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SOME CHARACTERIZATIONS OF STRONG LAWS FOR LINEAR FUNCTIONS OF ORDER STATISTICS

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Some necessary and sufficient conditions for strong laws of large numbers to hold for certain classes of linear functions of order statistics (L.F. of O.S.) are given. The results are used to extend and complement some sufficient conditions for strong laws for L.F. of O.S. derived by van Zwet [5] and Wellner [6]. Also a moment-like condition almost equivalent to the existence of an absolute pth moment is introduced.

1. Introduction and preliminaries. Let $U_1, U_2, \cdots$, be a sequence of uniform (0, 1) random variables. For each $n \geq 1$, let $U_{1n} \leq \cdots \leq U_{nn}$ be the order statistics of $U_1, \cdots, U_n$. Let H be a measurable real valued function defined on [0, 1]. The classical strong law of large numbers (C.S.L. of L.N.) says that

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^n H(U_i) = EH(U_1) \quad \text{(finite)} \quad \text{a.s.}$$

if and only if $E|H(U_1)| < \infty$.

Several workers, Helmers [2], Sen [4], van Zwet [5] and Wellner [6], have considered strong laws for a more general class of statistics, which includes the sample mean, called linear functions of order statistics (L.F. of O.S.). These are statistics of the form:

$$L_n = n^{-1} \sum_{i=1}^n J\left(\frac{i}{n+1}\right) H(U_{in}), \tag{1}$$

where J is a real valued function, often called a score function, defined on (0, 1).

[2], [4], [5], and [6] have shown that under a variety of conditions on J and H

$$\lim_{n\to\infty} L_n = \int_0^1 J(u)H(u)\, du \quad \text{(finite)} \quad \text{a.s.} \tag{L}$$

The strongest results are those obtained by van Zwet [5] and Wellner [6]. So far the problem of finding necessary and sufficient conditions on J and H such that (L) is true, analogous to the necessary and sufficient condition on H given by the C. S. L. of L. N., has not been solved. A partial solution to this problem will be presented in this paper. Necessary and sufficient conditions for (L) will be derived for two special subclasses of L.F. of O.S., which will be seen later on to arise naturally in the study of strong laws for L.F. of O.S. These are L.F. of O.S. of the following two forms:

Let $\mathscr{G}$ denote the class of nonnegative real valued functions defined on [0, 1], which are nonincreasing on (0, 1]. For each $g \in \mathscr{G}$ and $0 < p < \infty$ set

$$S_n(g, p) = \sum_{i=1}^n i^{1/p-1} n^{-1/p} g(U_{in}). \tag{2}$$

Let $\mathscr{H}$ denote the class of nonnegative real valued functions defined on [0, 1], which are nondecreasing on [0, 1). For each $h \in \mathscr{H}$ and $0 < q < \infty$ set

$$T_n(h, q) = \sum_{i=1}^n i^{1/q-1} n^{-1/q} h(U_{n+1-i,n}). \tag{3}$$

$S_n(q, p)$ and $T_n(h, q)$ are not quite in the form of L_n, but will be notationally more

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convenient. $S_n(g, p) = ((n+1)/n)^{1/p-1}L_n$ with $J(u) = u^{1/p-1}$ and $H = g$, and $T_n(h, q) = ((n+1)/n)^{1/q-1}L_n$ with $J(u) = (1-u)^{1/q-1}$ and $H = h$.

The asymptotic behavior of $S_n(g, p)$ and $T_n(h, q)$ respectively will be shown to be completely determined by whether the following quantities are finite or infinite: For $g \in \mathscr{G}$ and $0 < p < \infty$ set

$$E_p(g) = \int_0^1 g^p(u)\, du \quad \text{and} \quad M_p(g) = \int_0^1 u^{1/p-1} g(u)\, du;$$

and for $h \in \mathscr{H}$ and $0 < q < \infty$ set

$$E_q(h) = \int_0^1 h^q(u)\, du \quad \text{and} \quad N_q(h) = \int_0^1 (1-u)^{1/q-1} h(u)\, du.$$

Finiteness of $M_p(g)$ and $N_q(h)$ is almost a moment condition on g and h respectively. It is not too difficult to show that if $0 < p < 1$ and $E_p(g) < \infty$ then $M_p(g) < \infty$, though the converse is not true; for example, set $g(u) = (-u \ln(ue^{-1}))^{-1/p}$. When $p = 1$, $E_p(g) < \infty$ if and only if $M_p(g) < \infty$. Finally, if $\infty > p > 1$, $M_p(g) < \infty$ implies $E_p(g) < \infty$, but the converse is not true; for example, set $g(u) = u^{-1/p}(-\ln(ue^{-1}))^{-1}$. The analogous remarks are true for $E_q(h)$ and $N_q(h)$. Refer to the Appendix for proofs of these statements.

The characterizations obtained for the asymptotic behavior of $S_n(g, p)$ and $T_n(h, q)$ will be used to extend the results of Wellner [6] and complement the results of van Zwet [5]. In addition, information concerning the general problem stated above will be gained in the process.

2. Strong laws for linear functions of order statistics.

The following two theorems completely characterize strong laws for $S_n(g, p)$ and $T_n(h, q)$ respectively.

THEOREM 1. *Let $g \in \mathscr{G}$ and $\infty > p > 0$, then*
(A) $E_p(g) < \infty$ *and* $M_p(g) < \infty$ *if and only if*
(B) $\lim_{n\to\infty} S_n(g, p) = M_p(g) < \infty$ a.s.; *and*
(C) $E_p(g) = \infty$ *or* $M_p(g) = \infty$ *if and only if*
(D) $\limsup_{n\to\infty} S_n(g, p) = \infty$ a.s.

PROOF. Postponed until Section 3.

THEOREM 2. *Let $h \in H$ and $\infty > q > 0$, then*
(A′) $E_q(h) < \infty$ *and* $N_q(h) < \infty$ *if and only if*
(B′) $\lim_{n\to\infty} T_n(h; q) = N_q(h) < \infty$ a.s.; *and*
(C′) $E_q(h) = \infty$ *or* $N_q(h) = \infty$ *if and only if*
(D′) $\limsup_{n\to\infty} T_n(h, q) = \infty$ a.s.

PROOF. The proof follows from Theorem 1 by setting $g(u) = h(1-u)$ and observing that $1 - U_1, 1 - U_2, \cdots,$ is equal in distribution to $U_1, U_2, \cdots,$. □

Theorems 1 and 2 may be thought of as natural extensions of the strong law of large numbers for nonnegative random variables. The moment-like conditions $M_p(g)$ and $N_q(h)$ come into play only when $p > 1$.

The following corollary to Theorems 1 and 2 reveals the relationship between strong laws for $S_n(g, p)$ and $T_n(h, q)$ and more general strong laws for L_n.

COROLLARY I. (*A Strong Law for L_n*). *Assume that H and J are such that* (i) *for all*

$0 < a < 1/2$

$$n^{-1} \sum_{i=[na]}^{n-[na]} J\left(\frac{i}{n+1}\right) H(U_{in}) \to \int_a^{1-a} J(u)H(u)\, du \quad \text{(finite)} \quad \text{a.s.,}$$

($[x]$ *= greatest integer* $\leq x$.*)*

(ii) *there exist* $M > 0, p > 0, q > 0$ *and* $0 < b < 1/2$ *such that*

$$|J(u)| \leq Mu^{1/p-1} \quad \textit{and} \quad |J(1-u)| \leq Mu^{1/q-1}$$

for all $u \in (0, b)$; *and*
(iii) *there exist* $g \in \mathscr{G}$, $h \in \mathscr{H}$, *and* $0 < c < 1/2$ *such that* $|H(u)| \leq g(u)$ *and* $|H(1-u)| \leq h(1-u)$ *for all* $u \in (0, c)$, *where g satisfies* (A) *and h satisfies* (A′); *then*

$$\lim_{n\to\infty} L_n = \int_0^1 J(u)H(u)\, du \quad \text{(finite)} \quad \text{a.s.}$$

PROOF. Conditions (ii) and (iii) imply that for some $0 < M_0 < \infty$

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \left| n^{-1} \sum_{i=1}^{[n\varepsilon]} J\left(\frac{i}{n+1}\right) H(U_{in}) \right| \leq \tag{4}$$

$$M_0 \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \left| \sum_{i=1}^{[n\varepsilon]} i^{1/p-1} n^{-1/p} g(U_{in}) \right|, \tag{5}$$

which by (iii) and Theorem 1 is easily seen to equal

$$M_0 \lim_{\varepsilon\downarrow 0} \int_0^\varepsilon u^{1/p-1} g(u)\, du = 0 \quad \text{a.s.} \tag{6}$$

Similarly,

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \left| n^{-1} \sum_{i=1}^{[n\varepsilon]} J\left(\frac{n+1-i}{n+1}\right) H(U_{n+1-i,n}) \right| = 0 \quad \text{a.s.} \tag{7}$$

These two facts in combination with (i) complete the proof. □

REMARK 1. Wellner [6] and van Zwet [5] give sufficient conditions for assumption (i) to hold. In particular, the results of Wellner [6] can be shown to imply that if $|H(u)|$ is bounded on $(a, 1-a)$ for each $0 < a < 1/2$, and

$$J\left(\frac{[nu]}{n+1}\right) \to J(u) \quad \text{a.s.} \quad (0, 1),$$

then condition (i) holds. It would seem reasonable that $\int_0^1 |J(u)H(u)|\, du < \infty$ along with J and H satisfying the conditions just stated would be enough for a strong law to hold for L_n. The fact that $M_p(g) < \infty$ does not imply $E_p(g) < \infty$ for $0 < p < 1$ in combination with Theorem 1 and the example given in the introduction for $0 < p < 1$ shows that this conjecture is not true.

REMARK 2. Van Zwet [5] shows that if J is sufficiently smooth and for some $\infty > p > 1$ both $E|J(U_1)|^{p/(p-1)}$ and $E|H(U_1)|^p$ are finite, then the conclusion of Corollary 1 holds. Theorem 1 indicates that van Zwet's moment conditions, though sufficient, are not necessary. Consider for instance, for $\infty > p > 1$, $J(u) = u^{1/p-1}$ and $H(u) = u^{-1/p}(-\ln(ue^{-1}))^{-2}$. These choices of J and H do not satisfy the moment conditions of van Zwet [5], though Theorem 1 implies (L). On the other hand, there exist choices of J and H that satisfy van

Zwet's moment conditions but do not satisfy the conditions of Corollary 1. For example, for $\infty > p > 1$ set $J(u) = u^{1/p-1}(-\ln(ue^{-1}))^{-1}$ and $H(u) = u^{-1/p}(-\ln(ue^{-1}))^{-1}$. So the general problem posed in the introduction remains open.

REMARK 3. To see how Corollary 1 relates to the results in [6], in particular Theorem 4 of [6]; observe that if H and J satisfy Assumptions 1 and 2 of [6] the conditions of Corollary 1 hold automatically. The first example given in Remark 2 clearly does not satisfy Assumption 1 of [6], though (L) is true.

REMARK 4. A more general version of Corollary 1 could have been given here, where $J(i/n + 1)$ is replaced by the larger class of score functions considered in [5] and [6]. In this case, condition (i) and the conclusion would have to be suitably modified. The proof would be along the same lines as the proof just given. To simplify the presentation of the main ideas in this paper such a generalization is not presented here.

3. Proof of Theorem 1. The proof of Theorem 1 will require the following two lemmas. G_n will denote the empirical distribution based on $U_1, \cdots U_n$.

LEMMA 1.
(G) *Let* $g \in \mathscr{G}$ *and set* $D_n(g) =' \sup\{G_n(u)g(u): 0 < u \le 1\}$. *If* $Eg(U_1) < \infty$ *then* $\lim_{n\to\infty} D_n(g) = \sup\{ug(u): 0 < u \le 1\} < \infty$ a.s.; *and if* $Eg(U_1) = \infty$ *then* $\limsup_{n\to\infty} D_n(g) = \infty$ a.s.
(H) *Let* $h \in \mathscr{H}$ *and set* $E_n(h) = \sup\{(1 - G_n(u))h(u): 0 \le u < 1\}$. *If* $Eg(u_1) < \infty$ *then* $\lim_{n\to\infty} E_n(h) = \sup\{(1 - u)h(u): 0 \le u < 1\} < \infty$ a.s.; *and if* $Eh(U_1) = \infty$ *then* $\limsup_{n\to\infty} E_n(h) = \infty$ a.s.

PROOF. Refer to Remarks 1 and 4 of Mason [3].

LEMMA 2.
(G) *Let* $g \in \mathscr{G}$ *and* $0 < p < \infty$, *then*
(a) $E_p(g) < \infty$ *if and only if*
(b) $\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \max_{1\le i\le[n\delta]} i^{1/p} n^{-1/p} g(U_{in}) = 0$ a.s.; *and*
(c) $E_p(g) = \infty$ *if and only if*
(d) *for all* $0 < \delta \le 1$

$$\limsup_{n\to\infty} \max_{1\le i\le[n\delta]} i^{1/p} n^{-1/p} g(U_{in}) = \infty \quad \text{a.s.}$$

(H) *Let* $h \in \mathscr{H}$ *and* $0 < q < \infty$, *then*
(a′) $E_q(h) < \infty$ *if and only if*
(b′) $\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \max_{1\le i\le[n\delta]} i^{1/q} n^{-1/q} h(U_{n+1-i,n}) = 0$ a.s.; *and*
(c′) $E_q(h) = \infty$ *if and only if*
(d′) *for all* $0 < \delta \le 1$,

$$\limsup_{n\to\infty} \max_{1\le i\le[n\delta]} i^{1/q} n^{-1/q} h(U_{n+1-i,n}) = \infty \quad \text{a.s.}$$

PROOF. A proof will only be supplied for part (G); the proof of part (H) follows by symmetry considerations from part (G). First assume (a). Since for each $0 < \delta < 1$ $U_{[n\delta],n} \to \delta$ a.s., we have

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \max_{1\le i\le[n\delta]} i n^{-1} g^p(U_{in}) \le \lim_{\delta\downarrow 0} \limsup_{n\to\infty} \sup\{G_n(u)g^p(u): 0 \le u \le \delta\} \quad \text{a.s.},$$

which by Lemma 1 is almost surely equal to

$$\lim_{\delta\downarrow 0} \sup\{ug^p(u): 0 < u \le \delta\} \tag{8}$$

but (a) along with g nonincreasing implies that expression (8) is zero.

By Corollary 4.3.1 in Galambos [1], we have for all $\varepsilon > 0$

$$P(g^p(U_{1n}) > n\varepsilon \text{ i.o.}) = 0 \quad \text{or} \quad 1$$

according as

$$\textstyle\sum_{n=1}^{\infty} P(g^p(U_1) > n\varepsilon) < \infty \quad \text{or} \quad = \infty.$$

Thus $\lim_{n\to\infty} n^{-1} g^p(U_{1n}) = 0$ a.s. when $E_p(g) < \infty$, and $\limsup_{n\to\infty} n^{-1} g^p(U_{1n}) = \infty$ a.s. when $E_p(g) = \infty$.

This shows that (b) implies (a) and (c) implies (d). Finally it is not difficult to see that (d) implies that $\limsup_{n\to\infty} D_n(g^p) = \infty$ a.s., which by Lemma 1 implies (c).

PROOF OF THEOREM 1. The results of either [5] or [6] imply that for each $0 < \delta < 1$,

$$n^{-1} \textstyle\sum_{i=[n\delta]}^{n} (in^{-1})^{1/p-1} g(U_{in}) \to \displaystyle\int_{\delta}^{1} u^{1/p-1} g(u)\, du < \infty \quad \text{a.s.}$$

Therefore to complete the proof of Theorem 1 it will be sufficient to prove the following lemma. For each $g \in \mathscr{G}$, $0 < p < \infty$, and $0 < \delta < 1$, set

$$S_n(g, p, \delta) = \textstyle\sum_{i=1}^{[n\delta]} i^{1/p-1} n^{-1/p} g(U_{ni}).$$

LEMMA 3. *Let $g \in \mathscr{G}$ and $0 < p < \infty$, then*
(A) *if and only if*
(B_δ) $\lim_{\delta\downarrow 0} \limsup_{n\to\infty} S_n(g, p, \delta) = 0$ a.s.; *and*
(C) *if and only if*
(D_δ) *for every* $0 < \delta < 1$, $\limsup_{n\to\infty} S_n(g, p, \delta) = \infty$ a.s.

PROOF.

PART 1. (A) *if and only if* (B_δ). Observe that by the remarks in the introduction, when $0 < p \le 1$, (A) is equivalent to $E_p(g) < \infty$ and when $p \ge 1$, (A) is equivalent to $M_p(g) < \infty$.

CASE 1. $p = 1$.

PROOF. The proof in this case is elementary and left to the reader.

CASE 2. $0 < p < 1$.

PROOF. (A) *implies* (B_δ): For each $0 < \delta < 1$ and $n \ge \delta^{-1}$ set

$$m(n, \delta) = \max_{1\le i\le [n\delta]} in^{-1} g^p(U_{in}).$$

Now Lemma 2 in combination with $0 < p < 1$ implies that almost surely

$$\begin{aligned} &\lim_{\delta\downarrow 0} \limsup_{n\to\infty} n^{-1} \textstyle\sum_{i=1}^{[n\delta]} g^p(U_{in}) \\ &\ge \lim_{\delta\downarrow 0} \limsup_{n\to\infty} n^{-1} \textstyle\sum_{i=1}^{[n\delta]} (m(n, \delta))^{1/p-1} g^p(U_{in}) \ge \lim_{\delta\downarrow 0} \limsup_{n\to\infty} S_n(g, p, \delta). \end{aligned} \tag{9}$$

Since $E_p(g) < \infty$ the left hand side of (9) is a.s. zero by Case 1. (B_δ) *implies* (A): The proof follows directly from Lemma 2.

CASE 3. $p > 1$.

PROOF. (A) *implies* (B_δ): We will need the following fact.

CLAIM 1. When $p > 1$, for all $n \geq 1$ and $1 \leq i \leq n$

$$(in^{-1})^{1/p-1} g(U_{in}) n^{-1} \leq (in^{-1})^{1/p-1} g(in^{-1}) n^{-1} + (U_{in})^{1/p-1} g(U_{in}) n^{-1}.$$

PROOF. If $U_{in} \geq in^{-1}$ then $(U_{in})^{1/p-1} \leq (in^{-1})^{1/p-1}$, and since g is nonincreasing $g(U_{in}) \leq g(in^{-1})$. If $U_{in} < in^{-1}$, $(U_{in})^{1/p-1} > (in^{-1})^{1/p-1}$. So in either case the inequality is true.

By Claim 1, we see that

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} S_n(g, p, \delta) \leq$$

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} n^{-1} \textstyle\sum_{i=1}^{[n\delta]} (in^{-1})^{1/p-1} g(in^{-1}) + \tag{10}$$

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} n^{-1} \textstyle\sum_{i=1}^{[n\delta]} (U_{in})^{1/p-1} g(U_{in}). \tag{11}$$

Since $\int_0^1 u^{1/p-1} g(u)\, du < \infty$ and $u^{1/p-1} g(u)$ is nonincreasing, $u^{1/p-1} g(u)$ is Riemann integrable on (0, 1]. Thus expression (10) is zero. Expression (11) is zero by Case 1.

(B_δ) *implies* (A): Assume (B_δ) but $M_p(g) = \infty$. Choose any $0 < \delta' < \delta < 1$. The results of [5] or [6] show that

$$n^{-1} \textstyle\sum_{i=[n\delta']}^{[n\delta]} (in^{-1})^{1/p-1} g(U_{in}) \to \displaystyle\int_{\delta'}^{\delta} u^{1/p-1} g(u)\, du \quad \text{a.s.}$$

But since we assumed that $M_p(g) = \infty$, we have

$$\lim_{\delta'\downarrow 0} \lim_{n\to\infty} \textstyle\sum_{i=[n\delta']}^{[n\delta]} i^{1/p-1} n^{-1/p} g(U_{in}) = \infty \quad \text{a.s.},$$

which contradicts (B_δ). Thus (B_δ) implies (A).

PART 2. (C) *if and only if* (D_δ).

(D_δ) *implies* (C): Obvious from (A) implies (B_δ).

(C) *implies* (D_δ): Observe that when $0 < p \leq 1$, (C) is equivalent to $E_p(g) = \infty$; and when $1 \leq p < \infty$ (C) is equivalent to $M_p(g) = \infty$.

CASE 1. $0 < p < 1$.

PROOF. Since $E_p(g) = \infty$, $\limsup_{n\to\infty} n^{-1/p} g(U_{1n}) = \infty$ a.s.

CASE 2. $1 \leq p < \infty$.

PROOF. Note that for each $0 < \delta' < \delta < 1$

$$\limsup_{n\to\infty} S_n(g, p, \delta) \geq \lim_{n\to\infty} n^{-1} \textstyle\sum_{i=[n\delta']}^{[n\delta]} (in^{-1})^{1/p-1} g(U_{in}). \tag{12}$$

The right hand side of (12) converges a.s. to $\int_{\delta'}^{\delta} u^{1/p-1} g(u)\, du$. Since $M_p(g) = \infty$, the rest is obvious.

This completes the proof of Theorem 1. □

4. Appendix.

The following proposition shows the relationship between $E_p(g)$ and $M_p(g)$.

PROPOSITION. *Let* $g \in \mathscr{G}$.
(i) *If* $0 < p \leq 1$, $E_p(g) < \infty$ *implies* $M_p(g) < \infty$.
(ii) *If* $1 \leq p < \infty$, $M_p(g) < \infty$ *implies* $E_p(g) < \infty$.

PROOF. We can assume without loss of generality that $g(u) > 0$ for all u sufficiently close to 0. First assume $0 < p < 1$. $E_p(g) < \infty$ and g nonincreasing imply that both

$$\textstyle\sum_{n=1}^{\infty} g^p(n^{-1}) n^{-2} < \infty \quad \text{and} \quad \lim_{n\to\infty} g^p(n^{-1}) n^{-1} = 0.$$

Thus $\lim_{n\to\infty}(g(n^{-1})n^{-1/p})^{p-1} = \infty$. Hence there exists a $C > 0$ and an integer n_0 such that

$$\infty > \sum_{n=n_0}^{\infty} g^p(n^{-1})n^{-2} \geq C \sum_{n=n_0}^{\infty} g(n^{-1})n^{-1/p-1},$$

which implies that $M_p(g) < \infty$.

Now assume $1 < p < \infty$. $M_p(g) < \infty$ and $g(u)u^{1/p-1}$ nonincreasing imply that both

$$\sum_{n=1}^{\infty} g(n^{-1})n^{-1/p-1} < \infty \quad \text{and} \quad \lim_{n\to\infty} g(n^{-1})n^{-1/p} = 0.$$

Thus $\lim_{n\to\infty}(g(n^{-1})n^{-1/p})^{1-p} = \infty$. Hence there exists a $C > 0$ and an integer n_0 such that

$$\infty > \sum_{n=n_0}^{\infty} g(n^{-1})n^{-1/p-1} \geq C \sum_{n=n_0}^{\infty} g^p(n^{-1})n^{-2},$$

which implies that $E_p(g) < \infty$.

The proof for the case $p = 1$ is obvious. □

The same relationship exists between $E_q(h)$ and $N_q(h)$.

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REFERENCES

[1] GALAMBOS, J. (1978). *The Asymptotic Theory of Extreme Order Statistics*. Wiley, New York.

[2] HELMERS, R. (1977). A strong law of large numbers for linear combinations of order statistics. Report SW 50/77, Mathematisch Centrum, Amsterdam.

[3] MASON, D. M. (1982). Some characterizations of almost sure bounds for weighted multidimensional empirical distributions and a Glivenko-Cantelli theorem for sample quantiles. *Z. Wahrsch. verw. Gebiete* **59** 505–513.

[4] SEN, P. K. (1978). An invariance principle for linear combinations of order statistics. *Z. Wahrsch. verw. Gebiete* **42** 327–340.

[5] VAN ZWET, W. R. (1980). A strong law for linear functions of order statistics. *Ann. Probability.* **8** 986–990.

[6] WELLNER, J. A. (1977). A Glivenko-Cantelli theorem and strong laws of large numbers for functions of order statistics. *Ann. Statist.* **5** 473–480.

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