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CIRCUIT PROCESSES

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Circuit processes of order r are defined using a finite class of weighted circuits in a finite set S. The probability of the next value of the process is made proportional to the total weight of those circuits in the class which pass through the value in question *and* the last r values. The process is an order r Markov chain in S, and the stationary distribution is easily calculated. Also, it is shown that all stationary order r Markov chains in a finite set can be represented as circuit processes of that order.

1. Introduction. Consider the directed graph in Figure 1, which consists of three overlapping directed circuits, $c_1 = (t, a, b, c)$, $c_2 = (a, b, r, s)$ and $c_3 = (s, t, a, b, c)$. Imagine a particle moving among the points, or vertices, $S = \{a, b, c, r, s, t\}$, taking one step each unit of time, always traveling along one of the directed arcs chosen in a random way, thus giving rise to a stochastic process $X_1, X_2, \cdots$, with values in S. Given a history of two steps, say, $X_{n-1} = a$, $X_n = b$, the probability that $X_{n+1} = y$ is calculated as follows: first, fixed weights $w_c > 0$ are assigned to each circuit, for example, $w_{c_1} = 5$, $w_{c_2} = 3$, $w_{c_3} = 7$, as indicated in the figure. Then for each y in S, the circuits $\mathscr{C}(a, b, y)$ which *pass through* (a, b, y) in that order are located, such a circuit being one in which a occurs, followed by b and then by y. Thus $\mathscr{C}(a, b, c) = \{c_1, c_3\}$, and $\mathscr{C}(a, b, r) = \{c_2\}$, and there are no others, i.e., $\mathscr{C}(a, b, y)$ is empty for all other y in S. Next, the sum $w(a, b, y)$ of the weights w_c for $c \in \mathscr{C}(a, b, y)$ is calculated, taking the sum to be zero, of course, if $\mathscr{C}(a, b, y)$ is empty, and finally the probability that $X_{n+1} = y$ given $X_{n-1} = a$ and $X_n = b$ is set equal to

$$\frac{w(a, b, y)}{\sum_{x \in S} w(a, b, x)}. \tag{1}$$

For $y = c$ this gives $12/15$ and for $y = r$, $3/15$. Using this method for assigning probabilities to each point given each possible two step history (X_{n-1}, X_n) through which passes at least one of the circuits, determines a probability law for the process $X_1, X_2, \cdots$. We call it a circuit process of order two. Evidently it is a familiar stationary Markov chain of order two, i.e., the probability distribution of the next value of the process depends only on the last two values. That the earlier value may make a difference is seen by calculating as above, $P[X_{n+1} = t \mid X_n = s, X_{n-1} = c] = 1$ and $P[X_{n+1} = t \mid X_n = s, X_{n-1} = r] = 0$. Of course,

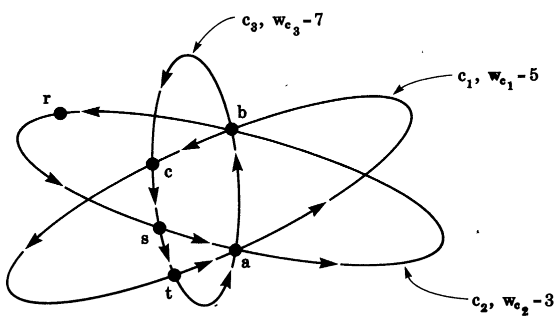

FIG. 1.

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the process is an ordinary Markov chain (of order one, if you will) in the space of such histories of length two.

It is easy to verify, as will be done below, that the stationary or invariant distribution for the process is given simply by $w(x, y) = \sum_{z \in S} w(x, y, z)$ calculated as above, but appropriately normalized on each ergodic[1] class. Thus, the order two Markov chain under consideration has for its states the set of ordered pairs (x, y) in which y follows x in at least one circuit, and in this instance, the class of such pairs for which $w(x, y)$ is positive is seen by inspection to form a single ergodic class. On this class $\mu(x, y) = w(x, y/\sum w(x, y)$, the sum in the denominator being taken over all pairs in the class, is the unique positive invariant distribution. For example $\mu(a, b) = 15/67$, where $\sum w(x, y) = 67$ is calculated simply by counting the number of arcs in each circuit, multiplying by the corresponding weight, and adding over circuits, viz., $4 \cdot 5 + 4 \cdot 3 + 5 \cdot 7 = 67$. In the case of multiple ergodic classes, $w(x, y)$ would be normalized on each class to give the invariant probability on that class. To illustrate, suppose b is removed from c_2 so $c_2 = (a, r, s)$. Then there are two ergodic classes, and if $X_1 = a$, $X_2 = r$, for example, the process is restricted thereafter to the single circuit (a, r, s) while if $X_1 = t$, $X_2 = a$, for example, the process is restricted thereafter to c_1 and c_3. On the latter class $\mu(a, b) = 12/55$. Multiple ergodic classes seem to be of potential interest here, whereas in the usual applications of Markov chains a single ergodic class will almost invariably suffice. For this reason the possibility of multiple ergodic classes is assumed throughout, which, as it happens, adds very little technical difficulty.

The above can be generalized to create for any given positive integer r, a general class of processes, called circuit processes, which are stationary and nontransient Markov chains of order r, with easily calculated stationary distributions. This is done in some detail in the section which follows. Moreover, it is shown that to every stationary Markov chain of order r in a finite set S, there is a circuit process with the same transition probabilities on the ergodic classes.

The latter result shows that without loss of generality the parameters of a stationary order r Markov chain may be taken to be a finite class of circuits and their weights, instead of the transition probabilities themselves. This fact may be useful in applications. Among other things, it suggests that model formulation might on occasion begin with a suitable class of circuits. If, by good fortune, the process at hand is seen to have a special structure such that a relatively small number of circuits will suffice to capture the qualitative behavior, subsequent exploration of the model and of the phenomena itself is facilitated because the stationary behavior is then directly accessible. For example, both transition probabilities, and average costs associated with visiting certain states, can be expressed explicitly in terms of the weights, which permits fairly easy sensitivity analysis of their relationship. A minimal use of circuit processes is heuristic and didactic: examples of order r processes with explicit stationary distributions are provided in abundance, which should help understand the value and limitations of such processes in describing natural phenomena.

2. Circuit processes. Here a *circuit* is a periodic function c on the integers $\mathcal{N}$ into a nonempty finite set S. We say c is a 'circuit in S'. Thus there is a smallest positive integer $k \geq 1$ such that $c(t + k) = c(t)$ for all $t \in \mathcal{N}$. We call $k = k(c)$ the *length*, or *period*, of c. Let c^j be the circuit defined by $c^j(t) = c(t + j)$, j any integer. A circuit c is also completely determined by any of the sequences of elements $(c(t), c(t + 1), \cdots, c(t + k - 1))$, or sequences of arcs $((c(t), c(t + 1)), (c(t + 1), c(t + 2)), \cdots, (c(t + k - 1), c(t)))$, where $k = k(c)$, and t is any integer. These alternative descriptions of a circuit will be used freely depending on convenience. For example, the ordered list of elements is used in the introduction. This is the most economical description for many purposes.

[1] Also called irreducible; all states in the class can be reached from one another following paths of positive probability.

A circuit has another representation when the elements of c are identified with points in the plane, as a closed, directed, continuous curve with a continuous directional derivative, which passes through each of the points in c, and in the same order as in c, but takes a distinct direction at each point. This definition sometimes enables one to draw a circuit on a piece of paper in a convenient and reasonably unambiguous fashion, because the continuity of direction enables one to follow the right path at each point where the circuits intersect, as is illustrated by Figure 1.

Given a sequence $u = (u(1), u(2), \cdots, u(n))$ where each $u(i) \in S$ and $n \geq 1$, which will be called a 'sequence in S,' let $J_c(u)$ be the number of distinct integers j, $0 \leq j \leq k(c) - 1$, such that $c^j(t) = u(t)$, $t = 1, 2, \cdots, n$. If there is at least one such j, we say c passes through u, and $J_c(u)$ is then just the number of times c passes through u. Obviously $J_{c^i}(u) = J_c(u)$ for all i. A property of the function J_c, basic to all that follows, is that it is *balanced* as defined by Lemma 1 below.

Let u be a sequence in S and y an element in S. Then (y, u) and (u, y) are, respectively, the sequences in S, $(y, u(1), \cdots, u(n))$ and $(u(1), \cdots, u(n), y)$.

Lemma 1. *For $u = (u(1), \cdots, u(n))$ any sequence in S, $\sum_{y\in S} J_c(y, u) = \sum_{y\in S} J_c(u, y) = J_c(u)$.*

Proof. If c does not pass through u, it does not pass through (y, u) for any y, so in this case $\sum_y J_c(y, u) = J_c(u) = 0$. On the other hand, for each distinct j such that $c^j(t) = u(t)$, $t = 1, 2, \cdots, n$, there is a unique y, namely $y = c^j(0)$, such that with the same and hence distinct j, we can say c^{-1} passes through (y, u), that is $c^{-1+j}(1) = c^j(0) = y$, $c^{-1+j}(2) = c^j(1) = u(1), \cdots, c^{-1+j}(n+1) = c^j(n) = u(n)$. Thus each j contributing a count of one to $J_c(u)$ contributes a count of exactly one to $\sum_{y\in S} J_{c^{-1}}(y, u) = \sum_{y\in S} J_c(y, u)$. A similar argument shows $J_c(u) = \sum_{y\in S} J_c(u, y)$. □

Let $\mathscr{C}$ be a finite class of circuits and for each $c \in \mathscr{C}$ let w_c be a given positive number. For u any sequence in S, let $w(u) = \sum_{c\in\mathscr{C}} w_c J_c(u)$. Then w is also balanced, as expressed by the following lemma.

Lemma 2. *For u any sequence in S, $\sum_{y\in S} w(y, u) = \sum_{y\in S} w(u, y) = w(u)$.*

Finally, consider the sequence of random variables $X_1, X_2, \cdots$ with values in S, satisfying for all $n \geq r$,

$$P[X_{n+1} = y \mid X_n = z_r, X_{n-1} = z_{r-1}, \cdots, X_{n-r+1} = z_1, X_{n-r} = y_{n-r}, \cdots, X_1 = y_1] = \frac{w(z_1, z_2, \cdots, z_r, y)}{w(z_1, z_2, \cdots, z_r)}, \tag{2}$$

whenever $w(z_1, z_2, \cdots, z_r)$ is positive, and whatever may be the sequence $y_1, y_2, \cdots, y_{n-r}$. The process $X_1, X_2, \cdots$ will be called a circuit process of order r. Evidently the process is a familiar Markov chain of order r, which is to say an ordinary Markov chain in the space of sequences in S of length r. The transition law is completely specified by the integer $r \geq 1$, the class of circuits $\mathscr{C}$, the positive circuit weights w_c, and the formula (2) for the transition probabilities.

To make the Markov structure clear, let $\mathscr{H}$ be the set of all sequences in S of the fixed length r, which will now be called histories, and let $\mathscr{H}^+$ be the subset of $\mathscr{H}$ where w is positive. For $u \in \mathscr{H}$, let $\Gamma(u)$ be the set of histories defined by $\Gamma(z_1, z_2, \cdots, z_r) = \{v : v = (z_2, z_3, \cdots, z_r, y), y \in S\}$, and for v a history $(z_1, z_2, \cdots, z_r)$ let $y(v) = z_r$, the last element of v. For $u \in \mathscr{H}^+$, and $v \in \Gamma(u)$ let

$$p(u, v) = \frac{w(u, y(v))}{w(u)} \tag{3}$$

and for $v \notin \Gamma(u)$ put $p(u, v) = 0$. Thus $p(u, v)$ is defined for all v but only for $u \in \mathscr{H}^+$. Then an equivalent definition of a circuit process, of order r, is a sequence of random variables $U_1, U_2, \cdots$ with values in $\mathscr{H}$, where

$$(4) \qquad P[U_{n+1} = v \mid U_n = u, U_{n-1} = u_{n-1}, \cdots, U_1 = u_1] = p(u, v)$$

whenever $u \in \mathscr{H}^+$. Although $p(u, v)$ is defined only for $u \in \mathscr{H}^+$, it will be assumed henceforth that U_1 is in $\mathscr{H}^+$ with probability one, and since $p(u, v) > 0$ only if $v \in \mathscr{H}^+$, any future history having positive probability given U_1, is also in $\mathscr{H}^+$. Thus the incompleteness of the definition causes no harm. The process $U_1, U_2, \cdots$, in $\mathscr{H}^+$ and the process $X_1, X_2, \cdots$ in S are directly related by $U_n = (X_n, X_{n+1}, \cdots, X_{n+r-1})$, $n = 1, 2, \cdots$.

The convenience in working with circuit processes comes largely from the following result:

LEMMA 3. *The function w is invariant with respect to p on $\mathscr{H}^+$, that is, for all $u \in \mathscr{H}^+$,*

$$(5) \qquad w(u) = \textstyle\sum_{v \in \mathscr{H}^+} w(v) p(v, u).$$

PROOF. Let $u = (z_1, z_2, \cdots, z_r)$. Since $p(v, u) \equiv 0$ if $u \notin \Gamma(v)$, only v of the form $(y, z_1, z_2, \cdots, z_{r-1})$ can contribute to the sum on the right in (5), which, from the definition of p, is then

$$\textstyle\sum_{y \in S} w(y, z_2, z_3, \cdots, z_{r-1}) \dfrac{w(y, z_1, z_2, \cdots, z_r)}{w(y, z_2, z_3, \cdots, z_{r-1})} = \textstyle\sum_{y \in S} w(y, z_1, z_2, \cdots, z_r)$$
$$= w(z_1, z_2, \cdots, z_r),$$

with the latter equality following from Lemma 2, and the proof is complete. □

To apply Lemma 3 in calculating stationary distributions, suppose $G \subseteq \mathscr{H}$ is an ergodic class for which $p(u, v) = 0$ if $u \in G$ and $v \notin G$. For such a class the unique stationary distribution p_G is given by $p_G(u) = 0$ for $u \notin G$, and for $u \in G$,

$$p_G(u) = \frac{w(u)}{\sum_{u \in G} w(u)}.$$

To see this it suffices to note that p_G so defined satisfies the equation $p_G(u) = \sum_{v \in G} p_G(v) p(v, u)$, for $u \in G$ by Lemma 3, and considering $\sum_{u \in G} p_G(u) = 1$, it is unique by a standard result (see Doob (1953), page 179 ff.). Given the circuits $\mathscr{C}$ and the weights w_c, determining the ergodic classes G and p_G for each class is computationally straightforward.

Consider now the problem of constructing a circuit process to represent any given order r Markov chain on the ergodic classes of the chain. It turns out to be convenient to actually do this construction first using circuits in $\mathscr{H}^+$, obtaining, in effect, a representation of the given transition probability p in terms of circuit processes of order one in $\mathscr{H}^+$. The construction using circuits in S then follows because the representations are isomorphic in a way made precise below. This approach, although perhaps not the shortest, has the advantage of making the relationship between the circuit processes of order r and the Markov chains in $\mathscr{H}^+$ completely transparent. Hence the following rather lengthy preliminaries, which only state formally what is readily apparent from study of a few examples.

A circuit in $\mathscr{H}$ is a periodic function γ on the integers $\mathscr{N}$ into the space $\mathscr{H}$ of sequences in S of length r, with the added property that $\gamma(t + 1) \in \Gamma(\gamma(t))$ for all t. The length or period of γ is the smallest positive integer $n = n(\gamma)$ such that $\gamma(t + n) = \gamma(t)$ for all t. A curcuit in $\mathscr{H}$ is *elementary* if for some t, and hence all t, the $n(\gamma)$ elements $\gamma(t), \gamma(t + 1), \cdots, \gamma(t + n(\gamma) - 1)$ are all different from one another. Also, for all integers j let γ^j be the circuit defined by $\gamma^j(t) = \gamma(t + j)$, and let $\gamma(t)(i)$, $i = 1, 2, \cdots, r$, be the ith element in $\gamma(t)$, that is if $\gamma(t) = (x_1, x_2, \cdots, x_r)$, $\gamma(t)(i) = x_i$, and $x_i \in S$, $i = 1, 2, \cdots, r$. Notice $\gamma(t)(i) = \gamma^j(t)(i - j)$ for $1 \leq j < i \leq r$, and specifically, $\gamma(t)(i) = \gamma^{i-1}(t)(1)$ for $i = 1, 2, \cdots, r$.

For γ a given circuit in $\mathscr{H}$ define the circuit c in S by the formula $c(t) = \gamma(t)(1)$. The period of c is exactly that of γ since if $c(t) = c(t+k)$ for all t, and hence $c(t+i-1) = c(t+i-1+k) = \gamma(t+i-1)(1) = \gamma^{i-1}(t)(1) = \gamma(t)(i) = \gamma(t+i-1+k)(1) = \gamma^{i-1}(t+k)(1) = \gamma(t+k)(i)$, $i = 1, 2, \cdots, r$, the period of γ is at most that of c, whereas since $c(t) = \gamma(t)(1) = \gamma(t+n)(1) = c(t+n)$ for all t the period of c is at most that of γ. Thus the two periods are equal.

If γ is elementary, the circuit c just defined has another property, that of being *r-elementary*, by which is meant, the smallest integer $k \geq 1$ such that for some t, $c(t+k+i) = c(t+i)$ for $i = 1, 2, \cdots, r$, is $k = k(c)$. Thus it is not surprising that the r-elementary circuits in S are isomorphic to the elementary circuits in $\mathscr{H}$. That is, for a given circuit c in S define γ by $\gamma(t)(i) = c(t+i-1)$, $i = 1, 2, \cdots, r$. Then $\gamma(t+1) \in \Gamma(\gamma(t))$, that is, $\gamma(t+1)(i) = c(t+i) = \gamma(t)(i+1)$, $i = 1, 2, \cdots, r-1$ (see the definition of Γ above). Also, if c is r-elementary, γ is elementary with the same period as c, since if $c(t+k) = c(t)$ for all t certainly $\gamma(t)(i) = \gamma(t+k)(i)$, $i = 1, 2, \cdots, r$, and the latter condition cannot hold for $k' < k$ if c is r-elementary. It will be convenient to let $c = \theta\gamma$ and $\gamma = \theta' c$ be the circuits in S and $\mathscr{H}$, respectively, defined for elementary circuits in $\mathscr{H}$ and r-elementary circuits in S, by the above formulae, that is, $c = \theta\gamma$ is defined for all t by $c(t) = \gamma(t)(1)$ and $\gamma = \theta' c$ is defined for all t by $\gamma(t)(i) = c(t+i-1)$, $i = 1, 2, \cdots, r$.

One other observation is needed before proceeding further. For γ an elementary circuit in $\mathscr{H}$, and u, v elements in $\mathscr{H}$ with $v \in \Gamma(u)$, let $J^*_\gamma(u, v) = 1$ if for some t, $1 \leq t \leq n(\gamma) - 1$, $\gamma(t) = u$ and $\gamma(t+1) = v$ and let $J^*_\gamma(u, v) = 0$ otherwise. Thus $J^*_\gamma(u, v)$ is just the number of times, necessarily zero or one, that γ passes through (u, v). Then for elementary γ and r-elementary c, $J^*_\gamma(u, v) = J_{\theta\gamma}(u, y(v))$ where $y(v) \in S$ is the last element in v defined previously and, conversely, $J_c(u, y) = J^*_{\theta' c}(u, v(u, y))$ where if $u = (x_1, \cdots, x_r)$, $v(u, y) = (x_2, \cdots, x_r, y)$. Both of these formulae are easily established. To illustrate, suppose $J^*_\gamma(u, v) = 1$, so for some t, $1 \leq t \leq n(\gamma) - 1$, $\gamma(t) = u$ and $\gamma(t+1) = v$. Then $\gamma(t)(i) = c(t+i-1)$, $i = 1, 2, \cdots, r$ and $c(t+r) = \gamma(t+r)(1) = \gamma(t+1)(r)$, so $c = \theta\gamma$ passes through u, $y(v)$, and exactly once at that, since c is r-elementary, so $J_{\theta\gamma}(u, y(v)) = 1$. The argument for the remaining cases is similar.

The above observations yield immediately the main result needed for the construction.

LEMMA 4. *Let $\mathscr{C}^*$ be a class of elementary circuits in $\mathscr{H}^+$ and for each γ in $\mathscr{C}^*$ let w^*_γ be a given positive number. Let $w^*(u, v) = \sum_{\gamma \in \mathscr{C}^*} w^*_\gamma J^*_\gamma(u, v)$. Then, letting $\mathscr{C} = \theta\mathscr{C}^*$ and $w_c = w^*_{\theta' c}$, we have $w^*(u, v) = w(u, y(v))$ where $w(u, y) = \sum_{c \in \mathscr{C}} w_c J_c(u, y)$.*

Now let $U_1, U_2, \cdots$ be a Markov chain of order r, that is, each U_n is a random variable with values in $\mathscr{H}$, and $P[U_{n+1} = v \mid U_n = u, U_{n-1} = u_{n-1}, \cdots, U_1 = u_1] = p(u, v)$ for all $n \geq 1$ and $\sum_{v \in \Gamma(u)} p(u, v) = 1$. As is well known, for each ergodic subclass $G \subseteq \mathscr{H}$ (see again, Doob (1953), page 179 ff.) there is a probability distribution p_G on $\mathscr{H}$ such that $p_G > 0$ on G, $\sum_{u \in G} p_G(u) = 1$, implying $p_G = 0$ outside G, and for $u \in G$, $p_G(u) = \sum_{v \in G} p_G(v) p(v, u)$. Also for $u \in G$, $p(u, v) = 0$ if $v \notin G$. Let $\mathscr{H}^+$ be the union of all such subsets G.

THEOREM 1. *There is a finite class $\mathscr{C}$ of r-elementary circuits in S, and circuit weights $w_c > 0$ such that for all $u \in \mathscr{H}^+$*

$$p(u, v) = \frac{w(u, y(v))}{w(u)}, \tag{6}$$

where as above $w(u, y) = \sum_{c \in \mathscr{C}} w_c J_c(u, y)$ and $w(u) = \sum_{c \in \mathscr{C}} w_c J_c(u)$.

PROOF. For each ergodic class G let α_G be a positive number, and define w^* for all elements u, v in $\mathscr{H}^+$ by

$$w^*(u, v) = \sum_G \alpha_G p_G(u) p(u, v). \tag{7}$$

Obviously,

(8) $$p(u, v) = \frac{w^*(u, v)}{w^*(u)},$$

where $w^*(u) = \sum_{v\in\mathscr{H}^+} w^*(u, v) = \sum_G \alpha_G p_G(u)$. Also notice w^* is balanced on $\mathscr{H}^+$, that is, $\sum_{v\in\mathscr{H}^+} w^*(u, v) = \sum_{v\in\mathscr{H}^+} w^*(v, u)$, because of the invariance of the p_G. It will be shown first, in Lemma 5, using just the balance, that there is a class $\mathscr{C}^*$ of elementary circuits in $\mathscr{H}^+$ and weights $w^*_\gamma > 0$ for $\gamma \in \mathscr{C}^*$ such that $w^*(u, v) = \sum_{\gamma\in\mathscr{C}^*} w^*_\gamma J^*_\gamma(u, v)$ for all u, v in $\mathscr{H}^+$.

LEMMA 5.[2] *Let w^* be a nonnegative function on the set of all ordered pairs (u, v) of elements from a finite set $\mathscr{H}^+$, with $\sum_{v\in\mathscr{H}^+} w^*(u, v) > 0$ for all $u \in \mathscr{H}^+$, and suppose $\sum_{v\in\mathscr{H}^+} w^*(u, v) = \sum_{v\in\mathscr{H}^+} w^*(v, u)$, $u \in \mathscr{H}^+$. Then $w^*(u, v) = \sum_{\gamma\in\mathscr{C}^*} w^*_\gamma J^*_\gamma(u, v)$ where $\mathscr{C}^*$ is a finite class of elementary circuits in $\mathscr{H}^+$, $J^*_\gamma(u, v) = 1$ if $\gamma(t) = u$ and $\gamma(t + 1) = v$ for some t and is zero otherwise, and w^*_γ is a positive number for each $\gamma \in \mathscr{C}^*$.*

PROOF. Here circuits are conveniently described in terms of their arcs. Consider the directed graph in $\mathscr{H}^+$ with a directed arc (u, v) for each pair u, v such that $w^*(u, v) > 0$. Since w^* is balanced, existence of an arc (u, v) entering v implied there is at least one arc leaving v. Thus starting at some element $u_1 \in \mathscr{H}^+$ we can find a sequence of arcs (u_1, u_2), $(u_2, u_3), \cdots$ for which $w^*(u_i, u_{i+1})$ is positive, the existence of each member of the sequence being guaranteed by the existence of the preceding. However, $\mathscr{H}^+$ being finite, there is a *smallest* integer $n \geq 2$ such that $u_n = u_i$ for some i, $1 \leq i < n$. Then $\gamma_1 = (u_i, u_{i+1}), (u_{i+1}, u_{i+2}), \cdots, (u_{n-1}, u_i)$ is an elementary circuit in $\mathscr{H}^+$. Let $w^*_{\gamma_1}$ be the minimum of $w^*(u, v)$ over the arcs in γ_1, and let $w_1(u, v) = w^*(u, v) - w^*_{\gamma_1} J^*_{\gamma_1}(u, v)$. Then w_1 is nonnegative for $u \in \mathscr{H}^+$ since if $J^*_{\gamma_1}(u, v) = 1$, that is, the arc (u, v) is in γ_1, $w^*(u, v) \geq w^*_{\gamma_1}$ by the choice of the latter. Since $J^*_{\gamma_1}$ is balanced, w_1 is balanced, and if w_1 is positive for some u, v, another elementary circuit γ_2 can be found and extracted, giving $w_2 = w_1 - w^*_{\gamma_2} J^*_{\gamma_2} = w^* - w^*_{\gamma_1} J^*_{\gamma_1} - w^*_{\gamma_2} J^*_{\gamma_2}$, etc. But note $w_1(u, v)$ is zero for at least one arc (u, v) for which $w^*(u, v)$ is positive, namely any arc in γ_1 yielding the minimum $w^*_{\gamma_1}$. Thus the initially finite number of arcs in the graph with positive weight is reduced at each step and finally $w_{n+1} = 0$ and then $w^* = \sum_{i=1}^n w^*_{\gamma_i} J^*_{\gamma_i}$. With $\mathscr{C}^* = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ and weights $w^*_{\gamma_i}$ the proof is complete.

The proof of Theorem 1 can now be finished. The function w^* in Lemma 5 is taken to be w^* given by (7). Then $w^* = \sum_{\gamma\in\mathscr{C}^*} w^*_\gamma J^*_\gamma$ and Lemma 4 provides the circuits $\mathscr{C} = \theta\mathscr{C}^*$ and the weights $w_c = w^*_{\theta' c}$ such that $w^*(u, v) = w(u, y(v))$ where $w(u, y) = \sum_{c\in\mathscr{C}} w_c J_c(u, y)$, and because of (8), $p(u, v) = w^*(u, v)/w^*(u) = w(u, y(v))/w(u)$ where obviously $w^*(u) = w(u)$, completing the proof.

The representation provided by Theorem 1 is not unique as is shown by the order two process with just two circuits (a, b, c, d, c, b) and (b, c), each with a weight of one, which process can be represented equally well by the two circuits (a, b, c, b) and (b, c, d, c) each with a weight of one again. All four circuits are 2-elementary, and either representation might arise from the construction of Lemma 5.

Theorem 1 shows that the circuits used in creating a circuit process of order r could be taken to be r-elementary without narrowing the class of processes, so far as their behavior is concerned. However, it is not necessary to do so and in fact it provides a certain sense of freedom to be able to put down any finite number of circuits and associated weights, and know that these define a circuit process of order r, for every $r \geq 1$. If the process so specified uses some circuits which are not r-elementary for a given r of interest, and an equivalent representation in terms of r-elementary circuits is desired, it can be obtained by

[2] This lemma and other elementary facts about Markov chains and balanced graphs were previously noted by the author (MacQueen (1977)). In spite of its usefulness, the lemma does not appear in any of a large number of works on Markov chains and graphs which were checked. The author was led to the lemma, eventually, because of an interesting note by S. K. Stein (1968).

using (3) and Lemma 3 to find the invariant distributions, and then applying Theorem 1 and the construction of Lemma 5.

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