mathcal{S}(\mu)$ is a compact group by Theorem 1 of Billingsley, $\mathcal{S}(\mu) = V^{-1}\mathcal{O}_0 V$ for some symmetric positive-definite matrix V, and for some subgroup $\mathcal{O}_0$ of $\mathcal{O}$, by Theorem 5 of Billingsley. Thus, $\mathcal{S}(V\mu) = \mathcal{O}_0$.

LEMMA 1. *Let μ be full and operator-stable on R^2 and suppose that $\mathcal{S}(\mu)$ is a subgroup of the orthogonal group. If A and B are two distinct exponents for μ, then every rotation is in $\mathcal{S}(\mu)$.*

PROOF. Since A and B are distinct exponents for μ,

$$\mu^t = t^A\mu.\delta(a(t)) = t^B\mu*\delta(b(t)), \qquad t > 0.$$

Thus

$$\mu = t^{-A}t^B\mu.\delta(t^{-A}(b(t) - a(t))), \qquad t > 0,$$

so $t^{-A}t^B \in \mathcal{S}(\mu)$ for $t > 0$. But, $\{t^{-A}t^B: t > 0\}$ is a connected subset of $\mathcal{S}(\mu)$ which contains operators arbitrarily close to the identity I. Since the operator norm of the difference between a reflection and the identity is 2, $\mathcal{S}(\mu)$ contains rotations arbitrarily close to I and so contains every rotation. □

LEMMA 2. *If $\mathcal{S}(\mu)$ contains all the rotations, then $\mathcal{S}(\mu)$ is the complete orthogonal group and there is an $x_0 \in R^2$ such that the characteristic function of $\mu*\delta(-x_0)$ at $y \in R^2$ depends only on $|y|$.*

PROOF. For each θ, $\mu = R(\theta)\mu*\delta(a(\theta))$ for some $a(\theta) \in R^2$. By Theorem 5 of Billingsley [1], there is a group $\mathcal{O}_0$ of orthogonal matrices, $x_0 \in R^2$, and a symmetric positive-definite matrix Z such that for some 0_θ in $\mathcal{O}_0$,

$$R(\theta)x + a(\theta) = Z^{-1}O_\theta Z(x - x_0) + x_0, \qquad \text{for all } x \in R^2.$$

Therefore, $R(\theta) = Z^{-1}O_\theta Z$ and $a(\theta) = x_0 - R(\theta)x_0$. Thus $\mu*\delta(-x_0) = R(\theta)(\mu*\delta(-x_0))$, for all θ. Let $\nu = \mu*\delta(-x_0)$. Then $\hat{\nu}(y) = \hat{\nu}(R(\theta)y)$, for all θ, y. Given y, select θ so that $R(\theta)y = (|y|, 0)$. Thus, $\hat{\nu}(y) = \hat{\nu}(|y|, 0)$. Let T be a reflection. Then

$$T\hat{\nu}(y) = \hat{\nu}(T*y) = \hat{\nu}(|T_y^*|, 0) = \hat{\nu}(|y|, 0) = \hat{\nu}(y).$$

Therefore, $\mu*\delta(-x_0)$ is invariant under all rotations and all reflections. This shows that its characteristic function depends only on $|y|$ and that $\mathscr{S}(\mu)$ contains the group of orthogonal matrices. Again, by Theorem 5 of Billingsley, this latter fact implies $\mathscr{S}(\mu)$ is the group of orthogonal matrices. □

LEMMA 3. *If $\mathscr{S}(\mu)$ contains the rotations and if A is an exponent of μ, then either $A = \alpha I$ or the eigenvalues of A are of the form $\alpha \pm i\beta$, $\alpha \geq 1/2$, $\beta > 0$.*

PROOF. Let $f(y) = |\hat{\mu}(y)|$; then $f^t(y) = f(t^{A^*}y)$ for all t and y. Suppose λ_1, λ_2 are real eigenvalues of A and suppose $\lambda_1 \neq \lambda_2$, say $\lambda_2 > \lambda_1$. Since $R(\theta)\mu^t * \delta(tb(\theta)) = \mu^t = t^A\mu * \delta(a(t))$, we have

$$f(t^{A^*}y) = f(t^{A^*}R(\theta)y), \qquad \text{for all } t, \theta, y.$$

Let S_i be the eigenspace corresponding to λ_i, $i = 1, 2$. Select θ so that $R(\theta)S_1 = S_2$. Then, for $y \in S_1$, $f(t^{\lambda_1}y) = f(t^{\lambda_2}R(\theta)y)$, for all t. Hence, $f(y) = f(t^{\lambda_2-\lambda_1}R(\theta)y)$, for all y in S_1. Letting $t \to \infty$ and applying the Riemann-Lebesgue lemma (which is applicable by a theorem of Sharpe, the proof of which may be found in Hudson [3]), we obtain $f(y) = 0$ for all y in S_1. But μ is infinitely divisible. This contradiction implies that if A has real eigenvalues λ_1, λ_2, then $\lambda_1 = \lambda_2$.

Now suppose that $t^{A^*} = t^\alpha(I + (\ln t)N)$, where $N \neq 0$, $N^2 = 0$. As before, $f(t^{A^*}y) = f(t^{A^*}R(\theta)y)$, for all t, θ, y. Select θ so that for $y \in K \equiv$ null space of N, $R(\theta)y \notin K$. Then

$$f(t^\alpha y) = f(t^\alpha(I + (\ln t)N)R(\theta)y), \qquad \text{for all } \quad y \in K, t.$$

Hence,

$$f(y) = f(R(\theta)y + (\ln t)NR(\theta)y), \qquad \text{for all } \quad y \in K, t.$$

As before, letting $t \to \infty$ yields a contradiction. Thus, t^{A^*} cannot be of the form $t^\alpha(I + (\ln t)N)$. □

Theorem 5 of Sharpe [4] characterizes the Lévy measure M of a full operator-stable measure μ as the mixture of measures concentrated on orbits of t^A, where A is an exponent of μ. In the following lemma we obtain the mixing measure when $A = \alpha I$.

LEMMA 4. *If αI is an exponent of μ, then the Lévy measure M of μ satisfies*

$$M(G) = \int_S \int_0^\infty I_G(t^\alpha x)t^{-2}\, dt\, dx,$$

for any Borel subset G of $R^2\backslash\{0\}$, where S is the unit circle in R^2.

PROOF. Define a measure K on S by

$$K(D) = M\{t^\alpha x : x \in D, t > 1\},$$

for any Borel subset D of S. For $x \in S$, define M_x to be the measure

$$M_x(G) = \int_0^\infty I_G(t^\alpha x)t^{-2}\, dt,$$

for any Borel subset G of $R^2\backslash\{0\}$. Note that $M_x\{t^\alpha x : t > t_0\} = 1/t_0$, so M_x satisfies the condition $t^{\alpha I}M_x = tM_x$ (see the proof of Lemma 6 in Sharpe). Set

$$M(G) = \int_S M_x(G)K(dx).$$

When G is of the form $\{t^\alpha x : x \in D, t > t_0\}$,

$$\tilde{M}(G) = (1/t_0)K(D) = (1/t_0)M\{t^\alpha x : x \in D, t > 1\}.$$

By Proposition 5 of Sharpe, $tM(G) = M(t^{-\alpha}G)$ for all $t > 0$, so $M(t_0^{-\alpha}G) = K(D) = t_0 M(G)$. Consequently, $\tilde{M}(G) = M(G)$ for such G. This implies $\tilde{M} = M$. □

4. Proofs of main results.

PROOF OF THEOREM 1. Assume $\mathscr{S}(\mu) = V^{-1}\mathscr{O}V$. We know that μ has at least one exponent. We will show that μ has infinitely many exponents.

For each $0 \in \mathscr{O}$, $\mu = V^{-1}0V\mu * \delta(a)$ for some a. Hence, $V\mu = 0V\mu * \delta(Va)$ which yields $\mathscr{S}(V\mu) = \mathscr{O}$. There is a one-to-one correspondence between $\mathscr{E}(\mu)$ and $\mathscr{E}(V\mu)$ given as follows. Let $A \in \mathscr{E}(V\mu)$. Then $V\mu^t = t^A V\mu * \delta(a(t))$, which implies $\mu^t = t^{V^{-1}AV}\mu * \delta(V^{-1}a(t))$. Thus, $V^{-1}AV \in \mathscr{E}(\mu)$. Let $B \in \mathscr{E}(\mu)$. Set $C = VBV^{-1}$. Then $(V\mu)^t = V\mu^t = V(t^B\mu * \delta(b(t)))$ $= V(t^{V^{-1}CV}\mu * \delta(b(t))) = t^C V\mu * \delta(Vb(t))$. Thus, $VBV^{-1} \in \mathscr{E}(V\mu)$. Hence, it suffices to show that μ has more than one exponent assuming $\mathscr{S}(\mu) = \mathscr{O}$.

Put $f(y) = |\hat{\mu}(y)|$. As in the proof of Lemma 3, $f(t^{A^*}y) = f(t^{A^*}R(\theta)y)$ holds for all θ, $t > 0$, and y.

Let $A \in \mathscr{E}(\mu)$. By Lemma 3, $t^{A^*} = t^\alpha W^{-1}R(-\beta \ln t)W$, for some symmetric positive-definite matrix W, where $\alpha \geq 1/2$, $\beta \geq 0$. We show that $W = \lambda I$ for some $\lambda > 0$. Suppose λ_1, λ_2 are the real eigenvalues of W and suppose $\lambda_1 < \lambda_2$. Let S_i be the eigenspace corresponding to λ_i, $i = 1, 2$. Since $f(t^{A^*}y) = f(t^{A^*}R(\theta)y)$, we have $f(t^\alpha W^{-1}R(-\beta \ln t)Wy) = f(t^\alpha W^{-1}R(-\beta \ln t)WR(\theta)y)$ and hence $f(W^{-1}R(-\beta \ln t)Wy) = f(W^{-1}R(-\beta \ln t)WR(\theta)y)$ for all t, θ, y. Select t so that $R(-\beta \ln t)y \in S_2$ for $y \in S_1$. Let $y' = R(-\beta \ln t)y$. Then $R(-\beta \ln t)y' = -y$, since S_1 and S_2 are orthogonal. Select θ so that $R(\theta)y = y'$, $y \in S_1$. Then $f((\lambda_1/\lambda_2)y') = f(-(\lambda_2/\lambda_1)y)$, $y \in S_1$. Since $|y| = |y'|$, $f((\lambda_1/\lambda_2)y') = f((\lambda_1/\lambda_2)y)$. Hence, $f((\lambda_2/\lambda_1)y) = f((\lambda_1/\lambda_2)y)$ for all $y \in S_1$. Therefore, $f(y) = f((\lambda_1/\lambda_2)^2 y)$, $y \in S_1$. This leads to the result that $f(y) = f((\lambda_2/\lambda_1)^{2n}y)$ for all $y \in S_1$, $n = 1, 2, \cdots$. By the Riemann-Lebesgue lemma, $f(y) = \lim_n f((\lambda_2/\lambda_1)^{2n}y) = 0$, for all $y \in S_1$. This contradiction yields $\lambda_1 = \lambda_2$.

Hence, we have that if A is an exponent for μ

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

with respect to the usual basis. This implies that αI is an exponent for μ. To see this, note that

$$\begin{aligned} \mu^t &= t^A\mu * \delta(a(t)) = t^{\alpha I}R(\beta \ln t)\mu * \delta(a(t)) \\ &= t^{\alpha I}(\mu * \delta(b(t))) * \delta(a(t)) = t^{\alpha I}\mu * \delta(t^\alpha b(t) + a(t)). \end{aligned}$$

Finally, we show that $\alpha I + \beta' J$ is an exponent for μ, for all $\beta' \geq 0$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, $t^{\beta' J}$ varies over the rotations as t varies, with β' fixed. Hence, $t^{\beta' J} \in \mathscr{S}(\mu)$, for all $t > 0$. Therefore,

$$\mu^t = t^{\alpha I}\mu * \delta(a(t)) = t^{\alpha I}t^{\beta' J}\mu * \delta(b(t)) = t^{\alpha I + \beta' J}\mu * \delta(b(t)).$$

We have shown that $\mathscr{S}(\mu) = V^{-1}\mathscr{O}V$ implies that μ has more than one exponent.

We now assume that μ has more than one exponent. We will show that $\mathscr{S}(\mu) = V^{-1}\mathscr{O}V$, for some symmetric positive-definite matrix V. By Theorem 1 of Billingsley [1], $\mathscr{S}(\mu)$ is a compact group. By Theorem 5 of Billingsley, there is a symmetric positive-definite matrix V such that $\mathscr{S}(V\mu)$ is a subgroup of $\mathscr{O}$. By Lemma 1, $\mathscr{S}(V\mu)$ contains all the rotations, so by Lemma 2, $\mathscr{S}(V\mu) = \mathscr{O}$. Therefore, $\mathscr{S}(\mu) = V^{-1}\mathscr{O}V$. □

PROOF OF THEOREM 3. We first show that μ does not have a Gaussian component. Suppose μ has both a Gaussian and a non-Gaussian component. By Theorem 4 of Sharpe [4], $\mu = \mu_1 * \mu_2$, where μ_i is concentrated on V_i, $i = 1, 2$, $R^2 = V_1 \oplus V_2$, μ_1 is a full Gaussian measure on V_1 and μ_2 is a full operator-stable measure on V_2 with no Gaussian component. Let F_1 and F_2 be projections of R^2 onto V_1 and V_2, respectively, so that $F_1F_2 = F_2F_1 = 0$. Clearly, $(1/2)F_1$ is an exponent for μ_1 on V_1. By Theorem 5 and Lemma 6 of Sharpe, we know there is an $\alpha > 1/2$ such that αF_2 is an exponent for μ_2 on V_2. Since

$$\begin{aligned}\mu^t = \mu_1^t * \mu_2^t &= t^{1/2}F_1\mu_1 * \delta(b_1(t)) * t^\alpha F_2\mu_2 * \delta(b_2(t))\\ &= (t^{1/2}F_1 + t^\alpha F_2)(\mu_1 * \mu_2) * \delta(b_1(t) + b_2(t)),\end{aligned}$$

we have that $(1/2)F_1 + \alpha F_2$ is an exponent for μ. But, $(1/2)F_1 + \alpha F_2$ has two real distinct eigenvalues which contradicts the corollary to Theorem 1. Therefore, μ does not have a Gaussian component. Since μ has more than one exponent, $\mathscr{S}(\mu) = V^{-1}\mathscr{O}V$, for some symmetric positive-definite matrix V, and $\mathscr{E}(\mu) = \{\alpha I + \beta V^{-1}JV : -\infty < \beta < \infty\}$, for some $\alpha > 1/2$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $\nu = V\mu$. Then $\mathscr{S}(\nu) = \mathscr{O}$ and $\mathscr{E}(\nu) = \{\alpha I + \beta J : -\infty < \beta < \infty\}$. Since αI is an exponent for ν, by Lemma 4, the Lévy measure M_1 of ν has the representation

$$M_1(G) = \int_S M_x(G)K(dx),$$

for any Borel subset G of $R^2\setminus\{0\}$, where S is the unit circle, $M_x(G) = \int_0^\infty I_G(t^\alpha x)t^{-2}\,dt$, and $K(D) = M_1\{t^\alpha x : x \in D, t > 1\}$, for D a Borel subset of S. Now since $\mathscr{S}(\nu)$ contains all the rotations, we have for $0 \le \theta < 2\pi$

$$\nu = R(\theta)\nu * \delta(c(\theta))$$

for some function $c : [0, 2\pi) \to R^2$. It follows easily that

$$M_1 = R(\theta)M_1, \qquad 0 \le \theta < 2\pi,$$

and hence

$$K = R(\theta)K, \qquad 0 \le \theta < 2\pi.$$

But, this implies that K is proportional to Lebesgue measure on the circle. Let

$$G = \left\{y : \frac{y}{\|y\|} \in D \quad \text{and} \quad \|y\| > s\right\},$$

where $D \subset S$. Then

$$\begin{aligned}M_x(G) &= \int_0^\infty I_G(t^\alpha x)t^{-2}\,dt\\ &= \begin{cases} s^{-1/\alpha} & \text{if } x \in D \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus

$$\begin{aligned}M_1(G) &= \int_S M_x(G)K(dx) = s^{-1/\alpha}\int_D c_1\,dx\\ &= \int_G \frac{c_1}{\alpha} r^{-(1+(1/\alpha))}\,dr\,d\theta \qquad \text{(in polar coordinates)}.\end{aligned}$$

It follows that $dVM = (c_1/\alpha)r^{-1-\gamma}\,dr\,d\theta$ in polar coordinates, where $\gamma = 1/\alpha$.

We now show that $\hat{\nu}(y) = \exp\{iy'v - c|y|^\gamma\}$, for some $v \in R^2$, some $c > 0$. Since ν is infinitely divisible with no Gaussian component,

$$\hat{\nu}(y) = \exp\left\{iy'v + \int_{R^2\setminus\{0\}} \left(e^{ix'y} - 1 - \frac{ix'y}{1+|x|^2}\right) dM_1(x)\right\},$$

for some $v \in R^2$. Hence, it suffices to show that

$$\int_{R^2\setminus\{0\}} \left(e^{ix'y} - 1 - \frac{ix'y}{1+|x|^2}\right) dM_1(x) = -c\,|y|^\gamma,$$

for some $c > 0$, i.e.,

$$\int_0^{2\pi}\int_0^\infty \left(e^{izr} - 1 - \frac{izr}{1+r^2}\right) kr^{-1-\gamma}\,dr\,d\theta = c\,|y|^\gamma,$$

where $y = (y_1, y_2)$, $k = c_1/\alpha$, $z = y_1 \cos\theta + y_2 \sin\theta$. First, we evaluate

$$\int_0^\infty \left(e^{izr} - 1 - \frac{izr}{1+r^2}\right)\frac{dr}{r^{1+\gamma}}, \qquad 0<\gamma<2.$$

This integral is a familiar one which arises in the canonical representation of univariate stable laws, for example, see pages 168–171 of Gnedenko and Kolmogorov [2]. Hence, the integral is

$$ic_1 z + ic_2|z|^\gamma (\operatorname{sgn} z) - c_3|z|^\gamma, \qquad \text{if} \quad \gamma \neq 1,$$

and

$$ic_4 z + ic_5|z| \ln|z| (\operatorname{sgn} z) - c_6|z|, \qquad \text{if} \quad \gamma = 1,$$

where $c_3 > 0$ and $c_6 > 0$.

Now each of these terms must be integrated over $(0, 2\pi)$ with respect to θ. To do this, one may use the substitution $z = |y| \cos(\theta + \phi)$, for an appropriate ϕ. One finds that the integrals are zero except for the term involving c_3 or c_6. This nonzero integral is $-c_7|t|^\gamma$. Therefore, $\hat{\nu}(y) = \exp\{iy'v - c|y|^\gamma\}$. Since $\nu = V\mu$, $\hat{\mu}(y) = \exp\{iy'V^{-1}v - c|V^{-1}y|^\gamma\}$. Replacing V by V^{-1} we obtain the stated representation for $\hat{\mu}$.

If $\hat{\mu}$ is a characteristic function of this form, then $\mathscr{S}(V^{-1}\mu) = \mathcal{O}$. Hence, $\mathscr{S}(\mu) = V\mathcal{O}V^{-1}$. This implies that $\mathscr{E}(\mu) = \{(1/\gamma)I + \beta VJV^{-1} : -\infty < \beta < \infty\}$ and the Lévy measure M of μ satisfies $dVM = kr^{-1-\gamma}\,dr\,d\theta$ in polar coordinates. □

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