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FINITE MARKOV CHAINS IN STATIONARY RANDOM ENVIRONMENTS

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A formulation of a Markov chain in a stationary random environment is given, and for the case of finite state space, necessary and sufficient conditions are found under which a state of the chain is inessential, positively essential or properly essential. It is shown that in this case improperly essential states cannot exist.

0. Following Cogburn (1980) and Bourgin, Cogburn (1981), a random sequence $(\alpha_n)_{n\ge1}$ is said to be a Markov chain in a random environment (with values in a countable set A and environments of a countable set X) if there exists a random sequence $(\xi_n)_{n\ge1}$ (the environment sequence with values in X) on the same probability space such that

$$\mathscr{P}(\alpha_{m+1} = a \mid \alpha_1, \cdots, \alpha_m, \xi_1, \xi_2, \cdots) = K_{\xi m}(\alpha_m, a) \qquad \text{a.s.}$$

for each $a \in A$ and $m \ge 1$, where $K_x = (K_x(a, a'))_{a,a' \in A}$, $x \in X$ is a family of stochastic matrices. The matrix K_x describes the evolution in the environment x.

Obviously, the distribution law of the random sequence $([\alpha_n, \xi_n])_{n\ge1}$ is determined by the family $(K_x)_{x\in X}$ and the distribution law $Q^+ = \mathscr{P}([\alpha_1, (\xi_n)_{n\ge1}] \in (\cdot))$:

$$\begin{aligned} &\mathscr{P}(\alpha_1 = a_1, \cdots, \alpha_m = a_m, \xi_1 = x_1, \cdots, \xi_m = x_m) \\ (1) \qquad &\qquad = Q^+(\alpha = a_1, \xi_1 = x_1, \cdots, \xi_m = x_m)K_{x_1}(a_1, a_2) \cdots K_{x_{m-1}}(a_{m-1}, a_m). \end{aligned}$$

Otherwise, it is easy to see that every family $(K_x)_{x\in X}$ and every distribution law Q^+ on $\mathscr{A} \times \mathscr{X}_1^\infty$ ($\mathscr{A}$ denotes the discrete σ-algebra to A, $\mathscr{X}$ the discrete σ-algebra to X, X_1^∞ the set of all sequences $(x_n)_{n\ge1}$ with $x_n \in X$ and $\mathscr{X}_1^\infty$ the σ-algebra of subsets of X_1^∞ generated by the cylinders) by (1) define a Markov chain in a random environment.

For such a Markov chain in a random environment, Cogburn (1980) and Bourgin, Cogburn (1981) assert that $([\alpha_n, \xi_n])_{n\ge1}$ is a Markov chain with transition probabilities $W(x, x')K_x(a, a')$ if $(\xi_n)_{n\ge1}$ is a time-homogeneous Markov chain (with transition probabilities $W(x, x')$). But, in general, this is not true. We give an example for this: Let $A = X$, $(K_x)_{x\in X}$ a family of stochastic matrices on A, $(\xi_n)_{n\ge1}$ a time-homogeneous Markov chain (the transition probabilities of which we denote by $W(x, x')$) and $Q^+ = \mathscr{P}([\xi_k, (\xi_n)_{n\ge1}] \in (\cdot))$ for any fixed $k > 1$. Then for the Markov chain in random environment defined by (1) for $m = k - 1$ the following holds

$$\mathscr{P}(\alpha_{m+1} = a, \xi_{m+1} = x \mid \alpha_1, \cdots, \alpha_m, \xi_1, \cdots, \xi_m) = \begin{cases} 0 & \text{if} \quad x \ne \alpha_1 \\ K_{\xi m}(\alpha_m, a), & \text{if} \quad x = \alpha_1, \end{cases} \qquad \text{a.s.,}$$

i.e. $([\alpha_n, \xi_n])_{n\ge1}$ is not a Markov chain.

It is remarkable that for a time-homogeneous Markovian environment $(\xi_n)_{n\ge1}$ the sequence $([\alpha_n, \xi_n])_{n\ge1}$ is a Markov chain if it is stationary (see Nawrotzki, 1981, Theorem 7).

From the point of view of representation (1), we describe a Markov chain in a stationary environment by a family $K_x = (K_x(a, a'))_{a,a' \in A}$, $x \in X$, of stochastic matrices and a stationary distribution P^+ on $\mathscr{X}_1^\infty$ (i.e. a distribution law P^+ on $\mathscr{X}_1^\infty$ with $P^+ \circ T^{-1} = P^+$,

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where T is the shift on X_1^∞: $T(x_n)_{n\geq 1} = (x_{n+1})_{n\geq 1}$). And a distribution law Q^+ on $\mathscr{A} \times \mathscr{X}_1^\infty$ we call an initial distribution if $Q^+(A \times (\cdot)) = P^+$.

This is the concept of open systems which are investigated in Nawrotzki (1981).

These investigations show that it is advantageous to describe the stationary environment as a sequence $(\xi_n)_{n\in\Gamma}$ (Γ is the set of integers), i.e. to consider the past environment sequences, too. We denote its distribution law by P. (Then P is a distribution law on $\mathscr{X}^\Gamma$, where $\mathscr{X}^\Gamma$ is the σ-algebra of subsets of $X^\Gamma = \{(x_n)_{n\in\Gamma} : x_n \in X\}$, generated by the cylinders, and we have $P \circ T^{-1} = P$ with the shift T on X^Γ.) And an initial distribution Q is a distribution law on $\mathscr{A} \times \mathscr{X}^\Gamma$ with $Q(A \times (\cdot)) = P$.

Such an initial distribution is called stationary if it is invariant with respect to the stochastic kernel

$$\mathbb{K}([a, (x_n)_{n\in\Gamma}], (\cdot)) = K_{x_1}(a, \ ) \times \delta_{(x_{n+1})_{n\in\Gamma}},$$

(where δ_z is the distribution law with $\delta_z(C) = 1$ iff $z \in C$). This kernel describes the joint motion of the states and the environment in a time unit.

The stochastic kernel q with

$$q(a, (x_n)_{n\in\Gamma}) = Q(\alpha = a \mid (\xi_n)_{n\in\Gamma} = (x_n)_{n\in\Gamma})$$

is called the density of Q. It is easy to prove (see Nawrotzki, 1977, Proposition 2.4) that such kernel q is a density of a stationary initial distribution if and only if for all $a' \in A$

(2) $$q(a', (x_{n+1})_{n\in\Gamma}) = \textstyle\sum_{a\in A} q(a, (x_n)_{n\in\Gamma}) K_{x_1}(a, a') \qquad P\text{-a.e.}$$

In Nawrotzki (1981) it is shown (Theorem 2) that for stationary Q the density q does not essentially depend on $(x_n)_{n\geq 1}$.

We define (see also Cogburn, 1980):

(1) A state $a \in A$ is positively essential if there exists a stationary initial distribution Q with $Q(\alpha = a) > 0$.

(2) A state $a \in A$ is inessential, if for all $a' \in A$

$$K_{a',(x_n)}(\alpha_m = a \text{ i.o.}) = 0 \qquad P\text{-a.e.},$$

where $K_{a',(x_n)}$ is the distribution of the inhomogeneous Markov chain with values in A, the initial state a' and the transition kernels $K_{x_1}, K_{x_2}, \cdots$, and i.o. stands for "infinitely often".

Otherwise the state a is essential.

(3) An essential state $a \in A$ is improperly essential if there exist sets $B_1, B_2, \cdots, \in \mathscr{X}^\Gamma$ with $\cup_{i=1}^\infty B_i = X^\Gamma$ and such that for every $a' \in A$, $i \geq 1$

$$K_{a',P}([\alpha_m, (\xi_{n+m-1})_{n\in\Gamma}] \in \{a\} \times B_i \text{ i.o.}) = 0,$$

where

$$K_{a',P} = \int (K_{a',(x_n)} \times \delta_{(x_n)_{n\in\Gamma}})(\cdot) P(d(x_n)_{n\in\Gamma}).$$

(4) An essential state $a \in A$ is properly essential if it is not improperly essential.

PROPOSITION. *Let* $a \in A$.
(1) *a is inessential if and only if*

$$K_{a,(x_n)}(\alpha_m = a \text{ i.o.}) = 0 \qquad P\text{-a.e.}$$

(2) *a is improperly essential if and only if a is essential and there exist sets $B_1, B_2, \cdots \in \mathscr{X}^\Gamma$ with $\cup_{i=1}^\infty B_i = X^\Gamma$ such that for every $i \geq 1$, holds*

$$K_{a,P}([\alpha_m, (\xi_{n+m-1})_{n\in\Gamma}] \in \{a\} \times B_i \text{ i.o.}) = 0.$$

PROOF. We prove the Assertion (1). The Assertion (2) follows in the same way. Obviously, the condition in (1) is necessary. Otherwise, by the stationarity of P from this condition, it follows that

$$K_{a,(x_{n+k-1})}(\alpha_m = a \text{ i.o.}) = 0 \quad P\text{-a.e.}$$

for every $k \geq 1$. And for every $a' \in A$ holds

$$K_{a',(x_n)}(\alpha_m = a \quad \text{i.o.}) = \sum_{k=1}^{\infty} K_{a',(x_n)}(\alpha_1 \neq a, \cdots, \alpha_{k-1} \neq a, \alpha_k = a, \alpha_{k+m} = a \quad \text{i.o.})$$

$$= \sum_{k=1}^{\infty} K_{a',(x_n)}(\alpha_1 \neq a, \cdots, \alpha_{k-1} \neq a, \alpha_k = a) K_{a,(x_{n+k})_{n\geq 1}}(\alpha_m = a \quad \text{i.o.}).$$

Hence the condition is sufficient, too.

1. Now assume the state space A to be finite. Then for every family $(K_x)_{x\in X}$ of stochastic matrices and every stationary distribution P on $\mathscr{X}^\Gamma$, there exists at least one stationary initial distribution Q (see Nawrotzki, 1975, Theorem 2).

For $x_1, \cdots, x_m \in X$ we denote the matrix product $K_{x_1} \cdots K_{x_m}$ by $K_{x_1\cdots x_m}$.

In the case of finite state space A the following holds:

THEOREM. *Let $a \in A$.*

(1) *a is positively essential if and only if a is properly essential.*

(2) *The following assertions are equivalent*:

(i) *a is not positively essential.*

(ii) *a is inessential.*

(iii) *For every $a' \in A$*

$$\sum_{m=0}^{\infty} K_{x_{-m}\cdots x_0}(a', a) < +\infty \qquad P\text{-a.e.}$$

(iv) *For every $a' \in A$*

$$\lim_{m\to\infty} K_{x_{-m}\cdots x_0}(a', a) = 0 \qquad P\text{-a.e.}$$

(v) *For every $a' \in A$*

$$\lim_{m\to\infty} K_{x_1\cdots x_m}(a', a) = 0 \qquad P\text{-a.e.}$$

(vi) *For every $a' \in A$*

$$\lim_{m\to\infty} \int K_{x_{-m}\cdots x_0}(a', a) P(d(x_n)_{n\in\Gamma}) = 0.$$

(vii) *For every $a' \in A$*

$$\lim_{m\to\infty} \int K_{x_1\cdots x_m}(a', a) P(d(x_n)_{n\in\Gamma}) = 0.$$

(3) *There exists at least one positively essential state and improperly essential states do not exist.*

PROOF. The first part of the Assertion (3) follows from the existence of at least one stationary distribution in the case of finite A and the second part from the assertions (1) and (2).

We prove the assertion (2). Obviously, (iii) is sufficient for (iv). Furthermore, (vi) follows from (iv) and (vii) from (v) by Lebesgue's theorem. The stationarity of P yields

$$\int K_{x_1\cdots x_m}(a', a) P(d(x_n)_{n\in\Gamma}) = \int K_{x_{-m+1}\cdots x_0}(a', a) P(d(x_n)_{n\in\Gamma})$$

for all $m \geq 1$ and $a' \in A$. Therefore, (vi) and (vii) are equivalent.

If a is inessential then for every $a' \in A$

$$\lim_{m\to\infty} K_{a',(x_n)}(\cup_{l=m}^{\infty} \alpha_l = a) = 0 \qquad P\text{-a.e.}$$

Otherwise,

$$K_{a',(x_n)}(\alpha_m = a) = K_{x_1\cdots x_{m-1}}(a', a).$$

Therefore from (ii) follows (v).

If Q is a stationary initial distribution and q its density, then from relation (2) we obtain

$$q(a, (x_{n+m})_{n\le 0}) = \sum_{a'\in A} q(a', (x_n)_{n\le 0}) K_{x_1\cdots x_m}(a', a) \quad P\text{-a.e.}$$

for all $m \ge 1$. And from the stationarity of P it follows that

$$\begin{aligned} Q(\alpha = a) &= \int q(a, (x_n)_{n\le 0}) P(d(x_n)_{n\in\Gamma}) \\ &= \int q(a, (x_{n+m})_{n\le 0}) P(d(x_n)_{n\in\Gamma}) \\ &= \int \sum_{a'\in A} q(a', (x_n)_{n\le 0}) K_{x_1\cdots x_m}(a', a) P(d(x_n)_{n\in\Gamma}) \\ &\le \sum_{a'\in A} \int K_{x_1\cdots x_m}(a', a) P(d(x_n)_{n\in\Gamma}). \end{aligned}$$

Therefore, (vii) implies (i).

If a is inessential then for all $a' \in A$

(3) $$K_{a',(x_n)}(\cup_{m=1}^{\infty} \cap_{l=m}^{\infty} \alpha_l \ne a) = 1 \qquad P\text{-a.e.}$$

Since

$$\begin{aligned} K_{a',(x_n)}(\cup_{m=1}^{\infty} \cap_{l=m}^{\infty} \alpha_l \ne a) &= K_{a',(x_n)}(\cap_{l=1}^{\infty} \alpha_l \ne a) + \sum_{m=1}^{\infty} K_{a',(x_n)}(\alpha_m = a, \cap_{l=m+1}^{\infty} \alpha_l \ne a) \\ &= K_{a',(x_n)}(\cap_{l=1}^{\infty} \alpha_l \ne a) + K_{a',(x_n)}(\alpha_1 = a, \cap_{l=2}^{\infty} \alpha_l \ne a) \\ &\quad + \sum_{m=1}^{\infty} K_{x_1\cdots x_m}(a', a) K_{a,(x_{n+m})}(\cap_{l=2}^{\infty} \alpha_l \ne a), \end{aligned}$$

by stationarity of P we obtain

$$\begin{aligned} &\int \sum_{m=1}^{\infty} K_{x_{-m+1+k}\cdots x_k}(a', a) K_{a(x_{n+k})}(\cap_{l=2}^{\infty} \alpha_l \ne a) P(d(x_n)_{n\in\Gamma}) \\ &= \sum_{m=1}^{\infty} \int K_{x_1\cdots x_m}(a', a) K_{a,(x_{n+m})}(\cap_{l=2}^{\infty} \alpha_l \ne a) P(d(x_n)_{n\in\Gamma}) \\ &= \int \sum_{m=1}^{\infty} \cdots P(d(x_n)_{n\in\Gamma}) \le 1, \end{aligned}$$

so that

(5) $$K_{a,(x_{n+k})}(\cap_{l=2}^{\infty} \alpha_1 \ne a) \sum_{m=1}^{\infty} K_{x_{-m+1+k}\cdots x_k}(a', a) < +\infty \qquad P\text{-a.e.}$$

for all $a' \in A$ and $k \in \Gamma$.

Furthermore, (3) and (4) imply in case of $a' = a$

$$K_{a,(x_n)}(\cap_{l=2}^{\infty} \alpha_l \ne a) + \sum_{m=1}^{\infty} K_{x_1\cdots x_m}(a, a) K_{a,(x_{n+m})}(\cap_{l=2}^{\infty} \alpha_l \ne a) = 1 \qquad P\text{-a.e.}$$

If the first term is positive then (5) for $k = 0$ yields

$$\sum_{m=1}^{\infty} K_{x_{-m+1}\cdots x_0}(a', a) < +\infty.$$

If the second term is positive there exists an integer $r \ge 1$ such that

(6) $$K_{x_1\cdots x_r}(a, a) K_{a,(x_{n+r})}(\cap_{l=2}^{\infty} \alpha_l \ne a) > 0$$

and (5) for $k = r$ yields

$$K_{a,(x_{n+r})}(\cap_{l=2}^{\infty} \alpha_l \neq a) \sum_{m=r+1}^{\infty} K_{x_{-m+r+1}\cdots x_0}(a', a) \cdot K_{x_1\cdots x_r}(a, a)$$

(7)

$$\leq K_{a,(x_{n+r})}(\cap_{l=2}^{\infty} \alpha_l \neq a) \sum_{m=1}^{\infty} K_{x_{-m+1+r}\cdots x_r}(a', a) < +\infty.$$

Now, (6) and (7) imply

$$\sum_{m=r+1}^{\infty} K_{x_{-m+1+r}\cdots x_0}(a', a) < +\infty.$$

Therefore, (ii) is sufficient for the validity of (iii).

Finally, we prove that from (i) follows (ii). In this part we use the notations and constructions from Nawrotzki (1980).

From these construction it follows that a is not positively essential iff

$$K_{(x_n)}(a \notin \gamma_0) = 1 \qquad P\text{-a.e.}$$

Furthermore, for all $m \geq 0$

$$K_{(x_n)}(a \notin \gamma_m) = K_{(x_{n+m})}(a \notin \gamma_0) \qquad P\text{-a.e.}$$

such that by stationarity of P for not positively essential a holds

(8) $$K_{(x_n)}(\cap_{m=0}^{\infty} a \notin \gamma_m) = 1 \qquad P\text{-a.e.}$$

Otherwise, in Nawrotzki (1980) it was shown

$$K_{(x_n)}(\cup_{l=1}^{\infty} \gamma_{-l,0} = \gamma_0) = 1 \qquad P\text{-a.e.}$$

and that from $\gamma_{-l,0} = \gamma_0$ it follows $\gamma_{-l,m} = \gamma_m$ for all $m \geq 1$.

Therefore, we obtain

$$K_{(x_n)}(\cup_{l=1}^{\infty} \cap_{m=0}^{\infty} \gamma_{-l,m} = \gamma_m) = 1 \qquad P\text{-a.e.}$$

and together with (8)

$$K_{(x_n)}(\cup_{l=1}^{\infty} \cap_{m=0}^{\infty} a \notin \gamma_{-l,m}) = 1 \qquad P\text{-a.e.}$$

Besides, the above cited construction yields

$$K_{a',(x_{n-l-1})}(\cap_{m=l+1}^{\infty} \alpha_m \neq a) \geq K_{(x_n)}(\cap_{m=0}^{\infty} a \notin \gamma_{-l,m})$$

and $\gamma_{-(l+1),m} \subseteq \gamma_{-l,m}$ for all $l \geq 0$.

Therefore, for all $a' \in A$ we obtain

$$\int K_{a',(x_n)}(\cup_{l=1}^{\infty} \cap_{m=l}^{\infty} \alpha_m \neq a)\, P(d(x_n)_{n\in\Gamma})$$

$$= \lim_{l\to\infty} \int K_{a',(x_n)}(\cap_{m=l+1}^{\infty} \alpha_m \neq a)\, P(d(x_n)_{n\in\Gamma})$$

$$= \lim_{l\to\infty} \int K_{a',(x_{n-l-1})}(\cap_{m=l+1}^{\infty} \alpha_m \neq a)\, P(d(x_n)_{n\in\Gamma})$$

$$\geq \lim_{l\to\infty} \int K_{(x_n)}(\cap_{m=0}^{\infty} a \notin \gamma_{-l,m})\, P(d(x_n)_{n\in\Gamma})$$

$$= \int K_{(x_n)}(\cup_{l=1}^{\infty} \cup_{m=0}^{\infty} a \notin \gamma_{-l,m})\, P(d(x_n)_{n\in\Gamma})$$

$$= 1,$$

i.e. a is inessential, and the second part of the theorem is proved.

Let us prove the first part. From the second part it follows immediately that a properly essential state $a \in A$ must be positively essential.

Otherwise, if $a \in A$ is positively essential then there exists a stationary initial distribution Q with $Q(\alpha = a) > 0$. Furthermore, the Markov chain $([\alpha_m, (\xi_{m,n})_{n\in\Gamma}])_{m\geq 1}$ with the state space $A \times X^\Gamma$, the initial distribution Q and the transition kernel $\mathbb{K}$ is stationary and it holds $(\xi_{m,n})_{n\in\Gamma} = (\xi_{1,n+m-1})_{n\in\Gamma}$ a.s. Therefore, by Poincaré's recurrence theorem we obtain

$$\mathscr{P}([\alpha_m, (\xi_{1,n+m-1})_{n\in\Gamma}] \in \{a\} \times B \quad \text{i.o.}) > 0$$

for every $B \in \mathscr{X}^\Gamma$ with $Q(\{a\} \times B) > 0$.

Now, from the constructions it follows that

$$\begin{aligned}&\mathscr{P}([(\alpha_n)_{n\geq 1}, (\xi_{1,n})_{n\in\Gamma}] \in (\cdot))\\ &\quad = \int \textstyle\sum_{a'\in A} (K_{a',(x_n)} \times \delta_{(x_n)_{n\in\Gamma}})(\cdot)\, q(a', (x_n)_{n\leq 0})\, P(d(x_n)_{n\in\Gamma}),\end{aligned}$$

where q is the density of Q. Therefore, a positively essential state $a \in A$ can be neither inessential nor improperly essential. □

The Assertion (3) of the theorem was proved in Cogburn (1980, Proposition 2.3) for Markovian environments.

REFERENCES

[1] Cogburn, R. (1980). Markov chains in random environments: the case of Markovian environments. *Ann. Probability* **8** 908–916.

[2] Bourgin, R. D. and Cogburn, R. (1981). On determining absorption probabilities for Markov chains in random environments. *Adv. in Appl. Probability* **13** 369–387.

[3] Nawrotzki, K. (1975). Stationäre Anfangsverteilungen stochastischer Automaten I. *Elektron. Informationsverarb. Kybernet.* **11** 151–164.

[4] Nawrotzki, K. (1977). Ein Grenzwertsatz für stationäre zufällige Folgen stochastischer Matrizen. *Math. Nachr.* **80** 133–150.

[5] Nawrotzki, K. (1980). Stationäre Verteilungen endlicher offener Systeme. *Elektron. Informationsverarb. Kybernet.* **16** 345–351.

[6] Nawrotzki, K. (1981). Discrete open systems or Markov chains in random environments. *Elektron. Informationsverarb. Kybernet.* **17** 569–599.

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