The Annals of Probability
1981, Vol. 9, No. 3, 468–481

COMPARING THE TAIL OF AN INFINITELY DIVISIBLE DISTRIBUTION WITH INTEGRALS OF ITS LÉVY MEASURE

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Let F be an infinitely divisible distribution on $[0, \infty)$, with Lévy measure ν. For all real r, define measures ν_r by $\nu_r(dx) = x^r\nu(dx)$ $(x > 1)$, $= 0$ $(x \le 1)$. For $0 < \alpha < \infty$, and $-\infty < r' < \alpha < r < \infty$, it is proved that $\nu_{r'}(x, \infty)$ is regularly varying (at ∞) with exponent $r' - \alpha$ if and only if $1 - F$ is regularly varying with exponent $-\alpha$ if and only if $\nu_r(0, x]$ is regularly varying with exponent $r - \alpha$. If any of this is the case there follow asymptotic relations between $1 - F$ and either of $\nu_{r'}(x, \infty)$ or $\nu_r(0, x]$. The paper characterises those distributions for which these asymptotic relations hold, some of the characterisations being complete and others assuming that not all moments of F are finite. The characterising classes involve regular variation, second order (de Haan) regular variation, rapid variation, and subexponentiality. An intermediate result is that when F has finite nth and infinite $(n + 1)$th moment, $\int_0^t x^{n+1}\{1 - F(x)\}\, dx \sim \int_0^t x^{n+1}\nu(x, \infty)\, dx$ as $t \to \infty$. The results are applied to generalised gamma convolutions.

1. Introduction and statement of results. Let F be an infinitely divisible distribution function on $[0, \infty)$ with Lévy measure ν determined by

$$f(s) \equiv \int_{0-}^{\infty} e^{-sx}F(dx) = \exp\left\{-\,as - \int_0^\infty (1 - e^{-sx})\nu(dx)\right\}, \qquad s \ge 0.$$

Here, $a \ge 0$ is constant and ν is a Borel measure on $(0, \infty)$ for which $\int_0^\infty \min(1, x)\nu(dx) < \infty$. Being interested in the tail of ν, we assume without further comment that ν has unbounded support. Our convention about intervals of integration is that they exclude {include} finite left {right} end-points, unless qualified as in the first integral above.

In Embrechts et al. (1979) we compared the tails of F and ν. Setting $\bar F \equiv 1 - F$ and $\bar\nu(x) \equiv \nu(x, \infty)$ we proved that $\bar F(x) \sim \bar\nu(x)$ as $x \to \infty$ if and only if F belongs to the subexponential class $\mathscr{S}$, that is, $1 - F*F(x) \sim 2\bar F(x)$. Now the spectral measure of an infinitely divisible F is often expressed other than in Lévy's form ν. For instance, Feller (1971) uses $M(dx) = x^2\nu(dx)$, and also $P(dx)$ given by $P[0, x] = a + \int_0^x y\nu(dy)$, $x \ge 0$. A natural follow-up to our previous paper is to compare $\bar F$ with the tails of these measures, but rather than treating special cases ad hoc we shall compare $\bar F$ with *all* tail integrals of ν weighted by powers of x. To this end we define for each real r the measure

$$\nu_r(dx) = \begin{cases} 0, & x \le 1 \\ x^r\nu(dx), & 1 < x < \infty, \end{cases}$$

and set $\bar\nu_r(x) = \nu_r(x, \infty)$ when finite, and $\nu_r(x) = \nu_r(0, x]$. For $-\infty < \rho < \infty$ let $\mathscr{R}_\rho^{(\infty)}$ denote the class of functions R regularly varying at ∞ with exponent ρ, that is, R on $(0, \infty)$ is measurable, eventually positive, and for each $t > 0$, $\lim_{x\to\infty} R(tx)/R(x) = t^\rho$.

Received April 25, 1979; revised October 15, 1980.
[1] On leave from Katholieke Universiteit Leuven
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AMS 1980 *subject classifications*. 60E07.

Key words and phrases. Generalised gamma convolutions, infinite divisibility, Lévy measures, rapid variation, regular variation, subexponentiality, tails of probability distributions.

Extrapolating from Feller (1971) pages 288, 572–573, one has

PROPOSITION 0. *For* $0 \le \alpha < \infty$,

$$\bar{\nu} \in \mathscr{R}_{-\alpha}^{(\infty)} \Leftrightarrow \bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)} \Rightarrow \lim_{x\to\infty} \bar{F}(x)/\bar{\nu}(x) = 1. \tag{1.1}$$

We give a direct proof of Proposition 0 in an appendix. It also follows immediately from Embrechts et al. (1979), since by Theorem 3 of Chistyakov (1964), $\mathscr{S}$ contains all F for which $\bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)}$, $0 \le \alpha < \infty$. From (1.1) one may deduce, using Karamata's theorem (Feller (1971) page 283),

COROLLARY. *For* $0 < \alpha < \infty$, *and for all* r, r' *satisfying* $-\infty < r' < \alpha < r < \infty$,

$$\bar{\nu}_{r'} \in \mathscr{R}_{r'-\alpha}^{(\infty)} \Leftrightarrow \bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)} \Leftrightarrow \nu_r(\cdot) \in \mathscr{R}_{r-\alpha}^{(\infty)}. \tag{1.2}$$

If any of the equivalent statements of (1.2) are in force, there follow tail comparisons of the forms

$$x^r\bar{F}(x)/\nu_r(x) \to l, \qquad x \to \infty, \tag{1.3}$$

$$x^{r'}\bar{F}(x)/\overline{\nu_{r'}}(x) \to l', \qquad x \to \infty, \tag{1.4}$$

for some limits l, l' depending on r, r' and α. How far can we enlarge the classes of (1.2) while keeping the two-way implications in force? Equivalently, what classes of distributions are characterised by the asymptotic relations (1.3), (1.4)? We call this the *characterisation problem*. Most of our conclusions about it are in Theorems 1 and 2 below. Since the characterising classes turn out to be the regularly varying $\bar{F}$, $\nu_r(\cdot)$ or $\bar{\nu}_r$, it is convenient to turn the problem over, widening it a little, and characterise instead the *classes* of regularly varying $\bar{F}$, $\nu_r(\cdot)$ and $\bar{\nu}_r$ by means of the *relations* (1.3), (1.4) *together with their second-order variants*. In all that follows, μ_k denotes the kth moment $\int_0^\infty x^kF(dx)$. Note that positive-integer moments of F and ν are finite or infinite together:

$$\int_0^\infty x^{k-1}\bar{F}(x)\,dx < \infty \Leftrightarrow \mu_k < \infty \Leftrightarrow \int_0^\infty x^k\nu(dx) < \infty, \qquad k = 1, 2, \cdots.$$

More generally, for $r > 0$ a result of Ramachandran (1969) gives $\nu_r(1, \infty) < \infty \Leftrightarrow \int_0^\infty x^rF(dx) < \infty$. For $r \le 0$ it is of course the case that $\nu_r(1, \infty) < \infty$.

THEOREM 1 *Fix* $r > 0$.
(a)

$$\left.\begin{matrix}\bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)}, \\ 0 \le \alpha < \infty\end{matrix}\right\} \Leftrightarrow \begin{cases} \text{for } 0 \le \alpha < r, & x^r\bar{F}(x)/\nu_r(x) \to r/\alpha - 1, \\ \text{for } \alpha = r, & \nu_r(x, tx]/\{rx^r\bar{F}(x)\} \to \log t, \\ \text{for } r < \alpha < \infty, & x^r\bar{F}(x)/\overline{\nu_r}(x) \to 1 - r/\alpha. \end{cases}$$

(b) *When* $\nu_r(1, \infty) = \infty$,

$$\left.\begin{matrix}\nu_r(\cdot) \in \mathscr{R}_{r-\alpha}^{(\infty)}, \\ 0 \le \alpha \le r\end{matrix}\right\} \Leftrightarrow \begin{cases} \text{for } \alpha = 0, & \{F(tx) - F(x)\}/\{rx^{-r}\nu_r(x)\} \to \log t, \\ \text{for } 0 < \alpha \le r, & x^r\bar{F}(x)/\nu_r(x) \to r/\alpha - 1. \end{cases}$$

(c) *When* $\nu_r(1, \infty) < \infty$,

$$\overline{\nu_r} \in \mathscr{R}_{r-\alpha}^{(\infty)},\ r \le \alpha < \infty \Leftrightarrow x^r\bar{F}(x)/\overline{\nu_r}(x) \to 1 - r/\alpha.$$

THEOREM 2 *Fix* $r < 0$.
(a)

$$\bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)},\ 0 \le \alpha < \infty \Leftrightarrow \mu_k = \infty \text{ for some } k, \text{ and } x^r\bar{F}(x)/\overline{\nu_r}(x) \to 1 - r/\alpha.$$

(b)

$$\left.\begin{matrix}\overline{\nu_r} \in \mathscr{R}^{(\infty)}_{r-\alpha}, \\ 0 \le \alpha < \infty\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}\textit{for } \alpha = 0,\ \{F(tx) - F(x)\}/\{-rx^{-r}\overline{\nu_r}(x)\} \to \log t, \\ \textit{for } 0 < \alpha < \infty,\ \mu_k = \infty \textit{ for some } k, \textit{ and } x^r\bar{F}(x)/\overline{\nu_r}(x) \to 1 - r/\alpha.\end{matrix}\right.$$

(All limits in the above are as $x \to \infty$, and where t appears the statement is to hold for each fixed $t > 1$. Here and subsequently, every statement about ν_r carries the unwritten statement with it that $\nu_r(1, \infty) < \infty$.)

The second-order statements above, involving $\log t$, bring in the class Π of de Haan (1970). Π is the class of nondecreasing functions R for which there exist functions $A(\cdot) > 0$ and $B(\cdot)$ such that for all $t \ge 1$,

$$\{R(tx) - B(x)\}/A(x) \to \log t, \qquad x \to \infty. \tag{1.5}$$

Clearly, B can be taken to be R itself. Then the *auxiliary function* A is slowly varying (Seneta (1976) pages 70–71), and it is obvious that (1.5) holds for all $t > 0$. The second-order statements of Theorems 1 and 2 assert that F, or $\nu_r(\cdot)$ as appropriate, belongs to Π with a specified auxiliary function. The next theorem shows that it suffices merely to ask for membership of Π. Also in Theorem 3, the equivalence $F \in \Pi \Leftrightarrow \nu_0(\cdot) \in \Pi$ (then trivially $\Leftrightarrow \nu(1, x] \in \Pi$) is new, not being a consequence of subexponentiality, as the class Π is not closed under asymptotic equality.

THEOREM 3.
(a) *For $r > 0$ the following are equivalent*:
(i) $\bar{F} \in \mathscr{R}^{(\infty)}_{-r}$, (ii) $\nu_r(\cdot) \in \Pi$,
(iii) $\lim_{x\to\infty} \nu_r(x, tx]/\{rx^r\bar{F}(x)\} = \log t$, *all* $t > 1$.
(b) *For $r > 0$ the following are equivalent*:
(i) $F \in \Pi$, (ii) $\nu_0(\cdot) \in \Pi$, (iii) $\nu_r(\cdot) \in \mathscr{R}^{(\infty)}_r$,
(iv) $\lim_{x\to\infty}\{F(tx) - F(x)\}/\{rx^{-r}\nu_r(x)\} = \log t$, *all* $t > 0$,
(v) $\lim_{x\to\infty} \nu_0(x, tx]/\{rx^{-r}\nu_r(x)\} = \log t$, *all* $t \ge 1$.
(c) *For $r < 0$ the following are equivalent*:
(i) $F \in \Pi$, (ii) $\nu_0(\cdot) \in \Pi$, (iii) $\overline{\nu_r} \in \mathscr{R}^{(\infty)}_r$,
(iv) $\lim_{x\to\infty} \{F(tx) - F(x)\}/\{-rx^{-r}\overline{\nu_r}(x)\} = \log t$, *all* $t > 0$,
(v) $\lim_{x\to\infty} \nu_0(x, tx]/\{-rx^{-r}\overline{\nu_r}(x)\} = \log t$, *all* $t \ge 1$.

The proofs of theorems 1 to 3 depend on the following 'Proposition 1', which asserts that when F has an infinite moment, $\bar{F}$ is asymptotic to $\bar{\nu}$ in a certain average sense, even though this may not be so in the pointwise sense that is characterised by subexponentiality. See also Remark 6.2. Proposition 1 extends to give the second half of proposition 2, both results being of interest generally in infinite divisibility. (The first half of Proposition 2 is essentially Feller's, and is included for completeness.)

PROPOSITION 1. *Suppose that for some nonnegative integer n, $\mu_n < \infty$ but $\mu_{n+1} = \infty$. Then*

$$\int_0^\infty e^{-sx}x^{n+1}\bar{F}(x)\,dx \sim \int_0^\infty e^{-sx}x^{n+1}\bar{\nu}(x)\,dx, \qquad s \downarrow 0, \tag{1.6}$$

$$\int_0^t x^{n+1}\bar{F}(x)\,dx \sim \int_0^t x^{n+1}\bar{\nu}(x)\,dx \sim \int_0^t x^{n+1}\overline{\nu_0}(x)\,dx, \qquad t \to \infty. \tag{1.7}$$

PROPOSITION 2. *For any infinitely divisible F on* $[0, \infty)$, $\liminf_{x\to\infty} \bar{F}(x)/\bar{\nu}(x) \ge 1$. *If F has an infinite moment then* $\liminf_{x\to\infty} \bar{F}(x)/\bar{\nu}(x) = 1$.

To return to the characterisation problem, when $r > 0$ Theorem 1 has solved (1.3) for all $l \in [0, \infty]$ and (1.4) for all $l' \in [0, 1)$. When $r < 0$, (1.3) is trivially satisfied, with $l = 0$, and the limit in (1.4) has to be at least 1, as a consequence of the first half of Proposition 2 and the obvious inequality $\overline{\nu_r}(x) \le x^r \overline{\nu_0}(x)$. Theorem 2 has solved (1.4) when $r' < 0$ for all $l' \in (1, \infty)$, under the assumption that not all moments of F are finite. We are left with the $l' = 1$ cases of (1.4), both for $r' > 0$ and $r' < 0$, and the $l' > 1$ cases of (1.4) for $r' > 0$. Remarks about the latter will be found in Section 6. For the $l' = 1$ cases we give some solutions, complete in case $r' < 0$, in Section 5. These involve so-called 'rapid variation'.

The contents of the ensuing sections are that Section 2 gives proofs of Propositions 1 and 2, Section 3 gives proofs of Theorems 1 and 2, and Section 4 proves Theorem 3. Section 5 is on 'rapid variation', as mentioned, and Section 6 contains complements and remarks, including an application to 'generalised gamma convolutions' which have recently become important in infinite divisibility following the work of Thorin and Bondesson.

2. Tail behaviour of the Lévy measure when not all moments finite. The connections between $\bar{F}$ and $\bar{\nu}$ used in this paper will hinge on the following formula. Define $\psi(s) = as + \int_0^\infty (1 - e^{-sx})\nu(dx)$, $s \ge 0$, so that $f = \exp(-\psi)$. Then for $k = 1, 2, \cdots$, and the suppressed argument nonnegative,

$$0 < (-)^k f^{(k)} = (-)^{k-1}\{\psi' \cdot f\}^{(k-1)}$$

$$= \sum_{j=0}^{k-1} \binom{k-1}{j} (-)^j \psi^{(j+1)} \cdot (-)^{k-1-j} f^{(k-1-j)}. \tag{2.1}$$

We need two elementary lemmas. The first uses an idea from Rudin (1973). The second is stated for a finite measure but can clearly be extended to accommodate measures such as ν.

LEMMA 1. *Let M be a Borel measure on* $(0, \infty)$. *Let $h > -1$ be such that* $\int_0^\infty \min(x^{h+1}, 1)\, M(dx) < \infty$. *Then*

$$\int_0^\infty e^{-sx} x^h M(x, \infty)\, dx = s^{-h-1} \int_0^s m_{h+1}(u) u^h\, du, \qquad s > 0,$$

where $m_{h+1}(u) = \int_0^\infty e^{-ux} x^{h+1} M(dx)$.

PROOF. The left side is $\int_0^\infty \int_0^y e^{-sx} x^h\, dx\, M(dy)$ by Fubini's theorem, the condition on M ensuring finite integrals. In the inner integral, transform to new variable of integration $u = sx/y$, and the result follows.

LEMMA 2. *Let M be a measure on* $[0, \infty)$ *satisfying* $M[0, \infty) < \infty$, $\int_0^\infty xM(dx) = \infty$, *and let* $m(s) = \int_{0-}^\infty e^{-sx} M(dx)$, $s \ge 0$. *Then* $\{m'(s)\}^2/m''(s) \to 0$ *as* $s \downarrow 0$.

PROOF. Fix $c > 0$. Then

$$0 < \{m'(s)\}^2/m''(s)$$

$$= \left\{\int_0^\infty e^{-sx} xM(dx)\right\}^2 \bigg/ \int_0^\infty e^{-sx} x^2 M(dx)$$

$$\sim \left\{\int_c^\infty e^{-sx} xM(dx)\right\}^2 \bigg/ \int_c^\infty e^{-sx} x^2 M(dx), \qquad s \downarrow 0$$

(since both numerator and denominator tend to ∞)

$$\leq \int_c^\infty e^{-sx} M(dx)$$

by the Cauchy-Schwarz inequality. Thus $\limsup_{s \downarrow 0} \{m'(s)\}^2/m''(s) \leq M(c, \infty)$ and the latter can be made as small as desired.

PROOF OF PROPOSITION 1. For $u > 0$, define $\Theta_j(u) = (-)^j \psi^{(j+1)}(u) \cdot (-)^{n+1-j} f^{(n+1-j)}(u) > 0$. We shall show that for $j = 0, 1, \cdots, n$,

$$\Theta_j(u) = o((-)^{n+2} f^{(n+2)}(u)), \qquad u \downarrow 0. \tag{2.2}$$

This will imply, by way of (2.1) with $k = n + 2$, that

$$(-)^{n+2} f^{(n+2)}(u) \sim \Theta_{n+1}(u) \sim (-)^{n+1} \psi^{(n+2)}(u) \tag{2.3}$$

since $f(u) \to 1$ as $u \to 0$.

To prove (2.2) consider first, for $n > 1$, those Θ_j with $1 \leq j \leq n$. These are bounded as $u \downarrow 0$ because

$$0 < (-)^j \psi^{(j+1)}(u) = \int_0^\infty e^{-ux} x^{j+1} \nu(dx) < \int_0^\infty x^{j+1} \nu(dx) < \infty \qquad \text{for} \quad j + 1 \leq n,$$

$$0 < (-)^{n+1-j} f^{(n+1-j)}(u) = \int_0^\infty e^{-ux} x^{n+1-j} F(dx) < \mu_{n+1-j} < \infty \qquad \text{for} \quad n + 1 - j \leq n.$$

Since $(-)^{n+2} f^{(n+2)}(u) \to \infty$, (2.2) is established for these j.

When $n \geq 1$, the positive quantity $-f'(u)$ is bounded by $\mu_1 < \infty$, so that $\Theta_n(u)$ is at most $\mu_1 (-)^n \psi^{(n+1)}(u)$. If we can show

$$(-)^n \psi^{(n+1)}(u) = o((-)^{n+1} \psi^{(n+2)}(u)), \qquad u \downarrow 0, \tag{2.4}$$

it will follow that $\Theta_n = o(\Theta_{n+1})$, which implies the $j = n \geq 1$ case of (2.2). But $(-)^{n-1} \psi^{(n)}(u) - a\delta_1^n$ is the Laplace-Stieltjes transform of the assumed-finite measure $x^n \nu(dx)$, and so (2.4) follows from Lemma 2, or indeed is easy to prove directly. (Here, δ_1^n equals 1 if $n = 1$, otherwise 0.)

There remains the $j = 0$ case of (2.2). When $n \geq 1$, $\Theta_0(u)$ is at most $\mu_1 (-)^{n+1} f^{(n+1)}(u)$, since $\psi'(u) < \psi'(0) = \mu_1 < \infty$. And $(-)^{n+1} f^{(n+1)}(u) = o((-)^{n+2} f^{(n+2)}(u))$, just as for (2.4), so (2.2) is proved for this case. The final sub-case is when $n = j = 0$, and here

$$\begin{aligned} \Theta_0(u) &= \psi'(u) \cdot \{-f'(u)\} \\ &= \{f'(u)\}^2/f(u) \sim \{f'(u)\}^2 = o(f''(u)) \end{aligned}$$

by Lemma 2. This completes the proof of (2.2).

From (2.3),

$$s^{-n-2} \int_0^s (-)^{n+2} f^{(n+2)}(u) u^{n+1} \, du \sim s^{-n-2} \int_0^s (-)^{n+1} \psi^{(n+2)}(u) u^{n+1} \, du, \qquad s \downarrow 0.$$

An application of Lemma 1 now proves (1.6). To deduce (1.7) we use the ratio Tauberian theorem of Korenblum (1955). Korenblum shows that if $h_i(\cdot)$ are positive nondecreasing functions on $(0, \infty)$, $h_i(0+) = 0$, $i = 1, 2$, such that h_1 satisfies

$$\lim_{\substack{x \to \infty \\ 1 < y/x \to 1}} h_1(y)/h_1(x) = 1. \tag{2.5}$$

then

$$\int_0^\infty e^{-sx} h_1(x) \, dx \sim \int_0^\infty e^{-sx} h_2(x) \, dx, \qquad s \downarrow 0, \tag{2.6}$$

implies $h_1(x) \sim h_2(x)$ as $x \to \infty$. In the present case we take $h_1(x) = \int_0^x t^{n+1}\bar{F}(t)\,dt$, $h_2(x) = \int_0^x t^{n+1}\bar{\nu}(t)\,dt$, then (2.5) holds because for $0 < x < y$,

$$h_1(y) = (y/x)^{n+2} \int_0^x t^{n+1}\bar{F}(ty/x)\,dt \le (y/x)^{n+2} h_1(x).$$

Since (1.6) may be written, on integrating by parts, as (2.6), Korenblum's theorem gives the first asymptotic equality of (1.7), and the second follows immediately.

Note. Theorem 3 of Feller (1963) implies a stronger version of Korenblum's result, in which (2.5) is replaced by a weaker condition. However Feller's proof is incorrect, in that in line 8 the statement $F^{(1)}(\cdot) = F^{(2)}(\cdot)$ holds only at continuity points, rather than everywhere as claimed, and that is not enough to justify the sequel. Stadtmüller and Trautner (1979) prove that among functions h_1 of dominated variation, condition (2.5) is *necessary* and sufficient for Korenblum's implication to hold. Thus Feller's result is in error, and indeed Example 1 of the last-mentioned paper provides a counterexample.

Proof of Proposition 2. As in Feller (1971) page 571, we may factorise $F = F_1 * F_0$, with F_1 and F_0 infinitely divisible, their Lévy measures being the restrictions of ν to [0, 1] and $(1, \infty)$ respectively. Let G be the distribution function obtained by normalising ν_0, viz. $G(dx) = \mu^{-1}\nu_0(dx)$, where $\mu = \bar{\nu}(1)$. Then F_0 is the compound Poisson distribution $\sum_{k=0}^{\infty} e^{-\mu}(\mu^k/k!)G^{(k)}$. For $x > 0$,

$$\bar{F}(x)/\bar{\nu}(x) \ge \bar{F}_0(x)/\{\mu\bar{G}(x)\} \ge \exp(-\mu\bar{G}(x)),$$

the second inequality from Feller (1971) VIII.10, Example 31. The first statement of the proposition follows.

For any finite $c < \liminf_{x\to\infty} \bar{F}(x)/\bar{\nu}(x)$ we have $\bar{F}(x) \ge c\bar{\nu}(x)$ for $x \ge x_0 = x_0(c)$. If F has an infinite moment then for suitable fixed n the integrals in (1.7) tend to infinity with t, and so are asymptotic to the corresponding integrals in which the lower limit of integration is x_0. But of the latter integrals, the one involving $\bar{F}$ is at least c times the one involving $\bar{\nu}$, hence $c \le 1$, and the second part of Proposition 2 is complete.

3. The characterisation problem. It is convenient to split the proofs of Theorems 1 and 2 into Lemmas 4, 5, and 6. First, though, Lemma 3 extends the monotone density theorem, cf. Seneta (1976) Example 2.7. For proof one may use a minor extension of the proof of de Haan (1977) *P5*.

Lemma 3. *Let U_2 be continuous and strictly increasing, and $U_2 \in \mathscr{R}_\delta^{(\infty)}$, $\delta > 0$. Let U_1 be eventually positive, monotone, and $\int_0^x U_1(t)U_2(dt) \in \mathscr{R}_{\gamma+\delta}^{(\infty)}$, $\gamma + \delta > 0$. Then $U_1 \in \mathscr{R}_\gamma^{(\infty)}$, and indeed*

$$U_1(x) \sim (1 + \gamma/\delta) \int_0^x U_1(t)U_2(dt)/U_2(x), \qquad x \to \infty. \tag{3.1}$$

Lemma 4. *Fix $r > 0$, and suppose there exists $\alpha \in [0, r]$ such that*

$$x^r\bar{F}(x)/\nu_r(x) \to r/\alpha - 1, \qquad x \to \infty. \tag{3.2}$$

Then (i) *if $\alpha \ne r$, $\bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)}$, and* (ii) *if $\alpha \ne 0$, $\nu_r(\cdot) \in \mathscr{R}_{r-\alpha}^{(\infty)}$.*

Proof. If $\alpha = 0$ then $x^r\bar{F}(x) \to \infty$, so that $\int_0^\infty x^{[r]}\bar{F}(x)\,dx = \infty$, $[\cdot]$ denoting integer part, whence $\mu_{[r]+1} = \infty$. If $\alpha = r$ then we may assume $\nu_r(1, \infty) = \infty$, for otherwise (ii) is trivially true. This assumption implies that ν have infinite $([r] + 1)$th moment, whence F also. If $0 < \alpha < r$ then since $\int_0^\infty x^{[r]-r}\nu_r(x)\,dx = \infty$, (3.2) gives $\int_0^\infty x^{[r]}\bar{F}(x)\,dx = \infty$, whence again $\mu_{[r]+1} = \infty$. So we may take it that the condition of Proposition 1 is in force for some

n satisfying $0 \le n \le [r]$.

Define $N(t) = \int_0^t x^{n+1-r}\nu_r(x)\,dx$, this being finite because $\nu_r(x) = 0$ for $x \le 1$. It will be important that

$$\int_1^\infty t^{-2}N(t)\,dt = \infty. \tag{3.3}$$

Equivalently, by Fubini's theorem, $\int_1^\infty x^{n-r}\nu_r(x)\,dx$ should be infinite. But this is established by employing $\nu_r(x) = \int_1^x t^r\nu_0(dt)$ in a further use of Fubini's theorem, and recalling that $\int_1^\infty t^{n+1}\nu_0(dt) = \infty$. Note that therefore $N(t) \to \infty$ as $t \to \infty$.

In (1.7), we shall substitute

$$\bar{\nu}_0(x) = -x^{-r}\nu_r(x) + r\int_x^\infty y^{-r-1}\nu_r(y)\,dy, \qquad x > 0 \tag{3.4}$$

for $\bar{\nu}_0(x)$. Now $t^{-n-2}N(t) \le t^{-1}\int_0^t x\cdot x^{-r-1}\nu_r(x)\,dx$, and the latter is $o(1)$ as $t \to \infty$ because $\int_x^\infty y^{-r-1}\nu_r(y)\,dy$ is finite. This justifies the integration by parts,

$$\begin{aligned}\int_x^\infty y^{-r-1}\nu_r(y)\,dy &= \int_x^\infty y^{-n-2}N'(y)\,dy \\ &= -x^{-n-2}N(x) + (n+2)\int_x^\infty y^{-n-3}N(y)\,dy.\end{aligned} \tag{3.5}$$

From (1.7), (3.4) and (3.5),

$$\begin{aligned}\int_0^t x^{n+1}\bar{F}(x)\,dx &\sim -N(t) + r\int_0^t \left\{-x^{-1}N(x) + (n+2)x^{n+1}\int_x^\infty y^{-n-3}N(y)\,dy\right\}dx \\ &= -N(t) + r\int_0^t (d/dx)\left\{x^{n+2}\int_x^\infty y^{-n-3}N(y)\,dy\right\}dx \\ &= -N(t) + rt^{n+2}\int_t^\infty x^{-n-3}N(x)\,dx,\end{aligned} \tag{3.6}$$

the last step being justified by the fact that $x^{n+2}\int_x^\infty y^{-n-3}N(y)\,dy \to 0$ as $x \downarrow 0$, which is an easy consequence of $N(y) \to 0$ $(y \downarrow 0)$. Both $\int_0^t x^{n+1}\bar{F}(x)\,dx$ and $N(t)$ tend to ∞ with t, and hence from (3.2), $\int_0^t x^{n+1}\bar{F}(x)\,dx \sim (r/\alpha - 1)N(t)$, with the appropriate interpretation when $r/\alpha - 1$ is 0 or ∞. Combining with (3.6),

$$t^{n+2}\int_t^\infty x^{-n-3}N(x)\,dx/N(t) \to 1/\alpha, \qquad t \to \infty. \tag{3.7}$$

Consider first the case $\alpha = 0$. By Karamata's theorem (Feller (1971) page 281), (3.7) gives $\int_t^\infty x^{-n-3}N(x)\,dx \in \mathscr{R}_0^{(\infty)}$, and $N(t) = o(t^{n+2}\int_t^\infty x^{-n-3}N(x)\,dx)$. By (3.6), $\int_0^t x^{n+1}\bar{F}(x)\,dx \in \mathscr{R}_{n+2}^{(\infty)}$, and so by Lemma 3, $\bar{F} \in \mathscr{R}_0^{(\infty)}$.

Now assume $\alpha > 0$, so that (3.7) and Karamata's theorem give $N \in \mathscr{R}_{n+2-\alpha}^{(\infty)}$. By another result of Karamata (Seneta (1976) page 18), and (3.3), we find $\alpha \le n + 1$. Therefore Lemma 3 may be applied to the defining formula of N, yielding $\nu_r(\cdot) \in \mathscr{R}_{r-\alpha}^{(\infty)}$. Finally, if $r \ne \alpha$, (3.2) give $\bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)}$.

LEMMA 5. *Fix $r \neq 0$. If $r > 0$, assume $\nu_r(1, \infty) < \infty$. If $r < 0$, assume F has an infinite moment. Whatever r, suppose that for some $\alpha \in [0, \infty)$,*

$$x^r \bar{F}(x)/\overline{\nu_r}(x) \to 1 - r/\alpha, \qquad x \to \infty. \tag{3.8}$$

Then (i) *if $\alpha \neq r$, $\bar{F} \in \mathscr{R}^{(\infty)}_{-\alpha}$, and* (ii) *if $\alpha \neq 0$, $\overline{\nu_r} \in \mathscr{R}^{(\infty)}_{r-\alpha}$.*

Note. If $\alpha > 0$ then we must have $r \leq \alpha$, as the left of (3.8) is positive. If $\alpha = 0$ then it must be the case that $r < 0$, and the right of (3.8) is to be read as $+\infty$.

PROOF. The proof will need F to have an infinite moment, and we show that for $r > 0$ this is implied by (3.8). For suppose otherwise, that is, $r > 0$, so that $\alpha > 0$, and all moments of $F(dx)$, and hence of $\nu_r(dx)$, are finite. Then for any fixed $c > 0$,

$$\int_0^\infty x^{k-1}\bar{F}(x)\,dx \sim \int_c^\infty x^{k-1}\bar{F}(x)\,dx \to \infty, \qquad k \to \infty,$$

$$\int_0^\infty x^{k-1-r}\overline{\nu_r}(x)\,dx \sim \int_c^\infty x^{k-1-r}\overline{\nu_r}(x)\,dx \to \infty, \qquad k \to \infty,$$

so that (3.8) implies

$$\int_0^\infty x^{k-1}\bar{F}(x)\,dx \sim (1 - r/\alpha)\int_0^\infty x^{k-1-r}\overline{\nu_r}(x)\,dx, \qquad k \to \infty,$$

$$k^{-1}\int_0^\infty x^k F(dx) \sim (1 - r/\alpha)(k - r)^{-1}\int_0^\infty x^{k-r}\nu_r(dx), \qquad k \to \infty,$$

$$\mu_k \sim (1 - r/\alpha)\int_0^\infty x^k \nu(dx). \qquad k \to \infty. \tag{3.9}$$

However, letting the argument in (2.1) tend to 0+, and retaining only the last term ($j = k - 1$) in the sum, we find that $\mu_k > \int_0^\infty x^k\nu(dx)$. This contradicts (3.9), in which $0 \leq 1 - r/\alpha < 1$.

Thus, whatever r, there exists an integer $n \geq 0$ such that $\mu_n < \infty$, $\mu_{n+1} = \infty$, and so Proposition 1 applies. Also, whatever r, $\nu_r(1, \infty) < \infty$, this being again by assumption in one case, automatically in the other. For $r > 0$ the assumption $\nu_r(1, \infty) < \infty$ implies $\nu_{[r]}(1, \infty) < \infty$, and hence $\mu_{[r]} < \infty$ as remarked in the introduction, and so $n \geq [r]$. Define $N(t) = \int_0^t x^{n+1-r}\overline{\nu_r}(x)\,dx$ and proceed by analogy with the previous lemma.

LEMMA 6. (a) *If $r > 0$, $\nu_r(1, \infty) = \infty$, $\nu_r(\cdot) \in \mathscr{R}_0^{(\infty)}$, then $\bar{F}(x) = o(x^{-r}\nu_r(x))$ as $x \to \infty$.*
(b) *If $r > 0$, $\nu_r(1, \infty) < \infty$, $\overline{\nu_r} \in \mathscr{R}_0^{(\infty)}$, then $\bar{F}(x) = o(x^{-r}\overline{\nu_r}(x))$ as $x \to \infty$.*

REMARK. This lemma may seem to treat a very special case. Intuitively, its separate proof is necessitated by the comparative lack of information available from assuming only slow variation of $\nu_r(\cdot)$ or $\overline{\nu_r}$. Thus $\bar{\nu}$ need not be regularly varying, or even subexponential, so the pointwise link between it and $\bar{F}$ is not present. Also, ratio Tauberian theorems are not relevant in part (a), so we resort instead to the Karamata Tauberian theorem (Feller (1971) page 445), henceforth denoted KT. Note that Lemmas 4 and 5 could likewise have been proved by multiple use of KT, starting from (1.6). Instead, we started from (1.7), obtaining that by merely a single use of Korenblum's *ratio* Tauberian theorem.

Let $\mathscr{R}_\rho^{(0)}$ denote the functions regularly varying at 0+ with index ρ, that is, $R(s) \in \mathscr{R}_\rho^{(0)} \Leftrightarrow R(1/x) \in \mathscr{R}_{-\rho}^{(\infty)}$.

PROOF OF LEMMA 6. (a). Slow variation implies that for any $\epsilon > 0$, $\nu_r(y) = o(y^\epsilon)$ as $y \to \infty$, so that

$$\infty > \int_x^\infty \nu_r(y)\, y^{-1-\epsilon}\, dy \geq \int_x^\infty \int_x^y u^r \nu_0(du)\, y^{-1-\epsilon}\, dy,$$

and the latter is seen to be $\epsilon^{-1}\overline{\nu_{r-\epsilon}}(x)$. Letting k be the smallest integer satisfying $k \geq r$, it follows that $\nu_{k-1}(x, \infty) < \infty$, so that $\mu_k = \infty$, $\mu_{k-1} < \infty$. In (2.1), all terms on the right are bounded as $s \downarrow 0$ except that for $j = k - 1$, which tends to ∞. Therefore

$$(3.10) \qquad (-)^k f^{(k)}(s) \sim (-)^{k-1}\psi^{(k)}(s)\cdot f(s) \sim (-)^{k-1}\psi^{(k)}(s) \sim \psi_k(s)$$

where for every real b we define $\psi_b(s) = \int_0^\infty e^{-sx}\nu_b(dx)$, $s \geq 0$. Consider first the case of integer r, for which $\psi_k \equiv \psi_r$. By KT, $\psi_r(s) \sim \nu_r(1/s)$. Therefore $(-)^k f^{(k)} \in \mathscr{R}_0^{(0)}$, and applying KT to it we find $F_r(t) \sim (-)^k f^{(k)}(1/t) \sim \nu_r(t)$ as $t \to \infty$, where $F_r(t) = \int_0^t x^r F(dx)$. But

$$\bar{F}(x) = -x^{-r} F_r(x) + r \int_x^\infty y^{-r-1} F_r(y)\, dy$$

which is $o(x^{-r}F_r(x))$ by Karamata's theorem since $F_r \in \mathscr{R}_0^{(\infty)}$. Thus $\bar{F}(x) = o(x^{-r}\nu_r(x))$.

When r is noninteger we consider instead $\int_0^x y^{k-r-1}\nu_r(y)\, dy$, which as $x \to \infty$ is $\sim (k - r)^{-1}x^{k-r}\nu_r(x)$ by Karamata's theorem, since $\nu_r(\cdot) \in \mathscr{R}_0^{(\infty)}$. By KT,

$$\int_0^\infty e^{-sx}x^{k-r-1}\nu_r(x)\, dx \sim \Gamma(k - r + 1)\cdot(k - r)^{-1}(1/s)^{k-r}\nu_r(1/s), \qquad s \downarrow 0$$

$$= \Gamma(k - r)s^{r-k}\nu_r(1/s).$$

Elementary calculations like those used for Lemma 1 reduce the left side to $s^{r-k}\int_s^\infty \psi_k(u)u^{k-r-1}\, du$. Thus

$$(3.11) \qquad \int_s^\infty \psi_k(u)u^{k-r-1}\, du \sim \Gamma(k - r)\nu_r(1/s), \qquad s \downarrow 0.$$

The right of (3.11) tends to ∞ as $s \downarrow 0$, and consequently the left also. We may therefore weight (3.10) and integrate:

$$\int_s^\infty (-)^k f^{(k)}(u)u^{k-r-1}\, du \sim \int_s^\infty \psi_k(u)u^{k-r-1}\, du, \qquad s \downarrow 0.$$

Again by the method of Lemma 1, the left side is equal to $s^{k-r}\cdot\int_0^\infty e^{-sx}x^{k-r-1}F_r(x)\, dx$. Combining with (3.11) and applying KT it follows that

$$\int_0^t x^{k-r-1}F_r(x)\, dx \sim \{1/\Gamma(k - r + 1)\}\cdot\Gamma(k - r)t^{k-r}\nu_r(t), \qquad t \to \infty$$

$$= (k - r)^{-1}t^{k-r}\nu_r(t).$$

By Lemma 3, $F_r(t) \sim \nu_r(t)$. As in the integer case we conclude $\bar{F}(x) = o(x^{-r}\nu_r(x))$.

(b). Set $k = [r] + 1$, $\bar{F}_r(t) = \int_t^\infty x^r F(dx)$. In this case one may use Korenblum's ratio Tauberian theorem—we omit the details—to show

$$\int_0^t x^{k-r-1}\bar{F}_r(x)\, dx \sim \int_0^t x^{k-r-1}\overline{\nu_r}(x)\, dx,$$

from which the conclusion follows by standard arguments.

PROOF OF THEOREMS 1 AND 2. All '$\Leftarrow$' statements have been proved in Lemmas 4 and 5, except those in which the right-hand side involves t, and the latter are dealt with by observing that they are of the form (1.5), and so their auxiliary functions are slowly varying. We turn to the '$\Rightarrow$' statements. Suppose first that $\bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)}$. Then from (1.1) and Karamata's theorem one finds that $x^r\bar{F}(x)/\overline{\nu_r}(x) \to 1 - r/\alpha$ when $r < \alpha$, and $x^r\bar{F}(x)/\nu_r(x) \to r/\alpha - 1$ when $0 \le \alpha < r$. In the case $\alpha = r > 0$ the conclusion $\nu_r(x, tx]/\{rx^r\bar{F}(x)\} \to \log t$ will be proved under Theorem 3.

Now suppose $\overline{\nu_r}$, or if it does not exist, $\nu_r(\cdot)$, is in $\mathscr{R}_{r-\alpha}^{(\infty)}$. When $\alpha \ne r$ and $\alpha \ne 0$ the relevant limit on the right of Theorem 1 or 2 follows from Karamata's theorem and Proposition 0. When $\alpha = 0$ the relevant results will follow from the more general Theorem 3 below. The cases $\alpha = r > 0$ were dealt with in Lemma 6, and so the proof is complete.

4. Second-order theory: proof of Theorem 3.

(a). By definition, (iii) $\Rightarrow$ (ii). By de Haan (1977) *P4*, $\nu_r \in \Pi$ implies $\overline{\nu_0} \in \mathscr{R}_{-r}^{(\infty)}$, which by (1.1) implies $\bar{F} \in \mathscr{R}_{-r}^{(\infty)}$; that is, (ii) $\Rightarrow$ (i). Conversely if $\bar{F} \in \mathscr{R}_{-r}^{(\infty)}$ then $\nu_0(x, \infty) \sim \bar{F}(x)$, again by (1.1). Adapting a technique of de Haan (1976) we set $L(x) = rx^r\overline{\nu_0}(x) \in \mathscr{R}_0^{(\infty)}$ then on integrating by parts, $\nu_r(x) = -L(x)/r + \int_0^x L(y)y^{-1}\,dy$. whence

$$\nu_r(x, tx]/L(x) = -r^{-1}\{L(tx) - L(x)\}/L(x) + \int_1^t \{L(ux)/L(x)\}u^{-1}\,du.$$

The integrand tends to u^{-1} uniformly, hence the whole right-hand side tends to $\log t$, and this gives (iii).

(b) and (c). Of course, (iv) and (v) are just more specific forms of (i) and (ii) respectively, and either of (iv) or (v) implies (iii). Suppose (i), that is,

$$\lim_{x\to\infty} \{F(tx) - F(x)\}/L(x) = \log t, \qquad t > 0, \tag{4.1}$$

for some $L \in \mathscr{R}_0^{(\infty)}$. By de Haan (1971) Theorem 4, (4.1) is equivalent to $\int_0^x yF(dy) \sim xL(x)$ as $x \to \infty$. The integral has Laplace-Stieltjes transform $-f'(s) = \psi'(s)f(s) \sim \psi'(s) \sim \int_0^\infty e^{-sx}\nu_1(dx)$, hence by two applications of KT, (4.1) is equivalent to $\nu_1(x) \sim xL(x)$. By the same theorem of de Haan, this is equivalent to

$$\lim_{x\to\infty} \nu_0(x, tx]/L(x) = \log t, \qquad t > 1.$$

However, by Karamata's theorem the relation $\nu_1(x) \sim xL(x)$ is equivalent for each $r > 0$ to $\nu_r(x) \sim r^{-1}x^rL(x)$, and for each $r < 0$ to $\overline{\nu_r}(x) \sim (-r^{-1})x^rL(x)$. After obvious details the proof of (b) and (c) is complete.

5. Rapid variation. Denote by $\mathscr{R}_{-\infty}^{(\infty)}$ the class of functions R that are measurable, eventually positive, and such that for all $t > 1$, $\lim_{x\to\infty} R(tx)/R(x) = 0$. We remind the reader that statements about $\overline{\nu_r}$ in what follows carry implicitly the statement that $\nu_r(1, \infty) < \infty$.

LEMMA 7. *For* $r \ne 0$,

$$\lim_{x\to\infty} x^{-r}\overline{\nu_r}(x)/\overline{\nu_0}(x) = 1 \Leftrightarrow \overline{\nu_0} \in \mathscr{R}_{-\infty}^{(\infty)} \Leftrightarrow \overline{\nu_r} \in \mathscr{R}_{-\infty}^{(\infty)}.$$

PROOF. Take the integration-by-parts formulae that give each of $\overline{\nu_r}$, $\overline{\nu_0}$ in terms of an integral of the other, and apply de Haan (1970) Theorem 1.3.1.

THEOREM 4. *Fix* $r < 0$. *Then*

$$\liminf_{x\to\infty} x^r\bar{F}(x)/\overline{\nu_r}(x) \ge 1. \tag{5.1}$$

Further, $x^r\bar{F}(x)/\overline{\nu_r}(x) \to 1$ *if and only if* $F \in \mathscr{S}$ *and* $\bar{F} \in \mathscr{R}_{-\infty}^{(\infty)}$.

PROOF.

$$x^r\bar{F}(x)/\overline{\nu_r}(x) = \{\bar{F}(x)/\overline{\nu_0}(x)\}\cdot\{x^r\overline{\nu_0}(x)/\overline{\nu_r}(x)\} \tag{5.2}$$

and the first quotient on the right has lim inf at least 1, by Proposition 2, while the second is obviously ≥ 1. This proves (5.1), and also shows that the left of (5.2) tends to 1 if and only if both quotients on the right tend to 1, that is, respectively $F \in \mathscr{S}$ (by Embrechts et al. (1979)), and $\bar{\nu} \in \mathscr{R}_{-\infty}^{(\infty)}$ (by Lemma 7). These conditions together are equivalent to $F \in \mathscr{S}$ and $\bar{F} \in \mathscr{R}_{-\infty}^{(\infty)}$.

THEOREM 5. *For* $r > 0$, *any two of the following imply the third*:

(a) $F \in \mathscr{S}$, (b) $\bar{F} \in \mathscr{R}_{-\infty}^{(\infty)}$, (c) $\lim_{x\to\infty} x^r\bar{F}(x)/\overline{\nu_r}(x) = 1$.

PROOF. Obvious from (5.2). This theorem is weaker than Theorem 4 because $\overline{\nu_r}(x) \geq x^r\,\overline{\nu_0}(x)$ for $r > 0$, as against the reverse inequality for $r < 0$.

6. Complements and remarks.

6.1. For $\gamma > 0$ we follow Chover et al. (1973) in saying that F (not necessarily infinitely divisible) belongs to $\mathscr{S}_\gamma$ if for all real t, $\lim_{x\to\infty}\bar{F}(x+t)/\bar{F}(x) = e^{-\gamma t}$, and $\lim_{x\to\infty}\{1 - F*F(x)\}/\bar{F}(x) = 2f(-\gamma) < \infty$. Clearly this implies $\bar{F} \in \mathscr{R}_{-\infty}^{(\infty)}$ and all moments finite. When such an F is infinitely divisible we can prove (Embrechts and Goldie (1980))that $\bar{F}(x)/\bar{\nu}(x) \to f(-\gamma)$ as $x \to \infty$. This implies $\bar{\nu} \in \mathscr{R}_{-\infty}^{(\infty)}$ and so by Lemma 7,

$$x^r\bar{F}(x)/\overline{\nu_r}(x) \to f(-\gamma) > 1, \qquad x \to \infty. \tag{6.1}$$

For $r > 1$ we have not so far met limits $l' > 1$ in (1.4), and indeed it is easy to see that such a limit is impossible if F has an infinite moment (use Proposition 2) or is in $\mathscr{S}$. Membership of $\mathscr{S}_\gamma$ gives one way of obtaining $l' > 1$; we conjecture it is the *only* way.

For $r < 0$, (6.1) gives an alternative way of obtaining any limit $l' \in (1, \infty)$ in (1.4), therefore the condition that $\mu_k = \infty$ for some k cannot be omitted from the right of theorem 2.

6.2. In Proposition 1 we could have proved, more easily, that (1.6) *and hence* (1.7) *remain valid when* $n + 1$ *is replaced by* n. This would suffice for Proposition 2, and for Lemmas 4 and 5 *except* in the case $\alpha = n + 1$, when it implies only $\int_0^t x^n\bar{F}(x)\,dx \in \mathscr{R}_0^{(\infty)}$, not enough to conclude $\bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)}$. Proposition 2 was suggested by Theorem 2* of Rudin (1973), and in this context it is worth remarking that Korenblum's ratio Tauberian theorem may be applied to Rudin's formula (3.7) just as we applied it to our (1.6). Noting that in Rudin's (3.7) the exponent p should read $p - 1$, the conclusion is, in his notation,

$$\lim_{t\to\infty} \int_0^t s^{p-1}T_s[g]\,ds \bigg/ \int_0^t s^{p-1}T_s[f]\,ds = \Phi'(f(1)).$$

6.3. Among the infinitely divisible F on $[0, \infty)$, Thorin (1977) names as the *generalised gamma convolutions* those for which

$$\int_{0-}^{\infty} e^{-sx}F(dx) = \exp\left\{-as - \int_0^{\infty} \log(1 + s/y)\,U(dy)\right\}, \qquad s \geq 0,$$

where $a \geq 0$ and $U(\cdot)$ is nondecreasing such that $U(0+) = U(0-) = 0$, $\int_0^1 |\log y|\,U(dy) < \infty$, $\int_1^\infty y^{-1}U(dy) < \infty$. From our Theorems 1 and 3 one may prove using similar methods the following consequences for the behaviour of U in the neighbourhood of 0:

$$F \in \Pi \Leftrightarrow U \in \mathscr{R}_0^{(0)} \Rightarrow \lim_{x\to\infty}\{F(tx) - F(x)\}/U(x^{-1}) = \log t, \qquad \text{for every } t > 0,$$

and for $0 < \alpha < \infty$,

$$\bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)} \Leftrightarrow U \in \mathscr{R}_{\alpha}^{(0)} \Rightarrow \lim_{x\to\infty} \bar{F}(x)/U(x^{-1}) = \Gamma(\alpha).$$

Note that the conclusions on the right suggest characterisation problems, to find the largest classes of F for which they hold. We shall not pursue them here.

6.4. As mentioned in the introduction, ν has been assumed throughout to have unbounded support. When ν has bounded support the tail behaviour of F is quite different: $\bar{F}(x) = O(\exp(-\delta x \log x))$ for some $\delta > 0$ (Kruglov (1970)). For further results see Steutel and Wolfe (1977).

6.5. Since Π is a proper subclass of $\mathscr{R}_0^{(\infty)}$ it follows that the $\alpha = r$ case of Theorem 1(b) has a proper inclusion in the corresponding case of Theorem 1(a), and similarly for the various $\alpha = 0$ cases. An example is instructive. Let ν be purely atomic with atoms of mass 2^{-k} at points 2^k, $k = 1, 2, \cdots$. Then $\nu_1(\cdot)$ is slowly varying and one may check that $x\bar{F}(x)/\nu_1(x) \to 0$. However, $\bar{\nu}$ is not regularly varying and so neither is $\bar{F}$.

6.6. From Theorem 3(b) and its proof we see that $F \in \Pi$ implies

$$\lim_{x\to\infty} \{F(tx) - F(x)\}/\nu(x, tx] = 1 \qquad \text{for each } t > 1,$$

$$\int_0^x yF(dy) \sim \nu_1(x) \to \infty, \qquad x \to \infty.$$

However, both of these hold more generally, for instance under regular variation. It would be interesting to know what classes of distributions they characterise.

APPENDIX

Proof of Proposition 0. Rather than specialising from Embrechts et al. (1979), (1.1) may be proved directly, making the paper more self-contained. The following proofs were kindly supplied by a referee. Write $\overline{F^{(n)}}$ for $1 - F^{(n)}$, where $F^{(n)}$ is the nth convolution power of F.

Lemma. *Let F be a distribution function on $[0, \infty)$, and let $\alpha \in [0, \infty)$. Then $\bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)}$ if and only if $\overline{F^{(n)}} \in \mathscr{R}_{-\alpha}^{(\infty)}$ for some (every) positive integer n, and then $\overline{F^{(n)}}(x) \sim n\bar{F}(x)$ as $x \to \infty$.*

Proof. We treat the case $n = 2$, the general case being similar. The only if part is stated and proved in Feller (1971) page 278 by means of the following inequalities: if Y_1, Y_2 are independent then for every $x > 0$ and $0 < \epsilon < 1$,

$$\text{(A.1)} \quad P(Y_1 + Y_2 > x) \geq P(Y_1 > x(1 \div \epsilon))P(|Y_2| \leq x\epsilon) + P(Y_2 > x(1 + \epsilon))P(|Y_1| \leq x\epsilon),$$

$$\text{(A.2)} \quad P(Y_1 + Y_2 > x) \leq P(Y_1 > x(1 - \epsilon)) + P(Y_2 > x(1 - \epsilon)) + P(Y_1 > x\epsilon)P(Y_2 > x\epsilon).$$

Assume now that each of Y_1, Y_2 has distribution function F and that $\overline{F^{(2)}} \in \mathscr{R}_{-\alpha}^{(\infty)}$. By nonnegativity, $P(Y_1 > x\epsilon)/P(Y_1 + Y_2 > x\epsilon) \leq 1$. Upon dividing (A.1) by $P(Y_1 + Y_2 > x(1 + \epsilon))$ and (A.2) by $P(Y_1 + Y_2 > x(1 - \epsilon))$ we find that for $0 < \epsilon < 1$,

$(1+\epsilon)^\alpha \geq \limsup_{u\to\infty} 2P(Y_1 > u)/P(Y_1 + Y_2 > u)$,

$(1-\epsilon)^\alpha \leq \liminf_{u\to\infty} 2P(Y_1 > u)/P(Y_1 + Y_2 > u) + \epsilon^{-\alpha}(1-\epsilon)^\alpha \lim_{u\to\infty} P(Y_1 > u)$,

that is, $\lim_{u\to\infty} P(Y_1 > u)/P(Y_1 + Y_2 > u) = \frac{1}{2}$, in particular showing the $\bar{F} \in \mathscr{R}_{-\alpha}^{(\infty)}$.

PROOF OF PROPOSITION 0. We use the notation and the factorisation of F given in the proof of Proposition 2. It suffices to prove (1.1) with F replaced by F_0, because then Feller (1971) XVII (4.11) gives $\bar{F}_0(x) \sim \bar{F}(x)$, whence (1.1) itself. Let $X_1, X_2, \cdots$ be independent, each with distribution function G, then for every $r \in N$, $x > 0$, $0 < t < 1$,

$$\begin{aligned}\bar{F}_0(x) &= \textstyle\sum_{k=1}^\infty e^{-\mu}(\mu^k/k!)P(X_1 + \cdots + X_k > x)\\ &\leq \textstyle\sum_{k=1}^{r-1} e^{-\mu}(\mu^k/k!)P(X_1 + \cdots + X_k > x)\\ &\quad + \textstyle\sum_{k=r}^\infty e^{-\mu}(\mu^k/k!)P(X_1 + \cdots + X_r > xt)\\ &\quad + \textstyle\sum_{k=r+1}^\infty e^{-\mu}(\mu^k/k!)P(X_{r+1} + \cdots + X_k > x(1-t))\\ &\leq \textstyle\sum_{k=1}^{r-1} e^{-\mu}(\mu^k/k!)\overline{G^{(k)}}(x) + \mu^r\overline{G^{(r)}}(xt)/r! + \mu^r\bar{F}_0(x(1-t))/r! \qquad \text{(A.3)}\end{aligned}$$

Suppose first that $\bar{\nu}$, and hence $\bar{G}$, is in $\mathscr{R}_{-\alpha}^{(\infty)}$. We may find y such that $\bar{G}(x)/\bar{G}(2x) \leq 2^{\alpha+1}$ for all $x \geq y$. Fixing $\epsilon > 0$, make r in (A.3) so large that $\mu^r 2^{\alpha+1}/r! \leq \epsilon$, and then taking $t = \frac{1}{2}$,

$$\frac{\overline{F_0}(2x)}{\bar{G}(2x)} \leq \sum_{k=1}^{r-1} e^{-\mu}\frac{\mu^k}{k!}\frac{\overline{G^{(k)}}(2x)}{\bar{G}(2x)} + \frac{\mu^r 2^{\alpha+1}}{r!}\frac{\overline{G^{(r)}}(x)}{\bar{G}(x)} + \epsilon\frac{\overline{F_0}(x)}{\bar{G}(x)}, \qquad \forall\, x \geq y.$$

The lemma gives $\lim_{x\to\infty} \overline{G^{(k)}}(x)/\bar{G}(x) = k$ for every $k \in N$, hence setting

$$C(y) = \sup_{x\in[y,\infty)} \{\textstyle\sum_{k=1}^{r-1} e^{-\mu}(\mu^k/k!)\overline{G^{(k)}}(2x)/\bar{G}(2n) + (\mu^r 2^{\alpha+1}/r!)\overline{G^{(r)}}(x)/\bar{G}(x)\},$$

it follows that $C(y) < \infty$ and

$$\overline{F_0}(2x)/\bar{G}(2x) \leq C(y) + \epsilon\overline{F_0}(x)/\bar{G}(x), \qquad \forall\, x \geq y.$$

Iterating the latter inequality gives that for $m = 0, 1, 2, \cdots$,

$$\sup_{x\in[2^m y, 2^{m+1}y]} \overline{F_0}(x)/\bar{G}(x) \leq C(y)(1 + \epsilon + \cdots + \epsilon^{m-1}) + \epsilon^m \sup_{x\in[y,2y]} \overline{F_0}(x)/\bar{G}(x),$$

and since the latter supremum is finite we conclude $\limsup_{x\to\infty} \overline{F_0}(x)/\bar{G}(x) \leq C(y)/(1-\epsilon)$. Let $y \to \infty$ and $\epsilon \to 0$ to get $\limsup \overline{F_0}(x)/\bar{G}(x) \leq \mu$. Thus $\limsup \overline{F_0}(x)/\bar{\nu}(x) \leq 1$, which together with the first half (essentially Feller's) of Proposition 2 gives $\overline{F_0}(x) \sim \bar{\nu}(x)$ and $\overline{F_0} \in \mathscr{R}_{-\alpha}^{(\infty)}$.

Let $F_0^{(1/n)}$ denote the convolution nth root of F_0, so that $F_0^{(1/n)} = \sum_{k=0}^\infty e^{-\mu/n}\{(\mu/n)^k/k!\}G^{(k)}$. Then (A.3) holds with F_0, μ replaced by $F_0^{(1/n)}$, μ/n respectively. Taking $r = 1$,

$$\overline{F_0^{(1/n)}}(x) \leq (\mu/n)\bar{G}(xt) + (\mu/n)\overline{F_0^{(1/n)}}(x(1-t)). \qquad \text{(A.4)}$$

If $\overline{F_0} \in \mathscr{R}_{-\alpha}^{(\infty)}$ then $\overline{F_0^{(1/n)}} \in \mathscr{R}_{-\alpha}^{(\infty)}$ and $n\,\overline{F_0^{(1/n)}}(x) \sim \overline{F_0}(x)$, by the lemma. With (A.4) this gives $\liminf_{u\to\infty} \mu\bar{G}(u)/\overline{F_0}(u) \geq \{1 - (\mu/n)(1-t)^{-\alpha}\}t^\alpha$. Let $n \to \infty$ and then $t \to 1$ to give $\liminf_{u\to\infty} \bar{\nu}(u)/F_0(u) \geq 1$. The regular variation of $\bar{\nu}$ now follows, again by Proposition 2.

In connection with this proof it should be remarked that the argument in Feller (1971) page 288 is not valid for the case $\alpha = 0$, although in XVII (4.10) the contrary seems to be assumed.

Acknowledgements. The authors are grateful to the referee for the provision of the content of the appendix. The problem that this paper deals with arose out of a discussion with Prof. J. L. Teugels and from a sight of part of a forthcoming book by him on regular variation.

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