The Annals of Probability
1982, Vol. 10, No. 4, 1085–1087

MARTINGALES WITH GIVEN CONVEX IMAGE

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Let φ be a convex function. A sufficient condition is given that a submartingale is equal in law to the φ-image of a martingale. It follows that each nonnegative submartingale, without any assumption on the regularity of the paths, can be obtained as the absolute value of a martingale.

A theorem of Gilat [3] states that each nonnegative right continuous submartingale $S = (S_t)_{t\geq 0}$ can be obtained as the absolute value of a martingale $M = (M_t)_{t\geq 0}$, possibly defined on a different probability space. An explicit construction of M was given by Barlow [1], [2], and in case of a strictly positive submartingale by Protter and Sharpe [5] and Maisonneuve [4]. As it was pointed out by Yor ([2], Theorem 3) not every nonnegative submartingale can be represented as the image of a martingale under a given nonnegative convex function φ with $\varphi(0) = 0$. In this paper we give a sufficient condition that a nonnegative submartingale S is equal in law to the φ-image of a martingale M. No assumption on the regularity of S, neither on the filtration nor on the paths, is needed. In particular it follows that any nonnegative submartingale is equal in law to the absolute value of a martingale.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. If φ is isotone then M is essentially determined by S. Therefore we may assume that φ is nonisotone with $\varphi \geq 0$ and $\varphi(0) = 0$. For each $t \geq 0$ set

$$\varphi_+^{-1}(t) := \sup\{r \in \mathbb{R} : \varphi(r) = t\} \in [0, +\infty[,$$

$$\varphi_-^{-1}(t) := \inf\{r \in \mathbb{R} : \varphi(r) = t\} \in]-\infty, 0].$$

We say that a process X is a martingale (submartingale) if it is a martingale (submartingale) w.r.t. the "natural filtration" for X.

THEOREM. *Let $(T, \leq)$ be a linearly ordered set and $(S_t)_{t\in T}$ a nonnegative process satisfying*

$$(*) \qquad (\varphi_+^{-1}\circ S_t)_{t\in T} \quad \textit{and} \quad (-\varphi_-^{-1}\circ S_t)_{t\in T} \quad \textit{are submartingales.}$$

Then there exists a martingale $(M_t)_{t\in T}$ (on a suitable probability space) such that $(S_t)_{t\in T}$ and $(\varphi\circ M_t)_{t\in T}$ have the same distribution.

Condition (*) is necessary if $\varphi_+^{-1} = c\cdot\varphi_-^{-1}$ holds for some $c > 0$. Especially if φ is symmetric, the distributions of $(\varphi_+^{-1}\circ S_t)_{t\in T}$, $(-\varphi_-^{-1}\circ S_t)_{t\in T}$ and $|M|$ have to be the same. But in general, condition (*) is not necessary, which can easily be seen by considering a positive martingale.

Note that if φ_+^{-1} and φ_-^{-1} are not "linear," a nondegenerate martingale S can not be the φ-image of a martingale. This implies that there is no condition on φ alone (not involving S) which ensures the existence of a suitable martingale M.

We prove the theorem for arbitrary T by using the validity of the assertion for finite T. Therefore we first establish the following lemma. To obtain the martingale $(M_t)_{t\in T}$ in this case we will essentially apply the procedure due to Gilat.

Received October 1980; revised August 1981.
AMS 1970 *subject classification.* 60G45.
Key words and phrases. Martingale, submartingale, convex function.

Henceforth let X_t^A be the projection of $\mathbb{R}^A$ on $\mathbb{R}^{\{t\}}$ for $t \in A \subset T$; instead of X_t^T we write X_t. The Borel-σ-algebra of a topological space Y is denoted by $\mathscr{B}(Y)$.

LEMMA. *The assertion of the theorem holds for* $T = \{1, \cdots, n\}$.

PROOF. 0. If λ is any probability measure on $\mathscr{B}([0, +\infty[)$ with

$$m_+ := \int \varphi_+^{-1}\, d\lambda < +\infty \quad \text{and} \quad m_- := \int \varphi_-^{-1}\, d\lambda > -\infty,$$

we define the new measure $\alpha(\lambda, x)$ on $\mathbb{R}$ by

$$\alpha(\lambda, x) := \frac{x - m_-}{m_+ - m_-}\varphi_+^{-1}(\lambda) + \frac{m_+ - x}{m_+ - m_-}\varphi_-^{-1}(\lambda) \quad \text{for} \quad \lambda \neq \varepsilon_0 \quad \text{and} \quad x \in [m_-, m_+];$$

$\alpha(\varepsilon_0, 0) := \varepsilon_0$, where ε_0 is the unit measure concentrated on 0. It is easily verified that

$$\int_{\mathbb{R}} id\, d\alpha(\lambda, x) = x; \tag{1}$$

$$\varphi(\alpha(\lambda, x)) = \lambda. \tag{2}$$

1. For $1 \leq k \leq n$ let σ_k be the distribution of $(S_t)_{t \leq k}$ on $\mathbb{R}^k$ and $\tilde{\sigma}_{k-1}$: $\mathbb{R}^{k-1} \times \mathscr{B}(\mathbb{R}) \rightarrow [0, 1]$ be the regular conditional distribution of X_k given $X_1, \cdots, X_{k-1}$ relative to σ_n. Then the assumption (*) is equivalent to

$$\left.\begin{aligned} \varphi_+^{-1}(y_k) &\leq \int \varphi_+^{-1}(t)\tilde{\sigma}_k(y_1, \cdots, y_k; dt) < +\infty \\ \varphi_-^{-1}(y_k) &\geq \int \varphi_-^{-1}(t)\tilde{\sigma}_k(y_1, \cdots, y_k; dt) > -\infty \end{aligned}\right\} \quad \sigma_k\text{-a.e. for} \quad 1 \leq k < n. \tag{3}$$

Without loss of generality we may assume that (3) holds for all $(y_1, \cdots, y_k) \in \mathbb{R}_+^k$. Hence we can define $\tilde{\mu}_0 := \alpha(\tilde{\sigma}_0, \gamma)$ with an arbitrary $\gamma \in [m_-(\tilde{\sigma}_0), m_+(\tilde{\sigma}_0)]$ and $\tilde{\mu}_k(x_1, \cdots, x_k; \cdot) := \alpha(\tilde{\sigma}_k(\varphi(x_1), \cdots, \varphi(x_k); \cdot), x_k)$ for $1 \leq k < n$ and $(x_1, \cdots, x_k) \in \mathbb{R}^k$.

2. If μ is the measure on $\mathscr{B}(\mathbb{R}^n)$ corresponding to $(\tilde{\mu}_k)_{0 \leq k < n}$ then clearly (1) implies that $(X_k)_{k \in T}$ is a martingale with respect to μ. Moreover from (2) it follows by induction on k that the distribution of $(\varphi \circ X_k)_{k \in T}$ is σ.

PROOF OF THE THEOREM. Let $\mathscr{T}$ be the collection of all finite nonvoid subsets of T and A an element of $\mathscr{T}$. Set σ_A to be the distribution of $(S_t)_{t \in A}$ on $\mathbb{R}^A$ and $\varphi_v^{-1} := (\varphi_{v_t}^{-1})_{t \in A}$ for $v \in \{+, -\}^A$.

1. The set $\mathscr{P}_A := \{\mu \mid \mathscr{B}(\mathbb{R}^A) : (\varphi, \cdots, \varphi)(\mu) = \sigma_A\}$ equipped with the weak topology is a compact space. Indeed

(i) $\mathscr{P}_A$ is uniformly tight, since for a compact subset K of $\mathbb{R}^A$ the set $K' := \cup_v \varphi_v^{-1}(K)$ is again compact and $\mu(K') \geq \sigma_A(K)$ holds for all $\mu \in \mathscr{P}_A$. Further, the continuity of φ and therefore of the map $\mu \rightarrow (\varphi, \cdots, \varphi)(\mu)$ implies that

(ii) $\mathscr{P}_A$ is closed in the space of all probability measures on $\mathscr{B}(\mathbb{R}^A)$.

2. $\mathscr{M}_A := \{\mu \in \mathscr{P}_A : (X_t^A)_{t \in A}$ is a martingale with respect to $\mu\}$ is a nonvoid and compact subspace of $\mathscr{P}_A$. In fact, by the preceding Lemma, $\mathscr{M}_A$ is nonvoid. To prove that $\mathscr{M}_A$ is closed and hence compact, we first show:

$$\text{(C)} \quad \left\{\begin{array}{l} \text{If } g \text{ is a continuous function on } \mathbb{R}^A \text{ satisfying} \\ \displaystyle\int |g \circ \varphi_v^{-1}|\, d\sigma_A < \infty \quad \text{for all} \quad v \in \{+, -\}^A \\ \text{then the map } \mu \rightarrow \displaystyle\int g\, d\mu \text{ is continuous on } \mathscr{P}_A. \end{array}\right.$$

Set $h := \Sigma_v |g \circ \varphi_v^{-1}|$. Then for $\varepsilon > 0$ there exists $c > 0$ such that $\int_{\{h>c\}} h \, d\sigma_A < \varepsilon$ and hence

$$\int_{\{|g|>c\}} |g| \, d\mu \le \int_{\varphi^{-1}(\{h>c\})} h \circ \varphi \, d\mu = \int_{\{h>c\}} h \, d\sigma_A < \varepsilon \quad \text{for all} \quad \mu \in \mathscr{P}_A.$$

It follows that $|\mu(g) - \nu(g)| < 3\varepsilon$ for all $\nu, \mu \in \mathscr{P}_A$ with $|\mu(-c \vee g \wedge c) - \nu(-c \vee g \wedge c)| < \varepsilon$. This proves (C).

If $f: \mathbb{R}^A \to \mathbb{R}$ is a continuous bounded function and r, s are elements of A with $r < s$, it follows from (C) by taking $g = f \cdot X_r^A$ resp. $g = f \cdot X_s^A$ that the set $\{\mu \in \mathscr{P}_A : \int f \cdot X_r^A \, d\mu = \int f \cdot X_s^A \, d\mu\}$ is closed. Consequently $\mathscr{M}_A$ is closed.

3. Let π_A^B denote the canonical projection of $\mathscr{M}_B$ on $\mathscr{M}_A$ for $A \subset B \in \mathscr{T}$ and

$$\mathscr{M}_{\mathscr{T}_0} := \{(\mu_A)_{A \in \mathscr{T}} \in \textstyle\prod_{A \in \mathscr{T}} \mathscr{M}_A : \mu_A = \pi_A^B(\mu_B) \quad \text{for all} \quad A, B \in \mathscr{T}_0, A \subset B\}$$

for finite $\mathscr{T}_0 \subset \mathscr{T}$. Evidently $\mathscr{M}_{\mathscr{T}_0}$ is a closed subspace of the product space $\prod_{A \in \mathscr{T}} \mathscr{M}_A$ and hence compact. Since we have $\cup \mathscr{T}_0 \in \mathscr{T}$, $\mathscr{M}_{\mathscr{T}_0}$ is also nonvoid. This shows

$$\mathscr{M} := \cap \{\mathscr{M}_{\mathscr{T}_0} : \mathscr{T}_0 \subset \mathscr{T} \ \text{ finite}\} \ne \varnothing.$$

Let $(\mu_A)_{A \in \mathscr{T}}$ be an element of $\mathscr{M}$ and $\mu | \otimes_{t \in T} \mathscr{B}(\mathbb{R})$ its projective limit. If $h: \mathbb{R}^T \to \mathbb{R}$ is a measurable bounded function depending only on the coordinates $t \in A$ with $A \in \mathscr{T}$, then by construction the equation $\int h \cdot X_r \, d\mu = \int h \cdot X_s \, d\mu$ holds for all $r, s \in T$ with $\max A \le r < s$. Thus the process $(X_t)_{t \in T}$ is a martingale with respect to μ. Clearly $(\varphi \circ X_t)_{t \in T}$ has the same distribution as $(S_t)_{t \in T}$. This completes the proof.

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