The Annals of Probability
1981, Vol. 9, No. 6, 1043–1044

ON THE JUMPS OF TIME CHANGED DIFFUSIONS[1]

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Assuming points to be polar, a time changed diffusion has zero or an infinity of jumps in any predictable set.

1. Let X_t be a Hunt process on a locally compact set E. (We shall use the notations of [1].) We shall say that X_t has *singular jumps* if and only if, for any Borel set $A \subseteq E$, and initial distribution μ, *the expectation of the number of jumps starting from A, up to a finite time t,* $E_\mu\{\#\{s \leq t, X_{t-} \neq X_t$ and $X_{t-} \in A\}\}$ *is always null or infinite.*

Proposition 1. *If X_t has singular jumps, for any predictable set H, the number of jumps in H,* $\Sigma_H(\omega) = \#\{t \in H(\omega), X_t(\omega) \neq X_{t-}(\omega)\}$ is a.s. *null or infinite.*

Proof. Let (N, K_t) be a Lévy system of the process. Since $E_\mu(\#\{s \leq t, X_{t-} \neq X_t$ and $X_{t-} \in A\}) = E_\mu(\int_0^t N1(X_s)1_A(X_s)\, dK_s)$, clearly, X_t has singular jumps iff the set $U = \{0 < N1 < \infty\}$ is such that $E_\mu(\int_0^\infty e^{-t}1_U(X_t)\, dK_t) = 0$. Therefore, if H is any predictable set, $E_\mu(\int_H N1(X_s)\, dK_s) = 0$ or $+\infty$. But this latter expression is equal to $E_\mu(\Sigma_H)$. Define $\tau_H(\omega) = \inf\{t \in H(\omega), X_{t-}(\omega) \neq X_t(\omega)\}$.

One checks easily that τ_H is a stopping time, since it can be written as the lower envelope of a countable set of stopping times.

Now, let σ_H be the restriction of τ_H to the set $V = \{X_{\tau_H} \neq X_{\tau_H-}\} \cap \{\tau_H(\omega) \in H(\omega)\}$. Since $V \in \mathscr{F}_{\tau_H-}$ (indeed, $\{X_{\tau_H} \neq X_{\tau_H-}\} = \{\tau_H = (\tau_H)_i\}$, $(\tau_H)_i$ being the totally inaccessible part of τ_H), there is a predictable set $L \subseteq [0\ \tau_H]$ such that $\{\tau_H(\omega) \in L(\omega)\} = V$. But since there is only one jump in $L \cap H$ if $\omega \in V = \{\sigma_H < \infty\}$, $E_\mu(\sigma_H < \infty) = E_\mu(\Sigma_{L\cap H})$. Therefore, $\{0 < \Sigma_H < \infty\}$, which is included in $\{\sigma_H < \infty\}$ is P_μ negligible for all μ.

2. Let X_t be a diffusion (continuous Hunt process.). Let C_t be a continuous additive functional of the diffusion with closed, regular support M. Let τ_t be the right continuous time change inverse of $C_t: \tau_t = \inf\{s, C_s > t\}$. Then by Proposition 4-11, page 232 of [1], we know that the time-changed process $\tilde{X}_t = X_{\tau_t}$ is a Hunt process on M. With these notations, we shall prove the following:

Theorem. *If the points of ∂M are polar, then $\tilde{X}_t$ has singular jumps.*

3. To prove this theorem, we need a few results in martingale theory, established by the author in [2] and [3]. Let us fix an initial distribution μ. Let $\mathscr{F}_t$ be the filtration generated by X_t, suitably completed and regularized. Since X_t is continuous, every $\mathscr{F}_t$-stopping time is predictable and every $\mathscr{F}_t$-martingale is continuous.

The $\tilde{\mathscr{F}}_t = \mathscr{F}_{\tau_t}$-square integrable martingales are all time changed $\mathscr{F}_t$ square integrable martingales $\tilde{M}_t = M_{\tau_t}$ (cf. [3] Proposition 2).

The purely discontinuous part of such a martingale is the limit (in quadratic norm) of martingales with finitely many values: if $]S_n^\epsilon, T_n^\epsilon[$ is the nth excursion of X_t outside M of duration larger than ϵ,

$$(\tilde{M})_t^d = \lim_{\epsilon\to 0} \sum_n (M_{T_n^\epsilon} - M_{S_n^\epsilon+\epsilon})1_{\{C_{T_n^\epsilon} \leq t\}}$$

Received February 7, 1980; revised June 26, 1980.

[1] This work was partially supported by the Canadian national research Council.

AMS 1970 *subject classifications.* 60J55, 60J40.

Key words and phrases. Time change.

(cf. the proof of Proposition 4 in [3].) Such martingales are called jump martingales. In [2], Proposition 1-2, it is proved that the stable space of martingales orthogonal to the space of jump martingales is the space of strict martingales, i.e., martingales which take value in F_{T-} at each stopping time T.

In the case of $\tilde{F}_t$, we can conclude that there are no purely discontinuous strict martingales, except 0, and therefore, no strict stopping times except predictable times. (cf. [2] Section 3-b).

4. Let $\mathscr{U}_t$ be the complete, right continuous filtration associated to $\tilde{X}_t$. $\mathscr{U}_t$ is imbedded in $\tilde{\tilde{\mathscr{F}}}_t$ and each $\mathscr{U}_t$ stopping time is therefore an $\tilde{\tilde{\mathscr{F}}}_t$ stopping time. If T is an $\mathscr{U}_t$-totally inaccessible stopping time, it is known that $\tilde{X}_{T-} \neq \tilde{X}_T$. We need to prove a stronger result:

LEMMA. *An $\tilde{\tilde{\mathscr{F}}}_t$ stopping time is totally inaccessible if and only if* $\tilde{X}_{T-} \neq \tilde{X}_T$ a.s.

Let T be a $\tilde{\tilde{\mathscr{F}}}_t$ predictable time. If S_n is an announcing sequence, $\tau_{T-} = \lim \uparrow \tau_{S_n}$ is an $\mathscr{F}_t$ stopping time.

Since M is regular, by the strong Markov property, the left extremities of the intervals of excursion outside M contain no graph of stopping time.

But if $\tau_{T-} < \tau_T$, $]\tau_{T-}\ \tau_T[$ is such an excursion, therefore, if T is a $\tilde{\tilde{\mathscr{F}}}_t$ predictable time, $\tau_{T-} = \tau_T$ and it implies that $\tilde{X}_{T-} = \tilde{X}_T$.

It follows that an $\tilde{\tilde{\mathscr{F}}}_t$ stopping time such that $X_T \neq X_{T-}$ has to be totally inaccessible.

Conversely, let T be a $\tilde{\tilde{\mathscr{F}}}_t$ stopping time. τ_T is a $\mathscr{F}_t$ stopping time and since C is continuous, $C_{\tau_T} = T$. If S_n is an announcing sequence of τ_T (every $\mathscr{F}_t$-stopping time is predictable), $T = \lim \uparrow C_{S_n}$. If $\tau_{T-}(\omega) = \tau_T(\omega)$, then $C_t(\omega)$ is strictly increasing in some interval $]\tau_T(\omega) - \epsilon\ \tau_T(\omega)[$, and therefore the times $C_{S_n}(\omega)$ announce $T(\omega)$. Therefore, if T is totally inaccessible, one has necessarily $\tau_{T-} < \tau_T$. This implies that $\tilde{X}_{T-} \neq \tilde{X}_T$ a.s. Indeed,

$$E_\mu(X_{\tau_{T-}} = X_{\tau_T}) = \lim_{\epsilon\to 0} \textstyle\sum_n E_\mu(X_{\tau_{T-}} = X_{\tau_T}, \tau_T = T_n^\epsilon) \leq \lim_{\epsilon\to 0} \sum_n E_\mu(X_{S_n^\epsilon} = X_{T_n^\epsilon})$$

but $E_\mu(X_{S_n^\epsilon} = X_{T_n^\epsilon})$

$$= \int P_\mu(d\omega_1) P_{X_{S_n^\epsilon+\epsilon}}(d\omega_2)\{X_{T_M}(\omega_2) = X_{S_n^\epsilon}(\omega_1)\}$$

by the strong Markov property applied at the stopping time $S_n^\epsilon + \epsilon$.

This last expression is equal to zero, since the points of ∂M are assumed to be polar.

We can now apply the results of the proposition III-2 of [2] to conclude. Indeed, if H is a $\mathscr{U}_t$-predictable set, it is a fortiori $\tilde{\tilde{\mathscr{F}}}_t$-predictable and by the lemma, Σ_H is the same for $\mathscr{U}_t$ and $\tilde{\tilde{\mathscr{F}}}_t$.

REMARK. The idea of the theorem appears after proposition III-2 of [2], but there is a confusion between $\mathscr{U}_t$ and $\tilde{\tilde{\mathscr{F}}}_t$.

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