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ON CONVERGENCE OF THE COVERAGE BY RANDOM ARCS ON A CIRCLE AND THE LARGEST SPACING[1]

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Consider n points taken at random on the circumference of a unit circle. Let the successive arc-lengths between these points be $S_1, S_2, \cdots, S_n$. Convergence of the momentgenerating function of $\max_{1 \le k \le n} S_k - \ln n$ is proved. Let each point be associated with an arc, each of length a_n, and let the length of the circumference which is not covered by any arc, the vacancy, be V_n. Convergence of the vacancy after suitable scaling is obtained. The methods used are general and can, e.g., be used to obtain asymptotic results for other spacings and coverage problems.

1. Introduction. Consider a circle of unit circumference and n points taken from a uniform distribution on it. Let the successive arc-lengths or spacings between these points be denoted by $S_1, S_2, \cdots, S_n$ with $S_1 + S_2 + \cdots + S_n = 1$. Such spacings have been widely studied, see the review papers by Pyke (1965), (1972). Some recent papers are Siegel (1978), (1979a), (1979b), Holst (1979), (1980a), (1980b), (1980c), Koziol (1980), where also further references can be found. In Chapter 4 of Solomon (1978) related problems are discussed.

Let $S_{(n)}$ be the largest spacing, i.e., $\max_{l \le k \le n} S_k$. In various ways it can be proved that

$$P(nS_{(n)} - \ln n \le x) \to \exp(-e^{-x}),$$

when $n \to \infty$; see Section 2 below. There we will give a rigorous proof of the convergence of the momentgenerating functions, which does not seem to have been done before.

Let each of the n points be the counterclockwise endpoints, say, of arcs on the circle, all of length a_n. It is easy to see that the whole circumference is covered if and only if $S_{(n)} \le a_n$, and that the uncovered part of the circumference, i.e., the vacancy, has length

$$V_n = \sum_{k=1}^n (S_k - a_n)_+.$$

Exact formulas for the distribution and moments of V_n are given in Siegel (1978) and Holst (1980c). Results on the asymptotic behavior of V_n are obtained in Siegel (1979a). By the general results by Le Cam (1958) convergence in distribution follows. Depending on how $n \to \infty$ and $a_n \to 0$, different cases occur. For the case $n \to \infty$, $a_n \to 0$ such that $P(V_n = 0) \to p$, $0 < p < 1$, it is proved in Section 3 that the momentgenerating function of $2nV_n$ is converging to that of the noncentral chi-square with zero degrees of freedom, i.e., a Poisson-mixture of chi-square distributions with even degrees of freedom and a one point distribution in zero, cf. Siegel (1979b) for further aspects of this distribution. This is a slight generalization of Siegel (1979a), Theorem 3.2, using quite different methods. In Section 4 the case when $n \to \infty$, $a_n \to 0$ such that $P(V_n = 0) \to 0$ and $\liminf na_n > 0$ is studied. The limiting distribution of $(nV_n - E(nV_n))/(\mathrm{Var}(nV_n))^{1/2}$ is a standard normal. Also, the moment generating functions converge in a neighborhood of zero implying convergence of

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all moments. This extends results by Siegel (1979a), who considered the special case $na_n = \lambda \ln(n/\beta)$, where $0 < \lambda < 1$ and $\beta > 0$ and proved convergence of moments and distributions. The methods used below are quite different from Siegel's.

The problems discussed above can obviously also be formulated as taking $n - 1$ points from a uniform distribution on the unit interval, [0, 1]. The endpoints correspond to one of the random points on the circumference.

2. The largest spacing. The exact distribution of $S_{(n)}$ was first obtained by Stevens (1939), cf. Solomon (1978) page 75. In Holst (1980b), Section 2, the distribution and expressions for the moments are also derived, e.g.,

$$E(nS_{(n)}) = \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1),$$

where

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) = \gamma,$$

is Euler's constant. Hence, $nS_{(n)}$ is of the order of magnitude $\ln n$ when $n \to \infty$. It is also known that

$$P(nS_{(n)} - \ln n \le x) \to \exp(-e^{-x}),$$

when $n \to \infty$ cf. Lévy (1939), Darling (1953), Le Cam (1958). For an (almost) elementary proof of this see Holst (1980b), Theorem 3.1. Before stating a theorem of convergence of the momentgenerating function of $nS_{(n)} - \ln n$ we will recall some facts about spacings.

Let $X_1, X_2, \cdots, X_n$ be i.i.d. exponential random variables with mean 1, and let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ denote the corresponding order statistic. Then the following representations hold

$$\mathscr{L}(nS_1, \cdots, nS_n) = \mathscr{L}(X_1, \cdots, X_n \mid \textstyle\sum_{k=1}^{n} X_k = n)$$

and

$$\mathscr{L}(nX_{(1)}, (n-1)(X_{(2)} - X_{(1)}), \cdots, 1(X_{(n)} - X_{(n-1)})) = \mathscr{L}(X_n, X_{n-1}, \cdots, X_1).$$

This is easily proved using simple properties of the Poisson process, or see, e.g., Feller (1971), pages 19, 75–76.

THEOREM 2.1. *Let $S_1, \cdots, S_n$ be the spacings of n points taken from the uniform distribution on the circumference of a unit circle and set $S_{(n)} = \max_{1\le k\le n} S_k$. Then for $t < 1$*

$$E(\exp(t(nS_{(n)} - \ln n))) \to \Gamma(1 - t),$$

when $n \to \infty$, where the gamma function can be written

$$\Gamma(1 - t) = \int_{-\infty}^{\infty} e^{tx}\, d(\exp(-e^{-x})).$$

From this Theorem we immediately have by the continuity theorem for momentgenerating functions that:

COROLLARY 2.1. *Let Y have the extreme value distribution* $\exp(-e^{-x})$. *Then*

$$\mathscr{L}(nS_{(n)} - \ln n) \to \mathscr{L}(Y),$$

and, for $r > 0$,

$$E((nS_{(n)} - \ln n)^r) \to E(Y^r),$$

when $n \to \infty$.

Before proving the Theorem we will obtain the following lemma which also is of some independent interest, at least the method of deriving it.

LEMMA 2.1. *For $t < 1$ and $n \geq 2$,*

$$E(\exp(tnS_{(n)})) = (2\pi n^{n+1}e^{-n}/n!)^{-1} \int_{-\infty}^{\infty} E(\exp(tX_{(n)} + iu(\bar{X} - 1)))\, du$$

where $X_{(n)} = \max_{1\leq k\leq n} X_k$, $\bar{X} = \sum_{k=1}^{n} X_k/n$, and $X_1, \cdots, X_n$ are i.i.d. *exponential random variables with mean* 1.

PROOF. From the representation of order statistics given above, it follows that

$$\mathscr{L}(X_{(n)}, \bar{X}) = \mathscr{L}(\textstyle\sum_{k=1}^{n} X_k/k, \sum_{k=1}^{n} X_k/n).$$

Thus

$$\begin{aligned} E(\exp(tX_{(n)} + iu\bar{X})) &= \textstyle\prod_{k=1}^{n} E(\exp(X_k(t/k + iu/n))) \\ &= \textstyle\prod_{k=1}^{n} (1 - t/k - iu/n)^{-1}, \end{aligned}$$

which clearly is an integrable function of u for $n \geq 2$. Using conditional expectation we can write

$$\begin{aligned} E(\exp(tX_{(n)} + iu\,\bar{X})) &= \int_{-\infty}^{\infty} E(\exp(tX_{(n)} + iu\,\bar{X}) \mid \bar{X} = x) \cdot f_{\bar{X}}(x)\, dx \\ &= \int_{-\infty}^{\infty} e^{iux} E(\exp(tX_{(n)}) \mid \bar{X} = x) \cdot f_{\bar{X}}(x)\, dx \end{aligned}$$

where $f_{\bar{X}}(x)$ is the density function of $\bar{X}$, which is $\Gamma(n, 1/n)$-distributed. By the integrability of $E(\exp(tX_{(n)} + iu\bar{X}))$ it follows by Fourier's inversion formula that

$$E(\exp(tX_{(n)}) \mid \bar{X} = x) \cdot f_{\bar{X}}(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} E(\exp(tX_{(n)} + iu\bar{X})) \cdot e^{-iux}\, du.$$

Thus by the representation of spacings we finally have

$$\begin{aligned} E(\exp(tnS_{(n)})) &= E(\exp(tX_{(n)}) \mid \bar{X} = 1) \\ &= (2\pi f_{\bar{X}}(1))^{-1} \int_{-\infty}^{\infty} E(\exp(tX_{(n)} + iu\bar{X})) \cdot e^{-iu}\, du \\ &= (2\pi n^{n+1}e^{-n}/n!)^{-1} \int_{-\infty}^{\infty} E(\exp(tX_{(n)} + iu(\bar{X} - 1)))\, du, \end{aligned}$$

proving the assertion.

PROOF OF THEOREM 2.1. From the lemma, the representation of order statistics of the exponential distribution, and Stirling's formula, we have

$$\begin{aligned} &E(\exp(t(nS_{(n)} - \textstyle\sum_{k=1}^{n} 1/k))) \\ &\sim (2\pi)^{-1/2} \int_{-\infty}^{\infty} \textstyle\prod_{k=1}^{n} E(\exp((t/k + iu/n^{1/2})(X_k - 1)))\, du \end{aligned}$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(iu(\textstyle\sum_{k=1}^{n} t/(k-t))/n^{1/2})$$

$$\cdot \textstyle\prod_{k=1}^{n} [\exp(-iu/(n^{1/2}(1-t/k)))/(1-iu/(n^{1/2}(1-t/k)))]\, du$$

$$\cdot \textstyle\prod_{k=1}^{n} (e^{-t/k}/(1-t/k)),$$

for $t > 1$. Now for fixed $t < 1$, when $n \to \infty$,

$$\textstyle\prod_{k=1}^{n} (e^{-t/k}/(1-t/k)) = E(\exp(t(X_{(n)} - \sum_{k=1}^{n} 1/k))) \to e^{\gamma t}\Gamma(1-t),$$

where γ is Euler's constant, cf. Holst (1980b), Theorem 3.3. The integrand in the integral above is dominated by

$$g_n(u) = (1 + cu^2/n)^{-n/2}$$

for some $c > 0$. Furthermore,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(u)\, du = \int_{-\infty}^{\infty} \lim_{n\to\infty} g_n(u)\, du.$$

For fixed $t < 1$, when $n \to \infty$,

$$\exp(n^{-1/2} \textstyle\sum_{k=1}^{n} t/(k-t)) \to 1,$$

and

$$\textstyle\prod_{k=1}^{n} [\exp(-iu/(n^{1/2}(1-t/k)))/(1-iu/(n^{1/2}(1-t/k)))] \to \exp(-u^2/2).$$

Thus it follows from the extended form of Lebesgue's convergence theorem, see, e.g., Rao (1973), page 136, that

$$\lim_{n\to\infty} E(\exp(t(nS_{(n)} - \ln n))) = e^{-\gamma t}(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-u^2/2}\, du\, e^{\gamma t}\Gamma(1-t) = \Gamma(1-t),$$

proving the assertion of the theorem.

Remark. With small modifications in the proof above the convergence of the momentgenerating function of any upper extreme value $nS_{(n-j)} - \ln n$ follows. Central order statistics are studied in Holst (1980b), Section 5. Barton and David (1956), page 84–86, also considered asymptotics of upper extreme values. But they could not obtain any useful form of the momentgenerating function. Instead they studied convergence of another generating function. Their results do not prove convergence of momentgenerating functions.

3. Positive coverage probability. In this section the coverage distribution, or equivalently the vacancy, of random arcs is studied, when the complete coverage probability stays strictly positive. With the notation of the introduction we can write

$$p_n = P(V_n = 0) = P(\textstyle\sum_{k=1}^{n} (S_k - a_n)_+ = 0)$$

$$= P(S_{(n)} \le a_n) = P(nS_{(n)} - \ln n \le na_n - \ln n).$$

From the results of the previous section we see that

$$p_n \to p, \quad 0 < p < 1 \Leftrightarrow na_n - \ln n \to \ln\ln(1/p),$$

i.e., a complete coverage probability strictly between 0 and 1 is equivalent to $na_n - \ln n = O(1)$. It also follows that

$$p_n \to 1 \Leftrightarrow na_n - \ln n \to +\infty,$$

and,

$$p_n \to 0 \Leftrightarrow na_n - \ln n \to -\infty.$$

Another way of stating the first two cases is

$$P(V_n = 0) \to e^{-\beta} \qquad 0 \le \beta < \infty.$$

The limit behavior of V_n is given by the following theorem. In the next section the case $p_n \to 0$ is considered.

THEOREM 3.1. *Let n arcs, each of length a_n, be placed at random on a unit circumference and V_n be the length of the uncovered part of the circumference. Assume that $n \to \infty$, $a_n \to 0$, such that $P(V_n = 0) \to e^{-\beta}$, $0 \le \beta < \infty$. Then, for $t < 1$, when $n \to \infty$,*

$$E(\exp(tnV_n)) \to e^{-\beta+\beta/(1-t)}.$$

In Siegel (1979b) the noncentral chi-square distribution with zero degrees of freedom is discussed. A consequence of Theorem 3.1 is the following corollary which is also proved in Siegel (1979a) by the method of moments. The convergence in distribution is a special case of general results by Le Cam (1958).

COROLLARY 3.1. *Let Z be a random variable with a noncentral chi-square distribution with zero degrees of freedom and noncentrality parameter β. Then, when $n \to \infty$,*

$$\mathscr{L}(2nV_n) \to \mathscr{L}(Z),$$

and,

$$E((2nV_n)^r) \to E(Z^r),$$

for all $r > 0$.

Before proving Theorem 3.1 the following lemma will be proved.

LEMMA 3.1. *Let $X_1, \cdots, X_n$ be* i.i.d. *exponential random variables with mean* 1. *Then for $t < 1$ and $n \ge 2$,*

$$E(\exp(tnV_n)) = (2\pi n^{n+1}e^{-n}/n!)^{-1} \int_{-\infty}^{\infty} E(\exp(t \textstyle\sum_{k=1}^n (X_k - na_n)_+ + iu(\bar{X} - 1)))\, du.$$

PROOF. By the independence between the X's and after some elementary calculation one obtains

$$\begin{aligned} &E(\exp(t \textstyle\sum_{k=1}^n (X_k - na_n)_+ + iu\bar{X})) \\ &\quad = [E(\exp(t(X_1 - na_n)_+ + iuX_1/n))]^n \\ &\quad = (1 - iu/n)^{-n}[1 + t\exp(-na_n(1 - iu/n))/(1 - t - iu/n)]^n, \end{aligned}$$

which is integrable in u for $n \ge 2$. Using the representation of spacings with exponential random variables we have

$$\mathscr{L}(nV_n) = \mathscr{L}(\textstyle\sum_{k=1}^n (X_k - na_n)_+ \,|\, \bar{X} = 1).$$

The rest of the proof proceeds like that of Lemma 2.1.

PROOF OF THEOREM 3.1. By the lemma above and Stirling's formula we get

$$E(\exp(tnV_n)) \sim (2\pi)^{-1/2} \int_{-\infty}^{\infty} [\exp(-iu/n^{1/2})/(1 - iu/n^{1/2})]^n$$

$$\cdot\, [1 + t\exp(-na_n(1 - iu/n^{1/2}))/(1 - t - iu/n^{1/2})]^n\, du.$$

As $na_n \gtrsim \ln n$ we get for fixed $t < 1$ uniformly in u that

$$|(1 + t\exp(-na_n(1 - iu/n^{1/2}))/(1 - t - iu/n^{1/2}))|^n$$
$$\leq (1 + K_1|t|/n|1 - t - iu/n^{1/2}|)^n \leq K_2 < \infty.$$

As

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |1 - iu/n^{1/2}|^n\, du = \int_{-\infty}^{\infty} \lim_{n\to\infty} |1 - iu/n^{1/2}|^n\, du = \int_{-\infty}^{\infty} e^{-u^2/2}\, du$$
$$= (2\pi)^{1/2},$$

and pointwise

$$[1 + te^{-na_n} \cdot \exp(iu\, na_n/n^{1/2})/(1 - t - iu/n^{1/2})]^n \to \exp(\beta t/(1 - t))$$

it follows by the extended form of Lebesgue's convergence theorem that

$$E(\exp(tnV_n)) \to (2\pi)^{-1/2} \cdot \int_{-\infty}^{\infty} \exp(-u^2/2) \cdot \exp(\beta t/(1 - t))\, du = e^{-\beta} e^{\beta/(1-t)},$$

for $t > 1$, which proves the theorem.

REMARK. The function

$$e^{-\beta} e^{\beta/(1-t)} = \textstyle\sum_{k=0}^{\infty} (e^{-\beta}\beta^k/k!) \cdot (1 - t)^{-k}$$

is the momentgenerating function of $\sum_{j=1}^{N} X_j$, where $N, X_1, X_2, \cdots$, are independent random variables, N Poisson with mean β and the X's exponential with mean 1. One can interpret N as the number of gaps, i.e., the number of regions on the circumference which are not covered by any of the arcs. This can be proved in a similar way as Theorem 3.1 using the indicator function $I(\cdot > na_n)$ instead of $(\cdot - na_n)_+$. Clearly $P(N = 0) = e^{-\beta}$ is the probability of complete coverage. It is intuitively clear that the lengths of the gaps (after scaling with n) should be independent exponential random variables. Because an arbitrary spacing nS_k converges in distribution to an exponential with mean 1 and depending on the lack of memory of the exponential distribution the excess (if any) over na_n has also in the limit an exponential distribution with mean 1. Theorem 3.1 is thus very reasonable. One can also say that the dependence structure between the spacings disappears in the case $na_n \gtrsim \ln n$. Actually the dependence is asymptotically negligible as soon as $na_n \to +\infty$ which will be apparent from the results of the next section.

4. Zero coverage probability. It is pointed out in the previous section that

$$p_n = P(V_n = 0) \to 0 \Leftrightarrow na_n - \ln n \to -\infty.$$

Two cases are of interest, namely, $na_n \to +\infty$, but $na_n - \ln n \to -\infty$, and $na_n \to \alpha$, $0 < \alpha < \infty$. The case $na_n \to 0$ means that the maximum covered length, na_n, is tending to zero and, therefore, $V_n \to 1$. Let us introduce

$$\sigma_n^2 = 2n(e^{-na_n} - e^{-2na_n}(1 + na_n + (na_n)^2/2)).$$

In the case $na_n \to +\infty$ we have $\sigma_n^2 \sim 2ne^{-na_n}$, and $\sigma_n \to +\infty$ if and only if $na_n - \ln n \to -\infty$.

THEOREM 4.1. *Suppose that* $n \to \infty$ *and* $a_n \to 0$ *in such a way that* $\sigma_n \to +\infty$, *and* lim inf $na_n > 0$. *Then, when* $n \to \infty$,

$$E(\exp(t(nV_n - ne^{-na_n})/\sigma_n)) \to e^{t^2/2},$$

for all sufficiently small $|t|$.

PROOF. As in the proof of Theorem 3.1, we find that

$$E(\exp(t(nV_n - ne^{-na_n})/\sigma_n)) \sim (2\pi)^{-1/2} \int_{-\infty}^{\infty} g_n(u)h_n(u, t)\, du,$$

where

$$g_n(u) = (\exp(-iu/n^{1/2})/(1 - iu/n^{1/2}))^n \to \exp(-u^2/2), \qquad n \to \infty,$$

and,

$$h_n(u, t) = \exp(-tne^{-na_n}/\sigma_n) \cdot (1 + (t \exp(-na_n(1 - iu/n^{1/2}))/\sigma_n)/(1 - t/\sigma_n - iu/n^{1/2}))^n.$$

For fixed t and u one finds after some calculation that

$$g_n(u)h_n(u, t) = \exp(-(u - ite^{-na_n}(na_n + 1)n^{1/2}/\sigma_n)^2/2 + t^2/2 + o(1)).$$

Thus one would expect

$$E(\exp(t(nV_n - ne^{-na_n})/\sigma_n) \sim (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-(u + O(1))^2/2 + o(1))\, du \cdot e^{t^2/2} \sim e^{t^2/2}.$$

The problem to justify these approximations is not trivial because $o(1)$ is not uniform in u.

We will consider the integral above over three different regions, namely, $I_1 = \{u\,;\, |u| \le n^{1/4}\}$, $I_2 = \{u\,;\, n^{1/4} < |u| \le \delta n^{1/2}\}$, and $I_3 = \{u\,;\, \delta n^{1/2} < |u|\}$ where $\delta > 0$ is a "sufficiently" small number. The idea is the same as that of proving local limit theorems using characteristic functions, see, e.g., Feller (1971), page 516.

In the interval I_1 one finds by expansion that uniformly in u

$$|h_n(u, t)| \le K_1 < \infty,$$

for some constant K_1. Thus

$$\lim_{A\to\infty} \limsup_{n\to\infty} \left| \int_A^{n^{1/4}} g_n(u)h_n(u, t)\, du \right|$$

$$\le \lim_{A\to\infty} \limsup_{n\to\infty} \int_A^{n^{1/4}} (1 + u^2/n)^{-n/2} K_1\, du = 0.$$

Using this it follows by the expansion above that

$$\lim_{n\to\infty} (2\pi)^{-1/2} \int_{I_1} g_n(u)h_n(u, t)\, du = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-u^2/2)\, du \cdot e^{t^2/2} = e^{t^2/2}.$$

In the region I_2 with $\delta > 0$ fixed sufficiently small one finds in a similar way that

$$|h_n(u, t)| \le K_2 \exp(K_2 |t| n^{1/2})$$

for some constant $K_2 < \infty$. Thus

$$\left| \int_{I_2} g_n(u)h_n(u, t)\, du \right| \le \int_{I_2} (1 + u^2/n)^{-n/2} K_2 \exp(K_2 |t| n^{1/2})\, du$$

$$\le K_3 n^{1/2} \exp(-K_4 n^{1/2}) \to 0, \qquad n \to \infty,$$

for some constants $0 < K_3, K_4 < \infty$, and $|t|$ sufficiently small.

Finally for I_3 we find for some constants $0 \le K_5, K_6, K_7 < \infty$ that

$$\left| \int_{I_3} g_n(u)h_n(u, t)\, du \right| \le K_5 \int_{\delta n^{1/2}}^{\infty} (1 + u^2/n)^{-1}\, du (1 + \delta^2)^{-n/2} \exp(K_6 n^{1/2})$$

$$\le K_7 n^{1/2} (1 + \delta^2)^{-n/2} \exp(K_6 n^{1/2}) \to 0, \qquad n \to \infty.$$

Combining the results above gives

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} g_n(u)h_n(u, t)\,du \to e^{t^2/2},$$

proving the assertion of the theorem.

In Siegel (1978) and Holst (1980c) formulas for the exact distribution and moments of the vacancy are obtained. Either by using these or by direct calculation it is not hard to show that

$$E(nV_n) - ne^{-na_n} = O(1),$$

and,

$$\operatorname{Var}(nV_n)/\sigma_n \to 1,$$

when $n \to \infty$, $a_n \to 0$, such that $\sigma_n \to +\infty$ and $\liminf na_n > 0$. Thus the following corollary follows.

COROLLARY 4.1. *If* $\sigma_n \to +\infty$ *and* $\liminf na_n > 0$, *then*

$$\mathscr{L}((nV_n - E(nV_n))/(\operatorname{Var}(nV_n))^{1/2}) \to N(0, 1),$$

and all moments and the momentgenerating function converge to those of the standard normal distribution.

REMARK. The convergence of distribution above is a special case of Le Cam (1958).

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