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SOME LIMIT THEOREMS ON REVERSED BROWNIAN MOTION

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Let B be the standard one-dimensional Brownian motion, $T_x = \inf\{s \mid s \geq 0; B(s) = x\}$, $x > 0$, $T = T_0 \wedge T_a$,

$$\gamma_x(t) = \begin{cases} \sup\{s \mid s \leq t; \quad B(s) = x\} \\ 0 \quad \text{if above set} = \phi \end{cases}, \quad \gamma(t) = \gamma_0(t) \vee \gamma_a(t)$$

where $a > 0$ is fixed and $W(s) = B(T - s)$, $0 \leq s \leq T$. Let $Z(s)$ be the restriction of B to the interval $[\gamma(t), t]$, that is, $Z(s) = B(\gamma(t) + s)$, $0 \leq s \leq L(t)$. In this paper we use a "time reversal" argument to study relations as $t \to \infty$ between the processes Z and W under $P^y(\cdot \mid B(t) = x)$ and to evaluate some limits related to $L(t)$.

0. Introduction. Let $B = (B(t))_{t \geq 0}$ be the standard one dimensional Brownian motion process, and let P^x denote the probability associated with the Brownian motion starting from x. P^0 will be written as P. θ_t is the shift operator.

Let $P^{x;t,y}$ denote the probability for the conditional probability under $B_0 = x$ and $B_t = y$. That is, $P^{x;t,y}$ is the unique probability on F_t^0 where $F_t^0 = \sigma(B(s), s \leq t)$ with the following property: If $o < t_1 < \cdots < t_n < t$ and $E_1, \cdots, E_n$ are Borel subsets of R^1 (let β denote the Borel field on R^1), then:

$$P^{x;t,y}(B(t_j) \in E_j; 1 \leq j \leq n) = \int_{E_1} \cdots \int_{E_n} \frac{p(t_1, x, x_1)p(t_2 - t_1, x_1, x_2) \cdots p(t - t_n, x_n, y)}{p(t, x, y)} \, dx_n \cdots dx_1$$

where $p(t, x, y) = (2\pi t)^{-1/2} e^{-(x-y)^2/2t}$ is the Brownian transition density.

For an arbitrary σ-field F, let bF denote the class of bounded, F-measurable functions.

It is immediate that if $Z \in bF_t^0$, then $E^{x;t,y}(Z)$ is a version of $E^x(Z \mid B(t) = y)$. Let us denote $E^x(Z \mid B(t) = y)$ by $E^{x;t,y}(Z)$.

We define

$$T_x = \inf\{s \mid s \geq 0; \quad B(s) = x\}, \quad x \geq 0.$$

T_x is called the hitting time of x. We will also write

$$T = T_0 \wedge T_a = \inf\{t : B(t) = 0 \quad \text{or} \quad a\},$$

where a is fixed and $a > 0$. If $t > 0$, define

$$\Delta_x(t) = \{\omega \in \Omega \mid 0 \leq s \leq t : B_s(\omega) = x\},$$

$$\gamma_x(t) = \begin{cases} \sup(s \mid s \leq t, B(s) = x) & \text{if} \quad \omega \in \Delta_x(t) \\ 0 & \text{if} \quad \omega \in \Omega - \Delta_x(t), \end{cases}$$

$$L_x(t) = t - \gamma_x(t),$$

$$L(t) = L_0(t) \wedge L_a(t),$$

$$\gamma(t) = \gamma_0(t) \vee \gamma_a(t).$$

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$\gamma_x(t)$ is the last exit time from x before time t.

For each $t > 0$, consider the process B restricted to the interval $[\gamma(t), t]$. Define a process Z as follow:

$$Z(s) = \begin{cases} B(\gamma(t)+s) & \text{for} \quad 0 \le s \le L(t) \\ \Delta & \text{for} \quad L(t) < s, \quad \text{where} \quad \Delta \bar{\in} R^1. \end{cases}$$

Set

$$\sigma_Z = \sigma(Z(s), \quad s \ge 0).$$

$$W(s) = \begin{cases} B(T-s) & \text{for} \quad 0 \le s \le T \\ \Delta & \text{for} \quad T < s, \end{cases}$$

$$\sigma_W = \sigma(W(s), \quad s \ge 0).$$

In Section 1, we shall prove some relations between the processes Z and W and that for arbitrary y and $0 < x < a$,

1. $\lim_{t\to\infty} P^y(L_0(t) < L_a(t) \mid B(t) = x) = \dfrac{a-x}{a}$,

   $\lim_{t\to\infty} P^y(L_a(t) < L_0(t) \mid B(t) = x) = \dfrac{x}{a}$.

2. $\lim_{t\to\infty} P^y(L(t) < s \mid B(t) = x) = P^x(T < s), \qquad s > 0.$

3. For any $k \ge 1$, $\lim_{t\to\infty} E^y(L^k(t) \mid B(t) = x) = E^x T^k$.

These results correspond to the following results which are well known:

$$P^x(T_0 < T_a) = \frac{a-x}{a}, \qquad P^x(T_a < T_0) = \frac{x}{a},$$

$$E^x T^k < \infty, \qquad E^x T = x(a-x) \quad \text{(see [2])}.$$

The central result of Section 1 is Theorem 1. The Result 2 is a direct corollary of Theorem 1, and the Result 3 follows from 2 after making some uniform integrability type estimates. The Result 1 is obtained in the process of proving Theorem 1.

In Section 2, we apply the above results to excursion of Brownian motion.

1. Limit theorems. We shall use a "time reversal" argument due to Getoor and Sharpe in [4] to study the relation between the processes Z and W and evaluate the limits in the Results 1, 2 and 3.

Let ϕ_t be the reversal from t operator; that is, $B(s)\circ\phi_t = B(t-s)$, $0 \le s \le t$. In [4], it is shown that:

$$\text{if} \quad Z \in bF_t^0, \quad \text{the} \quad E^{x;t,y}(Z) = E^{y;t,x}(Z\circ\phi_t). \tag{1.1}$$

LEMMA 1. *If $s < t$, the measure $P^{x;t,y}$ has a Radon-Nikodym derivative on F_s^0 with respect to the unconditional measure P^x which is uniformly bounded and tends to 1 as $t \to \infty$.*

PROOF. For any $A \in F_s^0$ we have

$$P^{x,t,y}(A) = E^x\left\{ I_A \frac{p(t-s, Bs, y)}{p(t,x,y)} \right\},$$

thus we obtain

$$\frac{dP^{x,t,y}}{dP^x} = \frac{p(t-S, B_s, y)}{p(t,x,y)} \quad \text{on} \quad F_s^0. \tag{1.2}$$

It is evident that the above right side converges boundedly to 1 as $t \to \infty$.

COROLLARY. *For $s < t$, we have*:

$$P^{x;t,y}(T_0 \in ds) = P^x(T_0 \in ds)\frac{p(t-s, 0, y)}{p(t, x, y)}, \tag{1.3}$$

$$P^{x;t,y}(T_0 < T_a, T_0 \in ds) = P^x (T_0 < T_a, T_0 \in ds)\frac{p(t-s, 0, y)}{p(t, x, y)}, \tag{1.4}$$

where

$$P^x(T_0 \in ds) = \frac{x}{\sqrt{2\pi s^3}} e^{-x^2/2s}\, ds,$$

$$P^x(T_0 < T_a, T_0 \in ds) = f(x, s)\, ds = -\textstyle\sum_{n=-\infty}^{\infty} p_x(s, 0, 2na + x)\, ds$$

(*see* [2] *page* 170).

As is well known, $\xi \in \sigma_Z$ is equivalent to the condition that there exists $g = g(x_1, x_2, \cdots, x_n, \cdots)$ where $g \in \beta_\Delta^\infty(\beta_\Delta^\infty = \beta_\Delta \times \beta_\Delta \times \cdots, \beta_\Delta = \beta(R' \cup \Delta))$ and $0 < s_i$, $i = 1, 2, \cdots, n, \cdots$ such that $\xi = g(Z(s_1), \cdots, Z(s_n), \cdots)$. Abbreviate the above expression as $\xi = g(Z(\cdot))$ (see [3]).

LEMMA 2. *If $g \in b\beta_\Delta^\infty(|g| < M_g)$ then for any y and $0 < x < a$ we have*:

$$\lim_{t\to\infty} E^y(g(Z(\cdot)); L_0(t) < L_a(t) \mid B(t) = x) = E^x(g(W(\cdot)); T_0 < T_a), \tag{1.5}$$

$$\lim_{t\to\infty} E^y(g(Z(\cdot)); L_a(t) < L_0(t) \mid B(t) = x) = E^x(g(W(\cdot)); T_a < T_0). \tag{1.6}$$

PROOF. Without loss of generality, we may suppose $0 \le g \le 1$. From (1.1), we obtain

$$L_x(t)\circ\phi_t = T_x \wedge t, \qquad B(\gamma(t) + s)\circ\phi_t = B(T\wedge t - s).$$

For $s < t$ we have by Lemma 1

$$E^{y;t,x}(g(Z(\cdot)); L_0(t) < L_a(t)) \ge E^{y;t,x}(g(Z(\cdot)); L_0(t) < L_a(t) \le s)$$

$$= E^{x;t,y}(g(W(\cdot)); T_0 < T_a \le s) = E^x\left(\frac{p(t-s, B_s, y)}{p(t, x, y)} g(W(\cdot))\,;\, T_0 < T_a \le s\right).$$

As $t \to \infty$, the right side tends to

$$E^x(g(W(\cdot)); T_0 < T_a \le s)$$

by the Dominated Convergence Theorem. As s is arbitrary, this shows

$$\liminf_{t\to\infty} E^{y;t,x}(g(Z(\cdot)); L_0(t) < L_a(t)) \ge E^x(g(W(\cdot)); T_0 < T_a). \tag{1.7}$$

Let $g = 1$; we obtain

$$\liminf_{t\to\infty} P^{y;t,x}(L_0(t) < L_a(t)) \ge P^x(T_0 < T_a) = \frac{a-x}{a} \tag{1.8}$$

and

$$\liminf_{t\to\infty} P^{y;t,x}(L_a(t) < L_0(t)) \ge P^x(T_a < T_0) = \frac{x}{a}; \tag{1.9}$$

since the right side of (1.8) plus the right side of (1.9) equals 1 and

$$P^{y;t,x}(L_0(t) < L_a(t)) + P^{y;t,x}(L_a(t) < L_0(t)) \le 1,$$

these imply that

$$\lim_{t\to\infty} P^y(L_0(t) < L_a(t) \mid B(t) = x) = \frac{a-x}{a}, \tag{1.10}$$

(1.11) $$\lim_{t\to\infty} P^y(L_a(t) < L_0(t) \mid B(t) = x) = \frac{x}{a},$$

(1.12) $$\lim_{t\to\infty} P^y(L_0(t) = L_a(t) \mid B(t) = x) = 0.$$

Replacing g by $1 - g$, we obtain from (1.10), (1.11), (1.12) that

(1.13) $$\limsup_{t\to\infty} E^{y;t,x}(g(Z(\cdot)); L_0(t) < L_a(t)) \le E^x(g(W(\cdot)); L_0(t) < L_a(t)).$$

Combining (1.7) with (1.13), we have (1.5). Using a similar argument, we can obtain (1.6). □

THEOREM 1. *If $g \in b\beta^\infty_\Delta$, then for any y and $0 < x < a$ we have:*

(1.14) $$\lim_{t\to\infty} E^y(g(Z(\cdot)) \mid B(t) = x) = E^x(g(W(\cdot))).$$

PROOF. The conclusion follows from Lemma 2. □

COROLLARY. *For $s > 0$, we have:*

(1.15) $$\lim_{t\to\infty} P^y(L(t) < s \mid B(t) = x) = P^x(T < s).$$

PROOF. Taking

$$g(x) = \begin{cases} 0 & x \in R^1 \\ 1 & x = \Delta \end{cases}$$

in (1.14), we obtain (1.15). □

THEOREM 2. *For any y, $0 < x < a$ and integer $k \ge 1$, we have:*

(1.16) $$\lim_{t\to\infty} E^y(L^k(t) \mid B(t) = x) = E^x T^k.$$

PROOF. Let

$$q_{s,t} = \frac{p(t-s, B_s, y)}{p(t, x, y)} = \frac{dP^{x;t,y}}{dP^x} \quad \text{on} \quad F^0_s.$$

We take $s = t/2$ and note $q_{t/2,t} < \sqrt{2}e^{(x-y)^2/t}$, so we have from Lemma 1:

(1.17) $$\begin{aligned} E^y(L^k(t) \mid B(t) = x) &= E^{x;t,y}((T \wedge t)^k) \le E^{x;t,y}\left(T^k; T < \frac{t}{2}\right) + t^k P^{x;t,y}\left(T > \frac{t}{2}\right) \\ &< \sqrt{2}e^{(x-y)^2/t}\left\{E^x\left(T^k, T < \frac{t}{2}\right) + t^k P^x\left(T > \frac{t}{2}\right)\right\}. \end{aligned}$$

Now $t^k P^x(T > t/2) \le 2^k E^x T^k$, hence if $t > 1$, then the right side of (1.17)

$$\le 2^{k+1}\sqrt{2}e^{(x-y)^2} E^x T^k < \infty.$$

This implies that for each k there exists a constant $M(x, y, k)$ such that

(1.18) $$\sup_t E^y(L^k(t) \mid B(t) = x) \le M(x, y, k) < \infty.$$

The conclusion of Theorem 2 follows from (1.15); and (1.18) (see [1], 4.5.2). □

2. Application to Brownian excursion. We use the following notations:

$$Y(t) = |B(t)|, \qquad h(t) = \inf\{s \mid s \ge t,\ B(t) = 0\},$$

$$M^1(t) = \max_{\gamma_0(t) \le s \le t} Y(s), \qquad M^2(t) = \max_{t \le s \le h(t)} Y(s),$$

$$M^3(t) = \max_{\gamma_0(t) \le s \le h(t)} Y(s),$$

$$g(t, 0, y) = \frac{|y|}{\sqrt{2\pi t^3}} e^{-y^2/2t}.$$

We can prove some results in Chung [2] by the above methods in an easier way. For example:

1. For $0 < s < t$, $y > 0$, we have:

$$P(\gamma_0(t) \in ds,\ Y(t) \in dy) = \frac{y}{\sqrt{s(t-s)^3}} e^{-y^2/2(t-s)}\, ds\, dy. \tag{2.1}$$

Noting $L_0(t) = t - \gamma_0(t)$, we obtain (2.1) from (1.1) and (1.3).

2.

$$P(M^1(t) \le a \mid L_0(t) = r) = 1 + 2 \textstyle\sum_1^\infty (-1)^n e^{-n^2a^2/2r}, \tag{2.2}$$

where $0 < r < t$.

Proof. We have

$$\begin{aligned} P(M^1(t) \le a,\ L_0(t) \in dr) &= 2P(L_0(t) < L_a(t),\ L_0(t) \in dr) \\ &= 2\int_0^a P^{0;t,y}(L_0(t) < L_a(t),\ L_0(t) \in dr) p(t, 0, y)\, dy. \end{aligned} \tag{2.3}$$

We know from (1.4) that the right side of (2.3) equals

$$2\int_0^a f(y, r) p(t - r, 0, 0)\, dy. \tag{2.4}$$

By (2.1) we have $P(L_0(t) \in dx) = dx/\pi\sqrt{r(t-r)}$, hence the left side of (2.3) equals

$$\sqrt{2\pi r} \int_0^a f(y, r)\, dy, \tag{2.5}$$

where $f(y, r) = -\sum_{n=-\infty}^{\infty} p_y(r, 0, 2na + y)$.
Integrating out dy in (2.5), we obtain (2.2). □

Let

$$H(s) = \begin{cases} Y(\gamma_0(t) + s) & \text{for } 0 \le s \le L_0(t) \\ \Delta & \text{for } L_0(t) < s \end{cases};$$

$$U(s) = \begin{cases} Y(T_0 - s) & \text{for } 0 \le s \le T_0 \\ \Delta & \text{for } T_0 < s \end{cases};$$

$$X(s) = \begin{cases} Y(s) & \text{for } 0 \le s \le T_0 \\ \Delta & \text{for } T_0 < s \end{cases};$$

$$Z_0(s) = \begin{cases} B(\gamma_0(t) + s) & \text{for } 0 \le s \le L_0(t) \\ \Delta & \text{for } L_0(t) < s \end{cases};$$

$$W_0(s) = \begin{cases} B(T_0 - s) & \text{for } 0 \le s \le T_0 \\ \Delta & \text{for } T_0 < s \end{cases}.$$

Next, we shall prove some relations among processes H, U and X.

Theorem 3. *If* $g \in b\beta_\Delta^\infty(R^+)(R^+ = [0, \infty))$ *and* $x > 0$, *we have*:

$$\lim_{t\to\infty} E(g(H(\cdot)) \mid Y(t) = x) = E^x(g(U(\cdot))). \tag{2.6}$$

PROOF. The proof of Lemma 2 is applicable if we delete $L_0(t) < L_a(t)$, $T_0 < T_a$ and replace $Z(\cdot)$ by $Z_0(\cdot)$, $W(\cdot)$ by $W_0(\cdot)$. We obtain

$$\lim_{t\to\infty} E(g(Z_0(\cdot)) \mid B(t) = x) = E^x g(W_0(\cdot)).$$

Since

$$E(g(H(\cdot)) \mid Y(t) = x) = E(g(Z_0(\cdot)) \mid B(t) = x)$$

and

$$E^x g(W_0(\cdot)) = E^x g(U(\cdot)),$$

we obtain (2.6). □

Using the same argument, we can obtain results similar to (1.15) and (1.16).

THEOREM 4. *If* $g_1, g_2 \in b\beta_\Delta^\infty(R^+)$ *and* $x > 0$, *we have*:

$$\lim_{t\to\infty} E(g_1(H(\cdot))\cdot g_2(X(\cdot)\circ\theta_t) \mid Y(t) = x) = E^x g_1(U(\cdot)) E^x g_2(X(\cdot)). \tag{2.7}$$

PROOF. This follows from the Markov property and Theorem 3. □

Theorem 4 shows that H and $X\circ\theta_t$ are asymptotically conditionally independent relative to $P^{0;t,x}$ as $t \to \infty$.

If we take $g_1(H(\cdot)) = I_{[0,a]}^{(M^1(t))}$, $g_2(X(\cdot)\circ\theta_t) = I_{[0,a]}^{(M^2(t))}$ and note that

$$M^3(t) = M^1(t) \vee M^2(t),$$

then we have

$$\lim_{t\to\infty} P(M^3(t) \le a \mid Y(t) = x) = \left(\frac{a-x}{a}\right)^2. \tag{2.8}$$

Taking

$$g_1 = g_2 = \begin{cases} 1 & x \in R^+ \\ 0 & x = \Delta \end{cases}$$

and noting that $(h(t) - t) = T_0\circ\theta_t$, we obtain

$$\lim_{t\to\infty} P(L_0(t) < s_1, (h(t) - t) < s_2 \mid Y(t) = x) = \int_0^{s_1} g(u, 0, x)\,du \int_0^{s_2} g(v, 0, x)\,dv. \tag{2.9}$$

We can obtain other important results if we take some special g_1 and g_2.

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