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GAUSSIAN MEASURABLE DUAL AND BOCHNER'S THEOREM

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Let E be a locally convex Hausdorff linear topological space, E' be the topological dual of E and γ be a nondegenerate, centered Gaussian-Radon measure on E. Then every nonnegative definite continuous functional on E is the characteristic functional of a Borel probability measure on E^γ, the closure of E' in $L_0(\gamma)$. In other words, identifying E^γ with the reproducing kernel Hilbert space $\mathscr{H}_\gamma$ of γ, we may say that for every continuous nonnegative definite function f on E there exists a Borel probability μ on $\mathscr{H}_\gamma$ such that f is the characteristic functional of μ.

1. Introduction. Let E be a locally convex Hausdorff linear topological space, E' be the topological dual and E^a be the algebraic dual of E. Let E and F be two linear spaces in duality with canonical bilinear form $\langle x, \xi \rangle$, $x \in E$, $\xi \in F$. Then the minimal σ-algebra of subsets of E that makes all functions $\{\langle x, \xi \rangle; \xi \in F\}$ measurable is denoted by $\mathscr{C}(E, F)$.

Let μ be a Radon probability measure on E and $L_0(\mu)$ be the linear metric space of all μ-measurable functions with metric

$$\rho(x, y) = \int_E \frac{|x(\xi) - y(\xi)|}{1 + |x(\xi) - y(\xi)|} d\mu(\xi), \qquad x, y \in L_0(\mu).$$

Then it is well-known that $L_0(\mu)$ is a complete metric space and the *canonical map* R_μ of E' into $L_0(\mu)$ defined by

$$R_\mu : x \in E' \to x(\xi) = \langle x, \xi \rangle \in L_0(\mu)$$

is continuous with respect to the compact convergence topology of E'. The *measurable dual* of $E = (E, \mu)$ is defined as the closure of $R_\mu(E')$ in $L_0(\mu)$ and denoted by E^μ. A Radon probability measure μ on E is called *nondegenerate* if the whole space E coincides with the minimal closed subspace of μ-measure 1. If μ is nondegenerate then R_μ is one-to-one. A Radon probability measure γ on E is called *centered Gaussian* if for every x in E' the real random variable $x(\xi) = \langle x, \xi \rangle$ on the probability space (E, γ) obeys a Gaussian law of mean 0.

In this paper, we will prove the following theorem.

Theorem 1. *Let E be a locally convex Hausdorff linear topological space, γ be a nondegenerate centered Gaussian Radon measure on E and f be a continuous nonnegative definite functional with $f(0) = 1$ on E. Then there exists a Radon probability measure μ on E^γ such that*:

(1) *The canonical map $R_\mu : (E^\gamma)' \to (E^\gamma)^\mu$ is extended to E.*

(2) *For every ξ in E we have*

$$f(\xi) = \int_{E^\gamma} e^{i\xi(x)} d_\mu(x),$$

where $\xi(x) = (R_\mu \xi)(x)$.

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In fact, identifying E^γ with the reproducing kernel Hilbert space $\mathscr{H}_\gamma$ of γ, we will explicitly construct such a probability measure μ on $\mathscr{H}_\gamma$.

Theorem 1 is claimed in a further general form in D. Xia [7], Theorem 4-3-11, but it is incorrect even in the case of a Hilbert space, and yet the author would like to mark that this work was greatly motivated by [7]. Theorem 1 is also a generalization of *the Duality Theorem* of L. Schwartz [6].

Recently, Y. Okazaki [4] has proved the theorem showing that u^*, the adjoint map of the canonical injection $u:\mathscr{H}_\gamma \to E$, is 0-Radonising. In this paper we will prove it in a purely probabilistic and constructive manner. The crucial point of our proof is to show the stochastic approximation property of a Gaussian Radon probability space (E, γ).

Throughout the paper we assume that the coefficients of linear spaces are real and that every nonnegative definite functional f satisfies $f(0) = 1$.

2. Stochastic approximation property. Let E be a locally convex Hausdorff linear topological space, γ be a nondegenerate centered Gaussian Radon measure on E and R_γ be the canonical map of E' into $L_0(\gamma)$. Then, since γ is nondegenerate, R_γ is injective and, since γ is Gaussian, R_γ transforms E' into $L_2(\gamma) \subset L_0(\gamma)$ and the measurable dual E^γ coincides with the closure of $R_\gamma(E')$ in $L_2(\gamma)$ as a linear topological space, which we denote by H_γ. Furthermore since γ is Gaussian Radon, $H_\gamma = E^\gamma$ is a separable Hilbert space and the adjoint map R_γ^* of R_γ is a linear injection of $H_\gamma = H_\gamma'$ into E (H. Sato and Y. Okazaki [5], C. Borell [1]). We translate the topology of H_γ onto $\mathscr{H}_\gamma = R_\gamma^*(H_\gamma)$ and call $\mathscr{H}_\gamma$ the reproducing kernel Hilbert space of γ. Obviously $\mathscr{H}_\gamma$ is isomorphic to $H_\gamma = E^\gamma$ and we may identify all of them.

Since H_γ is separable and $R_\gamma(E')$ is dense in H_γ, we can choose a sequence $\{x_n\}$ in E' such that $\{R_\gamma x_n]$ is a complete orthonormal system (CONS) of H_γ. Define

$$\xi_n = R_\gamma^* R_\gamma x_n, \qquad n = 1, 2, 3, \cdots,$$

and

$$\pi_n \xi = \textstyle\sum_{j=1}^n \langle x_j, \xi \rangle \xi_j, \qquad \xi \in E.$$

Then obviously π_n is a continuous linear map of E into itself so that an E-valued random variable on the probability space (E, γ).

LEMMA 1. *For every bounded continuous function f on E we have*

$$f(0) = \lim_n \int_E f(\xi - \pi_n \xi)\, d\gamma(\xi).$$

In other words the E-valued random variable $U_n(\xi) = \xi - \pi_n \xi$ converges to 0 *in law.*

PROOF. The idea of the proof is the same as that of Theorem 4-1 $(e) \to (d)$ of K. Itô and M. Nisio [2] but for completeness we state it below.

Since $\{R_\gamma x_n\}$ is a CONS of H_γ, $\{\langle x_n, \xi \rangle\}$ is an independent real random sequence with the same Gaussian distribution of mean 0 and variance 1 on the probability space (E, γ). Therefore $\{\langle x_n, \xi \rangle \xi_n\}$ is an independent symmetric E-valued random sequence.

Obviously we have for every y in E'

$$\begin{aligned}\langle y, \pi_n \xi \rangle &= \textstyle\sum_{j=1}^n \langle x_n, \xi \rangle \langle y, \xi_n \rangle \\ &= \textstyle\sum_{j=1}^n \langle x_n, \xi \rangle \langle y, R_\gamma^* R_\gamma x_n \rangle \\ &= \textstyle\sum_{j=1}^n (R_\gamma y, R_\gamma x_n) x_n(\xi) \\ &\to y(\xi) = \langle y, \xi \rangle, \qquad \text{a.s. } (\gamma) \qquad \text{as } n \to +\infty,\end{aligned}$$

where $(\cdot, \cdot)$ is the inner product of H_γ, and for every n, $\langle y, \pi_n \xi \rangle$ *and* $\langle y, \xi - \pi_n \xi \rangle$ are independent.

It is easy to show that for every A, B in $\mathscr{C}(E, E')$

$$\gamma(\xi \in E;\ \pi_n\xi \in A,\ \xi - \pi_n\xi \in B)$$
$$= \gamma(\xi \in E;\ \pi_n\xi \in A)\gamma(\xi \in E;\ \xi - \pi_n\xi \in B)$$

and since γ is Radon, by Lemma 3-2 of H. Sato and Y. Okazaki [5] we have

$$\gamma(\xi;\ \pi_n\xi \in C,\ \xi - \pi_n\xi \in D)$$
$$= \gamma(\xi;\ \pi_n\xi \in C)\gamma(\xi;\ \xi - \pi_n\xi \in D)$$

for all Borel subsets C, D of E. This means that $\pi_n\xi$ and $\xi - \pi_n\xi$ are independent as E-valued random variables so that for every compact subset K of E we have

$$\gamma(K) = \int_E \gamma^n(K - \eta)\, d\gamma_n(\eta)$$

where γ_n and γ^n are the distributions of $\pi_n\xi$ and $\xi - \pi_n\xi$, respectively. Therefore there exists η_0 in E such that

$$\gamma^n(K - \eta_0) \geq \gamma(K).$$

Furthermore, since $\xi - \pi_n\xi$ is symmetrically distributed, we have

$$\gamma^n(K - \eta_0) = \gamma^n(-K + \eta_0) \geq \gamma(K).$$

On the other hand, since γ is Radon, for every positive number ϵ there exists a compact subset K such that

$$\gamma(K) \geq 1 - \frac{\epsilon}{2}.$$

Then $K_\epsilon = \frac{1}{2}(K - K)$ is also compact and we have

$$\gamma^n(K_\epsilon) = \gamma^n(\tfrac{1}{2}[K - K])$$
$$\geq \gamma^n([K - \eta_0] \cap [-K + \eta_0])$$
$$\geq 1 - \gamma^n([K - \eta_0]^c) - \gamma^n([-K + \eta_0]^c)$$
$$\geq 1 - \frac{\epsilon}{2} - \frac{\epsilon}{2} = 1 - \epsilon.$$

Therefore $\{\gamma^n\}$ is weakly relatively compact and since we have

$$1 = \lim_{n\to\infty} \int_E e^{i\langle y,\xi\rangle}\, d\gamma^n(\xi), \qquad y \in E',$$

$\{\gamma^n\}$ converges weakly to the Dirac measure δ.

This proves the lemma.

REMARK. If E is a separable Fréchet space, then $\pi_n\xi$ converges to ξ almost surely (Nguen Zuy Tien [3]).

In a similar manner we can prove the following lemma.

LEMMA 2. *For every bounded continuous function f we have*

$$f(0) = \lim_{n,m\to+\infty} \int_E f(\pi_n\xi - \pi_m\xi)\, d\gamma(\xi).$$

3. Proof of the theorem. Let E be a locally convex Hausdorff linear topological space, γ be a nondegenerate centered Gaussian Radon measure on E and f be a continuous nonnegative definite function with $f(0) = 1$ on E. Then there exists a probability measure Q on $(E^a, \mathscr{C}(E^a, E))$ such that

$$f(\xi) = \int_{E^a} e^{i\langle x,\xi\rangle}\, dQ(x), \qquad \xi \in E,$$

where $\langle x, \xi\rangle$ is the canonical bilinear form on $E^a \times E$.

Define a sequence $\{x_n\}$ in E' such that $\{R_\gamma x_n\}$ is a CONS of H_γ,

$$\xi_n = R_\gamma^* R_\gamma x_n,$$

$$\pi_n \xi = \textstyle\sum_{j=1}^n \langle x_j, \xi\rangle \xi_j,$$

$$\psi_n(x, \xi) = \langle x, \pi_n\xi\rangle = \textstyle\sum_{j=1}^n \langle x, \xi_j\rangle\langle x_j, \xi\rangle \qquad x \in E^a, \xi \in E, n = 1, 2, 3, \cdots.$$

Then obviously $\psi_n(x, \xi)$ is $\mathscr{C}(E^a, E) \times \mathscr{B}(E)$-measurable, where $\mathscr{B}(E)$ is the Borel field of E, and continuous in ξ for every fixed x in E^a.

Since f is a bounded continuous function on E, by Lemma 2 we have

$$\int_E d\gamma(\xi) \int_{E^a} (1 - \exp[-|\psi_n(x, \xi) - \psi_m(x, \xi)|])\, dQ(x)$$

$$= \int_E d\gamma \frac{1}{\pi} \int_{R^1} \frac{dt}{1+t^2} \int_{E^a} (1 - e^{it\langle x, \pi_n\xi - \pi_m\xi\rangle})\, dQ(x)$$

$$= \frac{1}{\pi}\int_{R^1} \frac{dt}{1+t^2} \int_E [1 - f(t(\pi_n\xi + \pi_m\xi))]\, d_\gamma(\xi)$$

$$\to 0, \qquad \text{as} \quad n, m, \to +\infty.$$

Therefore $\psi_n(x, \xi)$ is a Cauchy sequence $L_0(Q \times \gamma)$ and there exists

$$\psi_\infty(x, \xi) = Q \times \gamma - \lim_n \psi_n(x, \xi).$$

The convergence is in probability so that we may extract an almost surely convergent subsequence.

On the other hand by Lemma 1 we have

$$\lim_n \int_E d\gamma(\xi) \int_{E^a} [1 - \exp(-|\langle x, \xi\rangle - \langle x, \pi_n\xi\rangle|)]\, dQ(x)$$

$$= \lim_n \int_E d\gamma(\xi) \int_{R^1} \frac{1}{1+t^2} [1 - f(t(\xi - \pi_n\xi))\, dt$$

$$= \lim_n \int_{R^1} \frac{dt}{1+t^2} \int_E [1 - f(t(\xi - \pi_n\xi))]\, d\gamma(\xi) = 0.$$

Since

$$F_n(\xi) = \int_{E^a} [1 - \exp(-|\langle x, \xi\rangle - \langle x, \pi_n\xi\rangle|)]\, dQ(x)$$

is nonnegative and $\mathscr{B}(E)$-measurable, $\{F_n\}$ converges to 0 in probability so that we can extract an almost surely convergent subsequence.

Therefore, for simplicity of notation, without loss of generality we may assume that

$$\psi_\infty(x, \xi) = \lim_n \psi_n(x, \xi), \qquad \text{a.e. } (Q \times \gamma)$$
$$\lim_n F_n(\xi) = 0, \qquad \text{a.e. } (\gamma).$$

Obviously $\psi_\infty(x, \xi)$ is $Q \times \gamma$-measurable.

Put

$$Z = \left\{ (x, \xi) \in E^a \times E; \quad \begin{matrix} \lim_n \psi_n(x, \xi) \text{ exists} \\ \lim_n F_n(\xi) = 0 \end{matrix} \right\}$$

and let

$$\psi(x, \xi) = \lim_n \psi_n(x, \xi), \qquad (x, \xi) \in Z.$$

Then Z and $\psi(x, \xi)$ are $Q \times \gamma$-measurable and we have

$$(Q \times \gamma)(Z) = 1$$
$$\psi(x, \xi) = \psi_\infty(x, \xi) \qquad \text{a.e. } (Q \times \gamma).$$

Since $\lim_n F_n(\xi) = 0$ implies the convergence in probability of $\psi_n(x, \xi)$ to $\langle x, \xi \rangle$, the set

$$\mathscr{F} = \{\xi \in E; \lim_n F_n(\xi) = 0\}$$
$$= \{\xi \in E; \langle x, \xi \rangle = Q - \lim_n \langle x, \pi_n \xi \rangle\}$$

is a $\mathscr{B}(E)$-measurable linear subspace of E such that $\gamma(\mathscr{F}) = 1$.

By Fubini's Theorem there exists a Q-measurable subset W of E^a such that $Q(W) = 1$, and for every x in W

$$E_x = \{\xi \in E; (x, \xi) \in Z\}$$

is a γ-measurable linear subspace of E, $\gamma(E_x) = 1$,

$$\psi(x, \xi) = \lim_n \psi_n(x, \xi)$$
$$= \lim_n \langle x, \pi_n \xi \rangle, \qquad \xi \in E_x,$$

and $\psi(x, \xi)$ is linear on E_x. Since $\psi_n(x, \xi)$ is continuous in ξ, this shows that $\psi(x, \xi)$ belongs to E^γ for every x in W.

Let $\mathscr{H}_\gamma$ be the reproducing kernel Hilbert space of γ. Then $\mathscr{H}_\gamma$ is included in every γ-measurable linear subspace F_0 and E such that $\gamma(F_0) = 1$ (Theorem 3-4 (2) of H. Sato and Y. Okazaki [5]). Therefore we have

$$\mathscr{H}_\gamma \subset \cap_{x \in W} E_x.$$

Define a map Ψ of W into $\mathscr{H}_\gamma$ by

$$[\Psi(x), \xi] = \psi(x, \xi), \qquad x \in W, \xi \in \mathscr{H}_\gamma,$$

where $[\ ,\ ]$ is the inner product of $\mathscr{H}_\gamma$. Obviously Ψ is a measurable linear map of $(W, \mathscr{C}_Q(W))$ into $(\mathscr{H}_\gamma, \mathscr{C}(\mathscr{H}_\gamma, \mathscr{H}_\gamma))$ where $\mathscr{C}_Q(W)$ is the σ-algebra of all Q-measurable subsets of W. Then $\mu = Q \circ \Psi^{-1}$ is a probability measure on $(\mathscr{H}_\gamma, \mathscr{C}(\mathscr{H}_\gamma, \mathscr{H}_\gamma))$ and, since $\mathscr{H}_\gamma$ is a separable Hilbert space, $\mathscr{C}(\mathscr{H}_\gamma, \mathscr{H}_\gamma)$ coincides with the Borel field of $\mathscr{H}_\gamma$.

On the other hand, for every ξ in $\mathscr{H}_\gamma$ we have

$$\langle x, \xi \rangle = Q - \lim_n \langle x, \pi_n \xi \rangle \qquad \xi \in \mathscr{H}_\gamma.$$
$$= \psi(x, \xi), \qquad \text{a.e. } (Q),$$

This implies that

$$\int_{\mathscr{H}_\gamma} e^{i[y, \xi]}\, d\mu(y) = \int_W e^{i[\Psi(x), \xi]}\, dQ(x)$$

$$= \int_W e^{i\psi(x,\xi)} \, dQ(x)$$

$$= \int_W e^{i\langle x,\xi\rangle} \, dQ(x) = f(\xi), \qquad \xi \in \mathscr{H}_\gamma.$$

Let R_μ be the canonical map of $\mathscr{H}_\gamma = (\mathscr{H}_\gamma)'$ into $L_0(\mu)$ and $\xi(y) = [y, \xi] = (R_\mu \xi)(y)$. Since $\mathscr{H}_\gamma$ is dense in E (Theorem 5-1 of [5], Corollary 8-2 of [1]), for every ξ in E there exists a net $\{\xi_\alpha\}$ of $\mathscr{H}_\gamma$ which converges to ξ in the topology of E. The continuity of f implies that

$$\lim_{\alpha,\beta} \int_{\mathscr{H}_\gamma} (1 - \exp[-| \xi_\alpha(y) - \xi_\beta(y)|]) \, d\mu(y)$$

$$= \lim_{\alpha,\beta} \frac{1}{\pi} \int_{R^1} \frac{1}{1+t^2} \{1 - f(t(\xi_\alpha - \xi_\beta))\} \, dt = 0.$$

Therefore there exists a limit of $R_\mu \xi_\alpha$ in $L_0(\mu)$ which does not depend on the choice of the net $\{\xi_\alpha\}$. We define $R_\mu \xi$ by this limit and consequently R_μ is extended to a map of E into $(\mathscr{H}_\gamma)^\mu$. Then the relation

$$f(\xi) = \int_{\mathscr{H}_\gamma} e^{i\xi(x)} \, d\mu(x), \qquad \xi \in E$$

is evident.

Since, by the definition, $\mathscr{H}_\gamma$ is isomorphic to $H_\gamma = E^\gamma$, this proves the theorem.

REMARK. The existence of a nondegenerate centered Gaussian Radon measure on an arbitrary locally convex Hausdorff space E is not trivial. But in the case where E is a separable Fréchet space, we have the following lemma.

LEMMA 3.[1] *On every separable Fréchet space E there exists a nondegenerate centered Gaussian measure.*

PROOF. Let $\{| \; |_n\}$ be a sequence of seminorms which defines the topology of E and $\{\xi_n\}$ be a countable dense subset of E. Without loss of generality, we may assume that $\xi_n \neq 0$ for every n and

$$| \xi |_1 \leq | \xi |_2 \leq | \xi |_3, \cdots,$$

for every ξ in E.

Let $\{g_n\}$ be a sequence of independent real random variables with the same Gaussian distribution of mean 0 and variance 1. Then

$$X = \textstyle\sum_n 2^{-n} a_n g_n \xi_n$$

converges almost surely in E where

$$a_n = (1 + | \xi_n |_n)^{-1}, \qquad n = 1, 2, 3, \cdots.$$

In fact for every natural number k we have

$$\begin{aligned} &E[| \textstyle\sum_n 2^{-n} a_n g_n \xi_n |_k] \\ &\leq \textstyle\sum_n 2^{-n} a_n | \xi_n |_k E[| g_n |] \\ &\leq \textstyle\sum_{n<k} 2^{-n} a_n | \xi_n |_k + \sum_{n>k} 2^{-n} < +\infty, \end{aligned}$$

where $E[\cdot]$ is the mathematical expectation. Therefore the series X converges in probability, so that almost surely (K. Itô and M. Nisio [1]), with respect to every seminorm $| \; |_k$. This implies the almost sure convergence of X in the topology of E.

Let γ be the distribution of X. Then obviously γ is a centered Gaussian Borel measure on E.

[1] Lemma 3 was suggested by Y. Okazaki.

In order to prove that γ is nondegenerate, it is enough to show that the minimal closed linear subspace E_0 of γ-measure 1 contains all the ξ_n's. Suppose that, say, ξ_1 does not belong to E_0 and put $Y = \sum_{n=2}^{+\infty} 2^{-n} a_n g_n \xi_n$. Then we have

$$\begin{aligned} 1 &= \gamma(E_0) = P(X \in E_0) \\ &= P\left(\frac{1}{2} a_1 g_1 \xi_1 + Y \in E_0\right) \\ &= P(Y \in E_0 - 2^{-1} a_1 g_1 \xi_1) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \nu(E_0 - q^{-1} a_1 t \xi_1) e^{-\frac{t^2}{2}}\, dt, \end{aligned}$$

where ν is the distribution of Y. Therefore we have

$$\nu(E_0 - t\xi_1) = 1$$

for at least infinitely many real numbers t.

On the other hand, since E_0 is a linear subspace, $E_0 - t\xi_1$ and $E_0 - s\xi_1$ are mutually disjoint unless $t = s$. This is a contradiction and the lemma is proved.

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