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TRANSITIVITY IN PROBLEMS OF OPTIMAL STOPPING

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In a sequential decision problem it is usually assumed that the available information is represented by an increasing family $\mathscr{F}$ of σ-algebras. Often a reduction, e.g., according to principles of sufficiency or invariance, is performed which yields a smaller family $\mathscr{G}$. The consequences of such a reduction for problems of optimal stopping are treated in this paper.

It is shown that $\mathscr{G}$ is transitive for $\mathscr{F}$ (in the Bahadur sense) if and only if for any stochastic process adapted to $\mathscr{G}$ the value (i.e., maximal reward by optimal stopping) under $\mathscr{G}$ and the value under $\mathscr{F}$ are equal.

1. Introduction and basic concepts. When we consider a sequential decision problem we usually assume that the amount of available information is increasing with time. We then represent the possible data which we can obtain up to the time $t \in T$ by a σ-algebra $\mathscr{F}_t$, and thus come to the formal requirement '$\mathscr{F}_s \subset \mathscr{F}_t$ for $s < t$'. But it is often hard to store and handle all the data as represented by $\mathscr{F} = (\mathscr{F}_t)_{t \in T}$ since their actual amount may be very large, so we want to apply a reduction procedure which leads to a smaller family $\mathscr{G} = (\mathscr{G}_t)_{t \in T}$ of σ-algebras. Such a reduction should of course take into account the underlying statistical structure and may, e.g., be accomplished according to the principles of sufficiency or invariance.

The consequences of reduction in sequential decision problems were treated by Bahadur (1954) who introduced the notion of transitivity. He showed (in the case $T = \mathbb{N}$) that for every sequential decision rule with respect to $\mathscr{F}$ there exists an equivalent sequential decision rule with respect to $\mathscr{G}$ if $\mathscr{G}$ is sufficient and transitive. The case that the parameter space of the decision problem contains only one element is trivial from the point of view of decision theory, and in this case the trivial family $(\{\varnothing, \Omega\})_{t \in T}$ is obviously sufficient and transitive. But from the point of view of optimal stopping there seems to emerge a problem of some significance.

Let us assume that all the available information is represented by a family $\mathscr{F} = (\mathscr{F}_t)_{t \in T}$ and that a data reduction has led to a smaller family $\mathscr{G} = (\mathscr{G}_t)_{t \in T}$, and furthermore that we are given a probability measure on the underlying sample space and a real-valued stochastic process $X = (X_t)_{t \in T}$ such that each X_t is $\mathscr{G}_t$-measurable. The problem of optimal stopping for X with respect to $\mathscr{G}$, resp $\mathscr{F}$, then is to maximize the reward EX_τ in a certain set of 'reasonable' stopping times τ with respect to $\mathscr{G}$, resp $\mathscr{F}$, i.e., find the supremum $v(X, \mathscr{G})$, resp $v(X, \mathscr{F})$, of possible rewards and look for optimal stopping times. It will usually be easier to determine $v(X, \mathscr{G})$ since there are fewer stopping times with respect to the smaller family $\mathscr{G}$, and so the problem arises to characterize those families $\mathscr{G}$, $\mathscr{F}$ for which $v(X, \mathscr{G})$ and $v(X, \mathscr{F})$ are equal. For a formalization of these considerations let us introduce some notions and notations which will be used from now on.

Let $(\Omega, \mathscr{A}, P)$ be the basic probaility space, $T \neq \varnothing$ an ordered set—the set of time parameters. Furthermore let

$$g(T) = \{\mathscr{G} = (\mathscr{G}_t)_{t \in T} \colon \mathscr{G}_t \text{ sub-}\sigma\text{-algebra of } \mathscr{A}\}$$

$$m(T) = \{\mathscr{G} = (\mathscr{G}_t)_{t \in T} \colon \mathscr{G} \in g(T),\ \mathscr{G}_s \subset \mathscr{G}_t \text{ for } s < t\}.$$

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For $\mathcal{G} \in g(T)$ define $\mathcal{G}^* \in m(T)$ by $\mathcal{G}_t^* = \sigma(\cup_{s\le t} \mathcal{G}_s)$, thus $\mathcal{G}_t \subset \mathcal{G}_t^*$.

Let there be given a point ∞ such that $\infty \notin T$ and $\infty > t$ for all $t \in T$ and set $T^* = T \cup \{\infty\}$ and for $\mathcal{G} \in g(T)$ $\mathcal{G}_\infty = \sigma(\cup_{t\in T} \mathcal{G}_t)$. A mapping $\tau: \Omega \to T^*$ is called a stopping time with respect to $\mathcal{G} \in g(T)$ iff for all $t \in T^*$ the following holds:

(i) $\{\tau = t\} = \{\tau \ge t\} \cap D_t$ for some $D_t \in \mathcal{G}_t$

(ii) $\{\tau \ge t\} \in \mathcal{G}_t^*$ and $P(\{\tau = \infty\}) = 0$.

The set of all stopping times with respect to $\mathcal{G}$ is denoted by $s(\mathcal{G})$. Condition (i) has the interpretation that, having observed up to the time t, the decision to stop the observation at this time is only to be based on events in $\mathcal{G}_t$. This definition yields the usual notion if $\mathcal{G}$ is an increasing family, but in the general case it may, e.g., happen that the maximum of two stopping times is no longer a stopping time (the minimum always is).

We now introduce some more notations: For $\mathcal{G} \in g(T)$ let $M(\mathcal{G})$ denote the set of all real-valued stochastic processes $X = (X_t)_{t\in T}$ on $(\Omega, \mathcal{A}, P)$ such that X_t is $\mathcal{G}_t$-measurable, and $M_1(\mathcal{G})$ all $X \in M(\mathcal{G})$ with $X_t(\Omega) \subset [0; 1]$ P-a.s. for all $t \in T$. We set

$$M = \cup \{M(\mathcal{G}): \mathcal{G} \in g(T)\}, \qquad M_1 = \cup \{M_1(\mathcal{G}): \mathcal{G} \in g(T)\}$$

and call $X \in M$ integrable iff EX_t is finite for all $t \in T$. Typical examples of stopping times with respect to a general family are (under suitable assumptions on measurability) hitting times of the form $\tau = \inf\{t \in T: X_t \in B_t\}$ with $X \in M(\mathcal{G})$.

For $X \in M$, $\mathcal{G} \in g(T)$ and $\tau \in s(\mathcal{G})$ the mapping X_τ is defined by $X_\tau(\omega) = I_{\{\tau<\infty\}}(\omega) X_{\tau(\omega)}(\omega)$.

The problem to maximize EX_τ in a certain set of 'reasonable' stopping times with respect to $\mathcal{G}$ is called the problem of optimal stopping for X with respect to $\mathcal{G}$.

For the following formal definition we denote by $d(\mathcal{G})$ the set of all stopping times with respect to $\mathcal{G}$ such that $\tau(\Omega) \cap T$ is order-isomorphic to a subset of the natural numbers.

(1.1) DEFINITION. For $X \in M$, $\mathcal{G} \in g(T)$ define

$$v(X, \mathcal{G}) = \sup\{EX_\tau: \tau \in d(\mathcal{G}), EX_\tau^- < \infty\} \qquad (\sup \varnothing = -\infty).$$

$v(X, \mathcal{G})$ is called value of X with respect to $\mathcal{G}$.

The quantity $v(X, \mathcal{G})$ thus gives the supremum which can be achieved in the problem of optimal stopping for X with respect to certain 'well-behaved' stopping times in $s(\mathcal{G})$ and obviously depends on $\mathcal{G}$ in general. To obtain a unified definition for arbitrary time sets T we only use stopping times in $d(\mathcal{G})$ for the definition of the value. This is certainly no restriction if already $T \subset N$ holds, but also in other interesting cases $v(X, \mathcal{G})$ is an upper bound of EX_τ for arbitrary $\tau \in s(\mathcal{G})$. We want to illustrate this fact in the case $T = [0; \infty)$ and for this and further use we introduce the following notion: Considering here and in the following order relations between random variables to hold a.s., we call a measurable process $X \in M$ $\mathcal{G}$-bounded from below (above) for $\mathcal{G} \in m(T)$ iff there exists an integrable random variable h with $X_\tau \ge E(h \mid \mathcal{G}_\tau)$ ($X_\tau \le E(h \mid \mathcal{G}_\tau)$) for all $\tau \in s(\mathcal{G})$.

(1.2) PROPOSITION. *Suppose $T = [0; \infty)$, $\mathcal{G} \in m(T)$ and X a measurable process. Assume that X is $\mathcal{G}$-bounded from below and that for every decreasing sequence $(\tau_n)_n$ in $s(\mathcal{G})$ $X_{\lim \tau_n} \le \liminf X_{\tau_n}$ holds. Then for every $\tau \in s(\mathcal{G})$ $EX_\tau^- < \infty$ and $EX_\tau \le v(X, \mathcal{G})$.*

The easy proof is omitted; compare Thompson (1971), page 310.

2. Bahadur-transitivity. In the definition of transitivity families $\mathcal{G}$ and $\mathcal{F}$ will be considered, which are ordered in the following way: We define $\mathcal{G} \le \mathcal{F}$ iff for every $t \in T$ $\mathcal{G}_t \subset \mathcal{F}_t$ holds P-a.s., i.e., for any $G_t \in \mathcal{G}_t$ there exists $F_t \in \mathcal{F}_t$ with $P(G_t \Delta F_t) = 0$. If $\mathcal{G} \le \mathcal{F}$ holds then obviously for any $X \in M$, $v(X, \mathcal{G}) \le v(X, \mathcal{F})$.

(2.1) DEFINITION. Suppose $\mathcal{G}, \mathcal{F} \in g(T)$. $\mathcal{G}$ is called Bahadur-(B-) transitive for $\mathcal{F}$ iff $\mathcal{G} \le \mathcal{F}$ and for all $s, t \in T$ with $s < t$

$$P(G_t \mid \mathcal{G}_s) = P(G_t \mid \mathcal{F}_s) \quad \text{for all} \quad G_t \in \mathcal{G}_t.$$

The following result shows the importance of B-transitivity for problems of optimal stopping. Let us remark that no assumptions are made on the ordered set T.

(2.2) THEOREM. *Assume $\mathscr{G} \in g(T)$, $\mathscr{F} \in m(T)$ with $\mathscr{G} \leq \mathscr{F}$. Then the following statements are equivalent:*

(i) *$\mathscr{G}$ is B-transitive for $\mathscr{F}$.*

(ii) *For every integrable $X \in M(\mathscr{G})$ we have*

$$v(X, \mathscr{G}) = v(X, \mathscr{F}).$$

(iii) *For every $X \in M_1(\mathscr{G})$ we have*

$$v(X, \mathscr{G}) = v(X, \mathscr{F}).$$

PROOF. (i) $\Rightarrow$ (ii). By the definition of the value it is obviously enough to show this for $T = \mathbb{N}$. But for $T = \mathbb{N}$ we may apply the arguments of Chow, Robbins and Siegmund (1971), page 103, page 111, which easily give the proof.

(ii) $\Rightarrow$ (iii) is obviously true.

(iii) $\Rightarrow$ (i). Consider $s, t \in T$ with $s < t$ and $G_t \in \mathscr{G}_t$. We define $X \in M_1(\mathscr{G})$ by

$$X_r = \begin{cases} 0 & r \notin \{s, t\} \\ I_{G_t} & r = t \\ P(G_t \mid \mathscr{G}_s) & r = s \end{cases}$$

Obviously $P(G_t) \leq v(X, \mathscr{G})$.
For $\tau \in d(\mathscr{G})$ there exists $D_s \in \mathscr{G}_s$ with $\{\tau = s\} = \{\tau \geq s\} \cap D_s$. This implies $\{\tau = s\} \subset D_s$ and $\{\tau = t\} \subset D_s^c$, thus

$$\begin{aligned} EX_\tau &= \int_{\{\tau=s\}} P(G_t \mid \mathscr{G}_s)\, dP + \int_{\{\tau=t\}} I_{G_t}\, dP \\ &\leq \int_{D_s} P(G_t \mid \mathscr{G}_s)\, dP + \int_{D_s^c} I_{G_t}\, dP = P(G_t). \end{aligned}$$

From this we have $v(X, \mathscr{G}) = P(G_t)$ and by (iii)

$$P(G_t) = v(X, \mathscr{F}) \geq \sup\{EX_\tau : \tau \in s(\mathscr{F}), \tau(\Omega) \subset \{s, t\}\} \geq P(G_t).$$

This implies

$$P(G_t) = E \max\{P(G_t \mid \mathscr{G}_s), P(G_t \mid \mathscr{F}_s)\},$$

thus $\max\{P(G_t \mid \mathscr{G}_s), P(G_t \mid \mathscr{F}_s)\} = P(G_t \mid \mathscr{F}_s)$. But this yields at once $P(G_t \mid \mathscr{G}_s) = P(G_t \mid \mathscr{F}_s)$.
□

Let us remark on the important special case that X is induced by a stochastic process, i.e., consider $\mathscr{F} \in m(T)$, $X \in M(\mathscr{F})$ and $\sigma(X) = (\sigma(X_t))_{t \in T} \in g(T)$. Then $\sigma(X)$ is B-transitive for $\mathscr{F}$ iff X has the Markov property with respect to $\mathscr{F}$, so that (2.2) gives a characterization of Markov processes by considering the behaviour in problems of optimal stopping. In this context the statement (i) $\Rightarrow$ (ii) is well-known, see, e.g., Chow, Robbins and Siegmund (1971), page 103.

For increasing families the following result is easily obtained, so that we omit the proof:

(2.3) PROPOSITION. *For $\mathscr{G}, \mathscr{F} \in m(T)$ with $\mathscr{G}_t \subset \mathscr{F}_t$ for all $t \in T$ the following statements are equivalent:*

(i) *$\mathscr{G}$ is B-transitive for $\mathscr{F}$.*

(ii) *Every supermartingale with respect to $\mathscr{G}$ is a supermartingale with respect to $\mathscr{F}$.*

(iii) *Every martingale with respect to $\mathscr{G}$ is a martingale with respect to $\mathscr{F}$.*

The definition of transitivity furthermore has the following measure-theoretical consequences.

(2.4) PROPOSITION. *Assume $\mathscr{G} \in g(T)$, $\mathscr{F} \in m(T)$ and T linearly ordered. If $\mathscr{G}$ is B-transitive for $\mathscr{F}$, then $\mathscr{G}^*$ is B-transitive for $\mathscr{F}$ and*

$$(\mathscr{G}_\infty \cap \mathscr{F}_t)_{t \in T} \leq \mathscr{G}^*.$$

PROOF. It is easily seen that $\mathscr{G}^*$ is B-transitive for $\mathscr{F}$ and thus by (2.3) every martingale with respect to $\mathscr{G}^*$ is a martingale with respect to $\mathscr{F}$. This implies, see Sekiguchi (1976), page 213, that for every $t \in T$, $\mathscr{G}_t^* \supset \mathscr{G}_\infty \cap \mathscr{F}_t$ P-a.s., thus $(\mathscr{G}_\infty \cap \mathscr{F}_t)_{t \in T} \leq \mathscr{G}^*$. □

A necessary condition for the B-transitivity of an increasing family $\mathscr{G}$ is thus given by $(\mathscr{G}_\infty \cap \mathscr{F}_t)_{t \in T} \leq \mathscr{G}$. It is easy to construct examples (already for $T = \{1, 2\}$) which show that this condition is not sufficient. Let us remark that in (2.4) the reverse relation, i.e., $\mathscr{G}^* \leq (\mathscr{G}_\infty \cap \mathscr{F}_t)_{t \in T}$, is trivially true.

The statements in (2.2 and (2.3)—given for arbitrary time sets—were essentially based on a discrete parameter argument. In the following we will treat the continuous parameter case—i.e., $T = [0; \infty)$, and consider arbitrary stopping times for well-measurable processes where in general an approximation by stopping times taking only countably many values does not seem to be possible. The following notions will be useful.

A family $\mathscr{G} \in m(T)$ is called regular iff $\mathscr{G}_\infty$ is P-complete, $\mathscr{G}_0$ contains all P-zero sets of $\mathscr{G}_\infty$ and $\mathscr{G}$ is right-continuous. These are the families usually encountered in the 'general theory of processes'. For a regular family $\mathscr{G} \in m(T)$ we define (compare Mertens (1972)): An integrable $X \in M(\mathscr{G})$ is called a strong $\mathscr{G}$-supermartingale iff X is $\mathscr{G}$-well-measurable and for all bounded $\rho, \tau \in s(\mathscr{G})$ with $\rho \leq \tau$

$$EX_\tau > -\infty \quad \text{and} \quad E(X_\tau \mid \mathscr{G}_\rho) \leq X_\rho.$$

X is called a strong $\mathscr{G}$-martingale iff X and $-X$ are strong $\mathscr{G}$-supermartingales. A strong $\mathscr{G}$-supermartingale is called regular iff for all $\rho, \tau \in s(\mathscr{G})$ with $\rho \leq \tau$

$$EX_\tau \ \text{exists and} \quad E(X_\tau \mid \mathscr{G}_\rho) \leq X_\rho.$$

It is well known that every $\mathscr{G}$-supermartingale with right-continuous paths (P-a.s.) is a strong $\mathscr{G}$-supermartingale and furthermore that every strong $\mathscr{G}$-supermartingale, which is $\mathscr{G}$-bounded from below, is regular. Conversely we have according to Mertens (1972): Every strong $\mathscr{G}$-supermartingale has (P-a.s.) paths which are upper semicontinuous from the right; every strong $\mathscr{G}$-martingale has (P-a.s.) right-continuous paths. We can now prove:

(2.5) THEOREM. *Assume $T = [0; \infty)$, $\mathscr{G}, \mathscr{F} \in m(T)$, $\mathscr{G}$ and $\mathscr{F}$ regular with $\mathscr{G}_t \subset \mathscr{F}_t$ for all $t \in T$. Then the following statements are equivalent:*

(i) *$\mathscr{G}$ is B-transitive for $\mathscr{F}$.*

(ii) $\sup\{EX_\tau : \tau \in s(\mathscr{G})\} = \sup\{EX_\tau : \tau \in s(\mathscr{F})\}$ *for all $\mathscr{G}$-well-measurable $X \in M_1(\mathscr{G})$.*

(iii) $\sup\{EX_\tau : \tau \in s(\mathscr{G})\} = \sup\{EX_\tau : \tau \in s(\mathscr{F})\}$ *for all $\mathscr{G}$-well-measurable $X \in M(\mathscr{G})$, which are $\mathscr{G}$-bounded from below.*

(iv) *Every strong $\mathscr{G}$-supermartingale is a strong $\mathscr{F}$-supermartingale.*

(v) *Every strong $\mathscr{G}$-martingale is a strong $\mathscr{F}$-martingale.*

PROOF. We will prove (i) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i); (i) $\Rightarrow$ (v): Let Y be a strong $\mathscr{G}$-martingale; then Y is a martingale with respect to $\mathscr{G}$ (in the usual sense), thus by (2.3) Y is a martingale with respect to $\mathscr{F}$. According to the above remarks Y has (P-a.s.) right-continuous paths, thus Y is a strong $\mathscr{F}$-martingale.

(v) $\Rightarrow$ (iv): Let X be a strong $\mathscr{G}$-supermartingale, then for $n \in \mathbb{N}$ $\min\{X, n\}$ defines a strong $\mathscr{G}$-supermartingale, for which we may apply the Doob-Meyer-decomposition theorem (see Mertens (1972)). Thus there is a strong $\mathscr{G}$-martingale Y^n and a process $A^n \in M(\mathscr{G})$ with increasing paths, such that $P(\cup_{t \in T}\{\min\{X_t, n\} \neq Y_t^n - A_t^n\}) = 0$. By (v) Y^n is a strong $\mathscr{F}$-martingale, which implies that $\min\{X, n\}$ is a strong $\mathscr{F}$-supermartingale. Letting n tend to infinity we obtain that X is a strong $\mathscr{F}$-supermartingale.

(iv) $\Rightarrow$ (iii): We need the following auxiliary statement:

(2.6). *If* (iv) *holds, then every $\mathscr{G}$-well-measurable $X \in M(\mathscr{G})$, which is $\mathscr{G}$-bounded from below, is also $\mathscr{F}$-bounded from below.*

PROOF OF (2.6): Consider a $\mathscr{G}$-well-measurable $X \in M(\mathscr{G})$ and an integrable random variable h such that $X_\tau \geq E(h \mid \mathscr{G}_\tau)$ for all $\tau \in s(\mathscr{G})$. From the regularity of $\mathscr{G}$ we may choose versions $E(h \mid \mathscr{G}_t)$, such that $(E(h \mid \mathscr{G}_t))_{t\in T}$ has right-continuous paths. Now the well-measurability of X implies together with the assumption of $\mathscr{G}$-boundedness that

$$P(\cup_{t\in T}\{X_t < E(h \mid \mathscr{G}_t)\}) = 0.$$

We may assume, without loss of generality, that h with this property is $\mathscr{G}_\infty$-measurable. Now as in (2.3), (iv) implies (v) and—using the regularity of $\mathscr{G}$—(v) implies (i).

From (i) one concludes by the usual extension procedure, that for every $G \in \mathscr{G}_\infty$ we have $P(G \mid \mathscr{G}_t) = P(G \mid \mathscr{F}_t)$ for every $t \in T$, thus also $E(h \mid \mathscr{G}_t) = E(h \mid \mathscr{F}_t)$ for every $t \in T$. Choosing versions $E(h \mid \mathscr{F}_t)$, such that $(E(h \mid \mathscr{F}_t))_{t\in T}$ has right-continuous paths, we may conclude

$$P(\cup_{t\in T}\{E(h \mid \mathscr{F}_t) \neq E(h \mid \mathscr{G}_t)\}) = 0,$$

and

$$P(\cup_{t\in T}\{X_t < E(h \mid \mathscr{F}_t)\}) = 0.$$

This yields $X_\tau \geq E(h \mid \mathscr{F}_\tau)$ for every $\tau \in s(\mathscr{F})$, thus X is $\mathscr{F}$-bounded from below.

To prove the implication '(iv) $\Rightarrow$ (iii)' consider X as in (2.6), thus $EX_\tau > -\infty$ for all $\tau \in s(\mathscr{F}) \supset s(\mathscr{G})$. Since for all $\tau \in s(\mathscr{F})$, $EX_\tau = \sup_{n\in\mathbb{N}} E \min\{X_\tau, n\}$, we may assume, without loss of generality, that $\sup_\tau EX_\tau \leq k$ for some $k \in \mathbb{N}$. According to Mertens (1972) there exists the minimal dominating regular $\mathscr{G}$-(resp $\mathscr{F}$) supermartingale $Z(\mathscr{G})$ (resp $Z(\mathscr{F})$) for X such that

$$\sup\{EX_\tau : \tau \in s(\mathscr{G})\} = EZ(\mathscr{G})_0, \quad \sup\{EX_\tau : \tau \in s(\mathscr{F})\} = EZ(\mathscr{F})_0,$$

see Mertens (1972), page 54–55 for details. (The condition '$\sup_\tau EX_\tau \leq k$' here ensures the integrability of $Z(\mathscr{G})$, $Z(\mathscr{F})$, but otherwise does not enter into the argument.) Since X is $\mathscr{F}$-bounded from below, $Z(\mathscr{G})$ is also $\mathscr{F}$-bounded from below and by (iv) $Z(\mathscr{G})$ is a strong $\mathscr{F}$-supermartingale. Thus $Z(\mathscr{G})$ is regular. Now the minimality of $Z(\mathscr{F})$ implies

$$Z(\mathscr{F})_0 \leq Z(\mathscr{G})_0,$$

thus

$$\sup\{EX_\tau : \tau \in s(\mathscr{G})\} = EZ(\mathscr{G})_0 \geq EZ(\mathscr{F})_0 = \sup\{EX_\tau : \tau \in s(\mathscr{F})\}.$$

Since the other inequality is obvious this proves (iii). (iii) $\Rightarrow$ (ii) is obvious and (ii) $\Rightarrow$ (i) follows as in (2.2). □

In the following we want to consider a situation, appearing rather naturally in statistical problems, in which we find B-transitivity.

(2.7) EXAMPLE. Let $T = [0; \infty)$ and P, Q Gaussian measures on $(\Omega, \mathscr{A})$ where $\Omega = \mathbb{R}^T$ and $\mathscr{A}$ the product-σ-algebra of the Borel sets of $\mathbb{R}$. For $t \in T$ let $\mathscr{F}_t$ be the σ-algebra generated by the coordinate mappings up to the time t. We denote the mean value function of P(resp Q) by m_p(resp m_Q); to simplify the exposition we assume $m_P = 0$ and $m_Q(0) = 0$.

Furthermore we assume that P and Q have a common covariance kernel K, K continuous and nonsingular, and that for every $t \in T$ $m_Q \mid [0; t]$ belongs to the reproducing kernel Hilbert space of $K \mid [0; t] \times [0; t]$. Then for every $t \in T$ $P \mid \mathscr{F}_t$ and $Q \mid \mathscr{F}_t$ are equivalent with density

$$\left.\frac{dQ}{dP}\right| \mathscr{F}_t = \exp(Y_t - \frac{1}{2} r(t)),$$

where $r: T \to T$ is increasing, and $(Y_t)_{t \in T}$ is with respect to P a Gaussian process with mean value function 0 and covariance kernel $r(\min\{s, t\})$; see Bhattacharya and Smith (1972); now $m_Q(0) = 0$ implies $r(0) = 0$ and $Y_0 = 0$ P-a.s. Assume now that r is continuous, strictly increasing and unbounded. Then (see, e.g., Irle (1980)) any separable version of $(Y_{r^{-1}(t)})_{t \in T}$ is a Wiener process with respect to the increasing right-continuous family $(\mathscr{F}^+_{r^{-1}(t)})_{t \in T}$ (under P). Setting $\mathscr{G}_t = \sigma(Y_t)$ it follows that $\mathscr{G}$ is B-transitive for $\mathscr{F}^+$

We remark that this can be used to obtain locally best tests for Gaussian processes; see Irle (1980).

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