DISCUSSION

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I would like to direct my comments to a few specific items in Professor Freed's paper.

I found the exact formula for $\langle |\mathbf{R}|^2 \rangle$, the first equation in the paper, to be quite striking even though peripheral to the main part of the paper. It turns out that it can easily be derived using conditional expectations. Since

$$\langle |\mathbf{R}|^2 \rangle = \langle |\sum_{j=1}^n \mathbf{b}_j|^2 \rangle = \sum_{i=1}^n \sum_{j=1}^n \langle \mathbf{b}_i \cdot \mathbf{b}_j \rangle \tag{1}$$

it suffices to compute $\langle \mathbf{b}_i \cdot \mathbf{b}_j \rangle$. Now $\langle \mathbf{b}_i \cdot \mathbf{b}_{i+1} \rangle = b^2 \cos \omega$ where $\omega = \pi - \theta$ since $\mathbf{b}_i \cdot \mathbf{b}_{i+1} \equiv b^2 \cos \omega$. Assume inductively that $\langle \mathbf{b}_i \cdot \mathbf{b}_{i+k-1} \rangle = b^2 \cos^{k-1} \omega$. It is clear geometrically that

$$E(\mathbf{b}_{i+1} \mid \mathbf{b}_i) = \mathbf{b}_i \cos \omega$$

for all i.

Let $\mathscr{F}_i = \mathscr{F}(\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_i)$. Then

$$\begin{aligned}\langle \mathbf{b}_i \cdot \mathbf{b}_{i+k} \rangle &= \langle E(\mathbf{b}_i \cdot \mathbf{b}_{i+k} \mid \mathscr{F}_{i+k-1}) \rangle \\ &= \langle \mathbf{b}_i \cdot E(\mathbf{b}_{i+k} \mid \mathscr{F}_{i+k-1}) \rangle = \langle \mathbf{b}_i \cdot E(\mathbf{b}_{i+k} \mid \mathbf{b}_{i+k-1}) \rangle \\ &= \langle \mathbf{b}_i \cdot \mathbf{b}_{i+k-1} \cos \omega \rangle = (b^2 \cos^{k-1} \omega) \cos \omega \\ &= b^2 \cos^k \omega\end{aligned}$$

by the properties of conditional expectation, the Markov property and the inductive hypothesis.

Therefore, $\langle \mathbf{b}_i \cdot \mathbf{b}_{i+k} \rangle = b^2 \cos^k \omega$ for all integers i and k by induction. Substituting in (1), the result follows by a straightforward calculation.

The quantity Z_N defined by equation (2.9) is proportional to the total number of walks given the excluded volume constraint. This can be understood as follows. For simplicity, let $J(r) = ((4/3)\,\pi\epsilon^3)^{-1} I_{B_\epsilon}$ where I_{B_ϵ} is the indicator function of the ball with center at the origin and radius ϵ. The essential thing is that J be an approximate δ function. Then

$$\begin{aligned} Z_n = &\left[\prod_{i=1}^N \int_{\{|r_i - r_j| > \epsilon \text{ all } i \neq j\}} d\mathbf{r}_i\right] G_0(\{\mathbf{r}_i\} \mid \mathbf{r}_0 \equiv 0) \exp\left[-\frac{v}{2} \sum_{i \neq j=0}^N J(\mathbf{r}_i - \mathbf{r}_j)\right] \\ &+ \left[\prod_{i=1}^N \int_{\{|r_i - r_j| \leq \epsilon \text{ some } i \neq j\}} d\mathbf{r}_i\right] G_0(\{r_i\} \mid \mathbf{r}_0 \equiv 0) \exp\left[-\frac{v}{2} \sum_{i \neq j=0}^N J(\mathbf{r}_i - \mathbf{r}_j)\right]\end{aligned}$$

The first term is $P\{|\mathbf{r}_i - \mathbf{r}_j| > \epsilon, \text{all } i \neq j\}$, the probability in terms of $G_0(\{r_i\})$. The second term is positive and less than $\exp[-v((4/3)\pi\epsilon^3)^{-1}]$. Thus, for the second term to be negligible we need $v = c((4/3)\pi\epsilon^3)$ for large enough c. This indicates why v is called the excluded volume. Thus

$$Z_N \approx P\{|r_i - r_j| > \epsilon, \quad \text{all} \quad i \neq j\}.$$

Now, if we let n_{EV} be the total number of walks under the Gaussian distribution satisfying the excluded volume restriction, and let n be the number of unrestricted walks (number of trials), then

$$\frac{n_{EV}}{n} \approx P\{|\mathbf{r}_i - \mathbf{r}_j| > \epsilon, \quad \text{all} \quad i \neq j\}$$

and Z_N is, except for a small error, proportional to n_{EV}.

Concerning Section 3, I know too little about the method of random fields to make any serious comments on the mathematics of the self consistent field approach. It is worthwhile, however, to point out that the use of simple finite dimensional analogues of equation (3.2) is a standard trick for evaluating partition functions in statistical mechanics problems. The simplest and best known is the Curie-Weiss or mean field model [6, page 99], but also see [5, page 254 and page 266]. Likewise, the use of the saddle point method is also standard [6, page 100], [5, page 272], [1]. Anyone with an aversion to using the saddle point method should also see [4]. Thus, operating on a purely formal level, it is natural to try equations (3.2) and (3.7) to see how much information can be obtained.

One might easily get bogged down reading through section 3 and not get to section 4, but then one would miss a very nice idea which is used in the scaling argument between equations (4.7) and (4.14). A rigorous proof of equation (4.14) would be a significant mathematical contribution.

Finally, it seems like it would be worthwhile to look at the work of Donsker and Varadhan [2], [3] in which they show that certain expectations involving Brownian motion have power law dependencies like $A_d t^{\gamma_d}$ for large time t, where A_d and γ_d depend on the space dimension d, to see if their techniques could be applied in this setting.

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