The Annals of Probability
1982, Vol. 10, No. 4, 992–1003

LIMIT THEOREMS FOR ESTIMATORS BASED ON INVERSES OF SPACINGS OF ORDER STATISTICS

BY PETER HALL

Australian National University

Let $X_{n1} < X_{n2} < \cdots < X_{nn}$ denote the order statistics of an n-sample from the distribution with density f. We prove the strong consistency and asymptotic normality of estimators based on the series

$$(\tfrac{1}{2}) \sum_1^{n-k} (X_{n,r+k} + X_{nr})/(X_{n,r+k} - X_{nr})^p \quad \text{and} \quad \sum_1^{n-k} (X_{n,r+k} - X_{nr})^{-p},$$

where $k > 2p > 0$ are fixed constants. These series may be used to estimate functionals of f. The ratio of the series was introduced by Grenander (1965) as an estimator of a location parameter, and he established weak consistency. In recent years several authors have examined such estimators using Monte Carlo experiments, but the lack of an asymptotic theory has prevented a more detailed discussion of their properties.

1. Introduction and summary. Let $X_1, X_2, \cdots$ be independent random variables with a common distribution function F and density f, and suppose $X_{n1} < X_{n2} < \cdots < X_{nn}$ are the order statistics of the sample $X_1, X_2, \cdots, X_n$. Define

$$\hat{a}_n = n^{-(p+1)} \sum_{r=1}^{n-k} (X_{n,r+k} - X_{nr})^{-p}, \qquad a = \mu \int_{-\infty}^{\infty} f^{p+1}(x)\, dx,$$

$$\hat{b}_n = n^{-(p+1)} \sum_{r=1}^{n-k} \{\rho X_{n,r+k} + (1-\rho) X_{nr}\}(X_{n,r+k} - X_{nr})^{-p} \quad \text{and} \quad b = \mu \int_{-\infty}^{\infty} x f^{p+1}(x)\, dx,$$

where $0 \le \rho \le 1$ and $\mu = \Gamma(k-p)/\Gamma(k)$ (which is finite if $k > p$). We shall prove that under appropriate conditions on f the quantities $\hat{a}_n$ and $\hat{b}_n$ are strongly consistent for a and b, and derive central limit theorems for the errors in this estimation procedure.

The ratio of the parameters b and a,

$$\theta_p = b/a = \left\{\int_{-\infty}^{\infty} x f^{p+1}(x)\, dx\right\} \Big/ \left\{\int_{-\infty}^{\infty} f^{p+1}(x)\, dx\right\},$$

was introduced by Grenander (1965) as a measure of location. It represents a true compromise between the mean and the mode. In the case $p = 0$, θ_p coincides with the population mean $E(X_1)$, and under mild regularity conditions on f, θ_p converges to the population mode as $p \to \infty$. Under the commonly made assumption that the distribution is symmetric about its centre, θ, we have $\theta_p \equiv \theta$ for all p. Grenander showed that under suitable conditions the ratio $\hat{b}_n/\hat{a}_n$ converges weakly to θ_p as $n \to \infty$. (Actually he considered the special case $\rho = \tfrac{1}{2}$. The case of a more general ρ appears to have been introduced by Andriano, Gentle and Sposito (1977b, 1978); see also Andriano, Gentle and Sposito (1977a).)

The performance of Grenander's estimate in small samples was complimented by Ekblom (1972, p. 184). One method of describing the estimator's performance in large

Received March 1981; revised March 1982.

AMS 1980 *subject classification.* Primary 60F05, 60F15; Secondary 62G30.

Key words and phrases. Central limit theorem, mean, mode, order statistics, spacings, strong consistency.

samples would be to derive asymptotic expressions for its distribution, bias and variance. However, there are no available results of this type. Indeed, the paucity of knowledge about the large sample behaviour of Grenander's estimator appears to have inhibited its discussion in the literature. It has been studied largely from the point of view of Monte Carlo trials; see Dalenius (1965), Ekblom (1972), and Andriano, Gentle and Sposito (1977b, 1978). One of our aims in the present paper is to remedy this situation.

Grenander confined attention to the case $p > 1$, and considered only the ratio $\hat{b}_n/\hat{a}_n$. We assume that $p > 0$, and study the estimators $\hat{a}_n$ and $\hat{b}_n$, as well as $\hat{c}_n = \hat{b}_n/\hat{a}_n$. These quantities are of independent interest because they provide simple, direct estimates of functionals of f. There is no difficulty in extending our techniques to estimators of the form

$$\hat{d}_n = n^{-(p+1)} \sum_{r=1}^{n-k} w\{\rho X_{n,r+k} + (1-\rho)X_{nr}\}(X_{n,r+k} - X_{nr})^{-p},$$

which estimates $d = \mu \int_{-\infty}^{\infty} w(x)f^{p+1}(x)\,dx$, where w is a general weight function. However it is simpler to discuss our conditions if we consider specific versions of w.

In order to obtain an intuitive understanding of the problems involved, let us consider the special case of the exponential distribution. Here we may write $X_{nr} = \sum_{i=1}^{r} Z_i/(n-i+1)$, where for each $n \geq 1$ the variables $Z_i = Z_i(n)$, $1 \leq i \leq n$, are independent and exponentially distributed (see David (1970, page 18)). In this case each k-spacing may be written as

$$X_{n,r+k} - X_{nr} = \sum_{i=r+1}^{r+k} Z_i/(n-i+1) \simeq \left(\sum_{i=r+1}^{r+k} Z_i\right)/(n-r),$$

and the quantities $\hat{a}_n$ and $\hat{b}_n$ are sums of $(k-1)$-dependent random variables. A central limit theorem is readily proved by adapting classical results for sums of m-dependent variables—see for example Theorem 7.3.1, page 214 of Chung (1974). More generally, Rényi's representation of order statistics may be used to show that

$$X_{n,r+k} - X_{nr} \simeq \left(\sum_{i=r+1}^{r+k} Z_i\right)/nf\{F^{-1}(r/n)\}, \tag{1.1}$$

and this property may be employed to derive a central limit theorem. Grenander used essentially this approximation to obtain a weak law of large numbers, and a conjecture he made (page 138) concerning the asymptotic normality of his estimator appears to be based on an estimate like (1.1). However, while this approximation is sufficiently accurate to give a law of large numbers, it yields a central limit theorem which can be in error. The reason is that the terms ignored in (1.1), while of a smaller order of magnitude than the primary term, tend to be of the one sign for many values of r and combine together to give the estimator a very different behaviour than would normally be expected. Paradoxically, the approximation (1.1) is quite adequate for a central limit theorem in the case of the exponential distribution, but inadequate in many other cases, such as the uniform. The special case of the uniform distribution is examined in a different context in Holst (1979), but Holst's results do not extend to other distributions.

In this paper we consider the case where p and k are fixed, which is the interesting situation if we are estimating a compromise between the mean and the mode, or if we are estimating the centre of a symmetric distribution. The case where p and k tend to infinity is examined in Hall (1981). The results obtained there are of a very different character—they depend crucially on behaviour of F in the neighbourhood of the mode, whereas in the present situation, behaviour in the tails is all-important. After stating our laws of large numbers we shall present several central limit theorems for the estimators, and discuss the conditions involved. These results are collected together in Section 2, and their proofs deferred until Section 3.

2. Limit theorems. Let $Z_1, Z_2 \cdots$ be independent exponential random variables, fix $k \geq 1$ and $p > 0$, and set $Y_r = (\sum_{i=r+1}^{r+k} Z_i)^{-p}$ and $\mu = E(Y_r)$. We shall adopt the notation of Section 1. The density f is said to be piecewise uniformly continuous if there exist numbers $x_0, x_1, \cdots, x_m$ satisfying $-\infty = x_0 < x_1 < \cdots < x_m = \infty$, such that f is uniformly continuous on (x_{i-1}, x_i) for $1 \leq i \leq m$. Note that this implies f is essentially bounded.

THEOREM 1. *(Strong consistency.) Let $k > 2p > 0$. If f is piecewise uniformly continuous then $\hat{a}_n \to a$ almost surely. If in addition $|x|f^p(x)$ is bounded, and $f(x)$ and $f(-x)$ are ultimately monotone as $x \to \infty$, then $\hat{b}_n \to b$ almost surely.*

Weak consistency may be proved by similar methods under the weaker constraint $k > p > 0$, and the same conditions on f.

The second order behaviour of these estimators, in terms of a central limit theorem, seems to depend crucially on the behaviour of f in the neighbourhood of points where $f(x) \to 0$. We shall shortly discuss the case of a density with unbounded support, but a density such as the uniform with compact support may be handled so much more easily that it is illuminating to consider it in isolation. Let us define

$$\sigma_w^2 = \left\{\int_{-\infty}^{\infty} g_w^2(x) f^{2p+1}(x)\, dx\right\}\{\textstyle\sum_{i=1}^{2k-1} \operatorname{cov}(Y_i, Y_k) - p^2\mu^2\}$$

$$+ \mu^2(p+1)^2 \int_{-\infty}^{\infty} \left[g_w(x) f^p(x) - \{F(x)\}^{-1} \int_{-\infty}^{x} g_w(y) f^{p+1}(y)\, dy\right]^2 f(x)\, dx$$

for $w = a$, b and c, where $g_a(x) \equiv 1$, $g_b(x) \equiv x$ and $g_c(x) \equiv x - b/a$. Let $c = b/a$ and $\hat{c}_n = \hat{b}_n/\hat{a}_n$.

THEOREM 2. *(Central Limit Theorem.) Let $k > 2p > 0$ and $-\infty < u < v < \infty$. If f vanishes outside (u, v) and is bounded away from zero on (u, v), and if f has two bounded derivatives on (u, v), then $n^{1/2}(\hat{a}_n - a) \to_{\mathscr{D}} N(0, \sigma_a^2)$, $n^{1/2}(\hat{b}_n - b) \to_{\mathscr{D}} N(0, \sigma_b^2)$ and $n^{1/2}(\hat{c}_n - c) \to_{\mathscr{D}} N(0, \sigma_c^2/a^2)$.*

Of considerable practical interest are those distributions like the normal which have unbounded support and are not covered by Theorem 2. Our next result is tailored to just this case. A similar argument may be applied to distributions for which the differentiation conditions below hold only in a piecewise sense, and to distributions like the exponential with support equal to a semi-infinite interval. Some distributions such as the gamma have an extra complicating factor in that the density $f(x)$ is nonzero for $x > 0$, but converges to zero as $x \downarrow 0$. However this problem can be overcome by imposing versions of the conditions below in a neighbourhood of the origin, rather than towards $-\infty$.

In Theorem 3 we shall assume that $f(x) > 0$ for all x; that f has two bounded derivatives on $(-\infty, \infty)$; and that the six functions $f(\pm x)$, $f^5(\pm x)/|f'(\pm x)|^2$ and $f^4(\pm x)/|f''(\pm x)|$ are ultimately nonincreasing as $x \to \infty$. (The $\pm$ signs are to be taken respectively, and so for example the monotonicity of $f(\pm x)$ requires that of both $f(x)$ and $f(-x)$.) We call these the "basic" conditions on f. We shall also ask that there exist positive constants $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$ and $\delta_\pm$ with the properties

$(A_\pm)$ $f(\pm x) \le C\{G_\pm(x)\}^{\alpha_\pm}$ for large x;

$(B_\pm)$ $\alpha_\pm \le \beta_\pm$ and $1/f(\pm x) + |f'(\pm x)|/f^3(\pm x) \le C\{G_\pm(x)\}^{-\beta_\pm}$ for large x;

$(C_\pm)$ $2\beta_\pm - \alpha_\pm \le \gamma_\pm$ and $|f''(\pm x)|/f^4(\pm x) + |f'(\pm x)|^2/f^5(\pm x) \le C\{G_\pm(x)\}^{-\gamma_\pm}$ for large x;

$(D_\pm)$ $2\beta_\pm - \gamma_\pm \ge \delta_\pm$ and $|\pm x| \le C\{G_\pm(x)\}^{-\delta_\pm}$ for large x,

where $G_+(x) = 1 - F(x)$, $G_-(x) = F(-x)$ and C denotes a positive generic constant. (The $\pm$ signs are to be taken respectively, and so each of these conditions stands for two conditions. For example, $(A_\pm)$ means $(A_+): f(x) \le C\{1 - F(x)\}^{\alpha_+}$, and $(A_-): f(-x) \le C\{F(-x)\}^{\alpha_-}$, as $x \to \infty$.)

These conditions are considerably less restrictive than their formulation makes them appear. The generality they confer is probably best described by considering several examples. Let c, $\ell > 0$.

(i) Suppose $F(x) = 1 - cx^{-\ell}$ for large x. The conditions above for the positive tail all hold if we set $\alpha_+ = (\ell + 1)/\ell$, $\beta_+ = \alpha_+ + 1$, $\gamma_+ = \alpha_+ + 2$ and $\delta_+ = \alpha_+ - 1$. The conditions $\gamma_+ - \alpha_+ < 2(\alpha_+ p + 1)$ and $\max(1, \gamma_+ - \alpha_+) < 2(\alpha_+ p + 1 - \delta_+)$ in Theorem 3 below are equivalent to the constraints $p > 0$ and $p > 1/(\ell + 1)$, respectively. This example is readily extended to distributions with more general regularly varying tails.

(ii) Suppose $F(x) = 1 - ce^{-\ell x}$ for large x. The conditions on the positive tail hold with $\alpha_+ = 1$, $\beta_+ = 2$, $\gamma_+ = 3$ and $\delta_+ = \varepsilon$, where $0 < \varepsilon \leq 1$. The constraints $\gamma_+ - \alpha_+ < 2(\alpha_+ p + 1)$ and $\max(1, \gamma_+ - \alpha_+) < 2(\alpha_+ p + 1 - \delta_+)$ revert to the single condition, $p > 0$.

(iii) If F is a normal distribution function, let $0 < \eta < 1$ and set $\alpha_+ = 1 - \eta$, $\beta_+ = 2$, $\gamma_+ = 3 + \eta$ and $\delta_+ = \varepsilon$, where $0 < \varepsilon < 1 - \eta$. Again the conditions on p collapse to $p > 0$.

THEOREM 3. *(Central limit theorem.) Assume f satisfies the basic conditions, and $(A_\pm)$, $(B_\pm)$ and $(C_\pm)$ hold. Suppose $k > 2p > 0$.*

(i) *If $\int_{-\infty}^{\infty} f^{p-1}(x)\,|f'(x)|\,dx < \infty$ and $\gamma_\pm - \alpha_\pm < 2(\alpha_\pm p + 1)$ then $n^{1/2}(\hat{a}_n - a) \to_{\mathscr{D}} N(0, \sigma_a^2)$ as $n \to \infty$.*

(ii) *If $\int_{-\infty}^{\infty} [x^2 f^{2p+1}(x) + f^p(x) + F(x)\{1 - F(x)\} f^{2p-1}(x)]\,dx < \infty$, $|x| f^p(x)$ is bounded, $(D_\pm)$ holds and $\max(\gamma_\pm - \alpha_\pm, 1) < 2(\alpha_\pm p + 1 - \delta_\pm)$ then $n^{1/2}(\hat{b}_n - b) \to_{\mathscr{D}} N(0, \sigma_b^2)$ and $n^{1/2}(\hat{c}_n - c) \to_{\mathscr{D}} N(0, \sigma_c^2/a^2)$ as $n \to \infty$.*

(The conditions of (ii) above imply that $\int_{-\infty}^{\infty} f^{p-1}(x)\,|xf'(x)|\,dx < \infty$.)

The parameter ρ has no direct influence on the asymptotic normality of $\hat{a}_n$, $\hat{b}_n$ and $\hat{c}_n$. Therefore any benefit to be gained by a specific choice of ρ would most likely be apparent only in small samples.

Note that $\int_{-\infty}^{\infty} [f^p(x) - \{F(x)\}^{-1} \int_{-\infty}^{\infty} f^{p+1}(y)\,dy]^2 f(x)\,dx = 0$ in the case of the uniform distribution on (0,1). Substituting this into the formula for σ_a^2 we deduce that $\kappa \equiv \sum_{i=1}^{2k-1} \operatorname{cov}(Y_i, Y_k) > p^2\mu^2$. The rough approximation (1.1) would suggest that the asymptotic variance of $n^{1/2}(\hat{a}_n - a)$ in the uniform case equals κ, which differs from the true value of $\kappa - p^2\mu^2$. The limiting variances also differ in the cases of $n^{1/2}(\hat{b}_n - b)$ and $n^{1/2}(\hat{c}_n - c)$.

3. Proofs. Let $H(x) = F^{-1}(e^{-x})$, $x > 0$. The following formulae are easily derived:

$$H'(x) = -e^{-x}/f\{H(x)\}, \tag{3.1}$$

$$H''(x) = e^{-x}/f\{H(x)\} - f'\{H(x)\}e^{-2x}/f^3\{H(x)\}, \quad \text{and} \tag{3.2}$$

$$\begin{aligned} H'''(x) = {}& -e^{-x}/f\{H(x)\} + f''\{H(x)\}e^{-3x}/f^4\{H(x)\} \\ & + 3f'\{H(x)\}e^{-2x}[1 - f'\{H(x)\}e^{-x}/f^2\{H(x)\}]/f^3\{H(x)\}. \end{aligned} \tag{3.3}$$

Rényi's representation of order statistics (see David (1970, page 18)) permits us to write $X_{n,n-r+1} = H\{\sum_{i=1}^{r} Z_i/(n - i + 1)\}$, $1 \leq r \leq n$, where for each n the variables $Z_i = Z_i(n)$ are independent and exponentially distributed with unit mean. The symbol C below denotes a positive generic constant, while if $\{\lambda_n\}$ is a sequence of positive constants, the identity $\xi_n = O_p(\lambda_n)$ for random variables ξ_n, means that $\lambda_n^{-1}\xi_n$ is tight:

$$\lim_{\ell \to \infty} \liminf_{n \to \infty} P(\lambda_n^{-1}|\xi_n| \leq \ell) = 1.$$

Furthermore, if $\Delta > 0$, $\lambda_{nr} > 0$ and ξ_{nr}, $1 \leq r \leq n\Delta$, are random variables, we use the phrase "$\xi_{nr} = O_p(\lambda_{nr})$ uniformly in $1 \leq r \leq n\Delta$" to mean that

$$\sup_{1 \leq r \leq n\Delta} \lambda_{nr}^{-1} |\xi_{nr}| = O_p(1).$$

PROOF OF THEOREM 1. Let $V_{n,n-r+1} = -\log F(X_{nr})$, $1 \leq r \leq n$, denote the order statistics from an exponential sample, and observe that

$$X_{n,n-(r+k)+1} - X_{n,n-r+1} = (V_{n,r+k} - V_{nr})H'\{V_{nr} + \phi(V_{n,r+k} - V_{nr})\} \tag{3.4}$$

where $0 < \phi(n, r) < 1$. Suppose $0 \leq \pi_1 < \pi_2 < 1$, and that if $\pi_1 \neq 0$, $F^{-1}(1-\pi_1)$ is a continuity point of f. Write $[n\pi_i]$ for the integer part of $n\pi_i$, with $[n\pi_1] = 1$ if $\pi_1 = 0$. Since $V_{n,[n\pi_i]} \to_{\text{a.s.}} -\log(1-\pi_i)$ then

$$\limsup_{n\to\infty} \sup_{[n\pi_1]\leq r\leq[n\pi_2]} | V_{nr} + \phi(V_{n,r+k} - V_{nr}) + \log(1-\pi_1) | \leq \log\{(1-\pi_1)/(1-\pi_2)\}$$

with probability one. From this result, (3.1) and the uniform continuity of f we see that

$$(3.5)\quad \limsup_{n\to\infty} \sup_{[n\pi_1]\leq r\leq[n\pi_2]} |[H'\{V_{nr} + \phi(V_{n,r+k} - V_{nr})\}]^{-1} + f(x)/(1-\pi_1)| \leq \varepsilon_{1\pi_1}(\pi_2)$$

with probability one, where $x = F^{-1}(1-\pi_1)$ and $\varepsilon_{iu}(v)$ stands for a function satisfying

$$(3.6)\quad 0 \leq \varepsilon_{iu}(v) \to 0 \quad \text{uniformly in} \quad u \in [0, 1-\Delta] \quad \text{as} \quad v \downarrow u, \qquad \text{for each} \quad \Delta > 0.$$

The next step is to prove that

$$(3.7)\quad \limsup_{n\to\infty} | n^{-(p+1)} \sum_{r=[n\pi_1]}^{[n\pi_2]} \{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-p} - \mu f^p(x)(\pi_2 - \pi_1) | \leq (\pi_2 - \pi_1)\, \varepsilon_{2\pi_1}(\pi_2)$$

with probability one, where $\varepsilon_{2u}(v)$ satisfies (3.6). This implies that whenever $0 < \pi < 1$,

$$n^{-(p+1)} \sum_{r=1}^{[n\pi]} \{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-p} \to_{\text{a.s.}} \mu \int_0^{\pi} f^p\{F^{-1}(1-x)\}\, dx = \mu \int_{F^{-1}(1-\pi)}^{\infty} f^{p+1}(x)\, dx,$$

and by symmetry it must also be true that

$$n^{-(p+1)} \sum_{r=[n\pi]+1}^{n} \{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-p} \to_{\text{a.s.}} \mu \int_0^{F^{-1}(1-\pi)} f^{p+1}(x)\, dx.$$

Therefore $\hat{a}_n \to_{\text{a.s.}} a$.

The inequality (3.7) will follow from (3.4) and (3.5) if we show that

$$(3.8)\quad | n^{-(p+1)} \sum_{r=[n\pi_1]}^{[n\pi_2]} (V_{n,r+k} - V_{nr})^{-p} - \mu\{(1-\pi_1)^{p+1} - (1-\pi_2)^{p+1}\}/(p+1) | \to_{\text{a.s.}} 0.$$

Using Rényi's representation, $V_{nr} = \sum_{i=1}^{r} Z_i(n)/(n-i+1)$, $1 \leq r \leq n$. Therefore the series in (3.8) is a sum of $(k-1)$-dependent variables, and may be broken up into k sums of independent variables, each sum similar to

$$S_{nj} = \sum_{r=[n\pi_1/k]}^{[n\pi_2/k]} (V_{n,rk+j+k} - V_{n,rk+j})^{-p}, \qquad 1 \leq j \leq k.$$

For each ε and $\Delta > 0$ we may obtain the following bound using Rosenthal's (1970) inequality:

$$\sum_n P(|S_{nj} - ES_{nj}| > n^{p+1}\varepsilon) \leq \varepsilon^{-(2+\Delta)} \sum_n n^{-(p+1)(2+\Delta)} E|S_{nj} - ES_{nj}|^{2+\Delta}$$
$$\leq CE\{(\sum_{i=1}^{k} Z_i)^{-(2+\Delta)p}\} \sum_n n^{-(1+\Delta)}.$$

Since $k > 2p$ we may choose $\Delta > 0$ such that $E\{(\sum_{i=1}^{k} Z_i)^{-(2+\Delta)p}\} < \infty$, and then it follows by the Borel-Cantelli lemma that $n^{-(p+1)}(S_{nj} - ES_{nj}) \to_{\text{a.s.}} 0$. It is readily proved that the mean of the quantity within modulus signs in (3.8) converges to zero as $n \to \infty$, and so (3.8) must be true.

Similar arguments may be used to prove that

$$n^{-(p+1)} \sum_{r=[n\pi_1]}^{[n\pi_2]} \{\rho X_{n,n-r+1} + (1-\rho) X_{n,n-(r+k)+1}\}\{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-p} \to_{\text{a.s.}} \mu \int_{F^{-1}(1-\pi_2)}^{F^{-1}(1-\pi_1)} x f^{p+1}(x)\, dx$$

whenever $0 < \pi_1 < \pi_2 < 1$. It is readily seen that the proof that $\hat{b}_n \to_{\text{a.s.}} b$ may be completed by showing that with probability one the quantity

$$\text{(3.9)} \qquad \limsup_{n\to\infty} n^{-(p+1)} \sum_{r=1}^{[n\Delta]} X_{n,n-(r+k)+1} \{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-p}$$

can be made arbitrarily small by choosing Δ small. Since $f(x)$ is ultimately nonincreasing then if Δ is small, and for all sufficiently large n with probability one,

$$|H'\{V_{nr} + \phi(V_{n,r+k} - V_{nr})\}|^{-1} \le \exp(V_{n,r+k})\, f\{H(V_{n,r+k})\} \le Cf(X_{n,n-(r+k)+1})$$

uniformly in $1 \le r \le [n\Delta]$. It now follows from (3.4) that the argument of (3.9) is dominated by a constant multiple of

$$n^{-(p+1)} \sum_{r=1}^{[n\Delta]} |X_{n,n-(r+k)+1}| f^p(X_{n,n-(r+k)+1})(V_{n,r+k} - V_{nr})^{-p} \le Cn^{-(p+1)} \sum_{r=1}^{[n\Delta]} (V_{n,r+k} - V_{nr})^{-p},$$

and we already know that $\limsup_{n\to\infty} n^{-(p+1)} \sum_{r=1}^{[n\Delta]} (V_{n,r+k} - V_{nr})^{-p}$ converges to zero as $\Delta \to 0$. This completes the proof.

The proof of Theorem 2 is very much like that of Theorem 3, and so is not given here.

PROOF OF THEOREM 3. We proceed via a sequence of lemmas.

LEMMA 1. *Let $\ell = 0$ or 1. Assume condition (A_+), and also (D_+) and $\delta_+ < \alpha_+ p + 1$ if $\ell = 1$. Then for each ε satisying $0 < \varepsilon < 1$,*

$$\sum_{r=1}^{n^\varepsilon} |\rho X_{n,n-r+1} + (1-\rho)X_{n,n-(r+k)+1}|^{\ell} \{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-p} = O_p\{n^{p+1-(1-\varepsilon)(\alpha_+ p+1-\delta_+\ell)}\}.$$

PROOF. (Here and below we shall drop the subscripts $\pm$ whenever no ambiguity arises.) From Rényi's representation, the Taylor expansion

$$X_{n,n-(r+k)+1} - X_{n,n-r+1} = \{\textstyle\sum_{i=r+1}^{r+k} Z_i/(n-i+1)\}$$
$$\cdot H'\{\textstyle\sum_{i=1}^{r} Z_i/(n-i+1) + \phi \sum_{i=r+1}^{r+k} Z_i/(n-i+1)\},$$

and the result (3.1) we may write

$$\{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-1} \le (n-r)\, (\textstyle\sum_{i=r+1}^{r+k} Z_i)^{-1} \exp\{\sum_{i=1}^{r+k} Z_i/(n-i+1)\}$$
$$\times f[H\{\textstyle\sum_{i=1}^{r} Z_i/(n-i+1) + \phi \sum_{i=r+1}^{r+k} Z_i/(n-i+1)\}],$$

where $0 < \phi < 1$. In view of the monotonicity of f, with probability tending to one as $n \to \infty$ the last factor is dominated by

$$f[H\{\textstyle\sum_{i=1}^{r+k} Z_i/(n-i+1)\}] = f(X_{n,n-(r+k)+1}) \le C\{1 - F(X_{n,n-(r+k)+1})\}^{\alpha} = C\, U_{n,r+k}^{\alpha}$$

uniformly in $1 \le r \le n^\varepsilon$, where the U_{nj}'s are order statistics from a sample uniform on (0,1). Also,

$$\text{(3.10)} \qquad \sum_{i=1}^{r} (n-i+1)^{-1} - \log\{n/(n-r)\} = O(n^{-1}) \quad \text{and}$$
$$\sum_{i=1}^{r} (Z_i - 1)/(n-i+1) = O_p(n^{-1/2})$$

uniformly in $1 \le r \le n\Delta$ for any $0 < \Delta < 1$, and it may also be shown that

$$\text{(3.11)} \qquad \exp\{\textstyle\sum_{i=1}^{r+k} Z_i/(n-i+1)\} = O_p(1) \quad \text{and} \quad U_{nr} = O_p(r/n)$$

uniformly in $1 \le r \le n\Delta$. Observe that by Theorem 0 of Wellner (1978),

$$\sup_{\log n \le r \le n} |U_{nr}/(r/n) - 1| \to_p 0.$$

Rényi's representation may be used to show that

$$\sup_{1 \le r \le \log n} U_{nr}/(r/n) = O_p(1).$$

This proves the latter part of (3.11).) Combining these estimates we obtain,

$$\text{(3.12)} \qquad \{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-1} = O_p(1) n (r/n)^{\alpha} (\textstyle\sum_{i=r+1}^{r+k} Z_i)^{-1}.$$

It follows from work of Daniels (1945) and Robbins (1954) that the variable $\inf_{1\le r\le n} U_{nr}(r/n)$ is uniform on (0,1), and so

$$U_{nr}^{-1} = O_p(n/r) \tag{3.13}$$

uniformly in $1 \le r \le n\Delta$. Therefore with probability tending to 1 as $n \to \infty$,

$$|\rho X_{n,n-r+1} + (1-\rho)X_{n,n-(r+k)+1}| \le X_{n,n-r+1} \le C\,\{1 - F(X_{n,n-r+1})\}^{-\delta} = O_p\{(n/r)^{\delta}\}.$$

Lemma 1 follows from this estimate and (3.12); note that $\sum_{r=1}^{n^\varepsilon} r^{\alpha p-\delta\ell}(\sum_{i=r+1}^{r+k} Z_i)^{-p} = O_p(n^{\varepsilon(\alpha p+1-\delta\ell)})$, by Markov's inequality.

LEMMA 2. *Under conditions* (A_+) *and* (B_+) *there exists* $\Delta > 0$ *such that*

$$X_{n,n-(r+k)+1} - X_{n,n-r+1} = \{\textstyle\sum_{i=r+1}^{r+k} Z_i/(n-i+1)\}H'\{\sum_{i=1}^{r} Z_i/(n-i+1)\} \times [1 + O_p\{n^{-1}(r/n)^{\alpha_+-\beta_+}\log n\}]$$

uniformly in $1 \le r \le n\Delta$, *as* $n \to \infty$.

PROOF. Observe first that

$$X_{n,n-(r+k)+1} - X_{n,n-r+1} = \{\textstyle\sum_{i=r+1}^{r+k} Z_i/(n-i+1)\}H'\{\sum_{i=1}^{r} Z_i/(n-i+1)\}(1 + R_{1nr})$$

where

$$R_{1nr} = \tfrac12\{\textstyle\sum_{i=r+1}^{r+k} Z_i/(n-i+1)\}H''\{\sum_{i=1}^{r} Z_i/(n-i+1) + \phi \sum_{i=r+1}^{r+k} Z_i/(n-i+1)\}/H'\{\sum_{i=1}^{r} Z_i/(n-i+1)\},$$

and $0 < \phi < 1$. From (3.2) and the nondecreasing nature of $1/f$ and $|f'|/f^3$ we see that if $0 \le \varepsilon < x$ and x is sufficiently small,

$$|H''(x)| \le 1/f\{F^{-1}(e^{-x+\varepsilon})\} + |f'\{F^{-1}(e^{-x+\varepsilon})\}|/f^3\{F^{-1}(e^{-x+\varepsilon})\}.$$

Setting $\varepsilon = \phi \sum_{i=r+1}^{r+k} Z_i/(n-i+1)$, $y = \sum_{i=1}^{r} Z_i/(n-i+1)$ and $x = y + \varepsilon$ we deduce that

$$|H''(x)| \le C[1 - F\{F^{-1}(e^{-y})\}]^{-\beta} = CU_{nr}^{-\beta} = O_p\{(n/r)^{\beta}\}$$

uniformly in $1 \le r \le n\Delta$, and using techniques from the proof of Lemma 1 we obtain

$$|H'\{\textstyle\sum_{i=1}^{r} Z_i/(n-i+1)\}|^{-1} = O_p\{(r/n)^{\alpha}\}$$

uniformly in $1 \le r \le n$. Finally observe that

$$\textstyle\sum_{i=r+1}^{r+k} Z_i/(n-i+1) \le k(\max_{1\le i\le n} Z_i)/(n - n\Delta - k) = O_p(n^{-1}\log n)$$

uniformly in $1 \le r \le n\Delta$. Lemma 2 follows on combining these estimates.

LEMMA 3. *Under conditions* (A_+) *and* (C_+) *there exists* $\Delta > 0$ *such that*

$$\begin{aligned} H'\{\textstyle\sum_{i=1}^{r} Z_i/(n-i+1)\} &= H'\{\textstyle\sum_{i=1}^{r} (n-i+1)^{-1}\} \\ &\times [1 + \{\textstyle\sum_{i=1}^{r} (Z_i-1)/(n-i+1)\}H''\{\sum_{i=1}^{r}(n-i+1)^{-1}\}/H'\{\sum_{i=1}^{r}(n-i+1)^{-1}\} \\ &\qquad + O_p\{n^{-1}(n/r)^{\gamma_+-\alpha_+}\}] \end{aligned} \tag{3.14}$$

uniformly in $1 \le r \le n\Delta$, *as* $n \to \infty$.

PROOF. Observe first that a precise form for the relation (3.14) is given by a Taylor expansion in which the remainder $O_p\{n^{-1}(n/r)^{\gamma-\alpha}\}$ is replaced by

$$\begin{aligned} R_{2nr} = \tfrac12\{\textstyle\sum_{i=1}^{r} (Z_i-1)/(n-i+1)\}^2 H'''\{&\textstyle\sum_{i=1}^{r}(n-i+1)^{-1} \\ &+ \phi \textstyle\sum_{i=1}^{r}(Z_i-1)/(n-i+1)\}/H'\{\sum_{i=1}^{r}(n-i+1)^{-1}\}. \end{aligned} \tag{3.15}$$

The function $\xi \equiv 3\{1/f(H) + |f'(H)|/f^3(H) + |f'(H)|^2/f^5(H) + |f''(H)|/f^4(H)\}$ is nonincreasing for small values of x, and dominates $|H'''|$ (see (3.3)). Let

$$\eta = |H'''\{\textstyle\sum_{i=1}^r (n-i+1)^{-1} + \phi \sum_{i=1}^r (Z_i - 1)/(n-i+1)\}|$$

and $\zeta = \sum_{i=1}^r (Z_i - 1)/(n-i+1)$, and suppose Δ is small. If $\zeta \geq 0$ then with probability tending to one as $n \to \infty$, η is dominated by $\xi(x)$ with $x = \sum_{i=1}^r (n-i+1)^{-1}$, for all $1 \leq r \leq n\Delta$. Moreover,

$$\xi(x) \leq C[1 - F\{H(x)\}]^{-\gamma} = C[1 - \exp\{-\textstyle\sum_{i=1}^r (n-i+1)^{-1}\}]^{-\gamma} = O\{(n/r)^\gamma\}.$$

A similar argument shows that if $\zeta < 0$,

$$\eta \leq C[1 - F\{H(\textstyle\sum_{i=1}^r Z_i/(n-i+1))\}]^{-\gamma} = CU_{nr}^{-\gamma} = O_p\{(n/r)^\gamma\},$$

by (3.13). Therefore $\eta = O_p\{(n/r)^\gamma\}$. It is easily proved that $|H'\{\sum_{i=1}^r (n-i+1)^{-1}\}|^{-1} = O\{(r/n)^\alpha\}$ uniformly in $1 \leq r \leq n\Delta$, and substituting these two estimates and (3.10) into (3.15) we obtain (3.14). The next result is proved almost identically.

LEMMA 4. *Under conditions* (A_+) *and* (B_+) *there exists* $\Delta > 0$ *such that*

$$\{\textstyle\sum_{i=1}^r (Z_i - 1)/(n-i+1)\}H''\{\sum_{i=1}^r (n-i+1)^{-1}\}/H'\{\sum_{i=1}^r (n-i+1)^{-1}\}$$
$$= O_p\{n^{-1/2}(n/r)^{\beta_+ - \alpha_+}\}$$

uniformly in $1 \leq r \leq n\Delta$, *as* $n \to \infty$.

LEMMA 5. *Suppose* $0 < \varepsilon < 1$ *and* $(\gamma_+ - \alpha_+)(1-\varepsilon) < 1$. *Under conditions* (A_+), (B_+) *and* (C_+) *there exists* $\Delta > 0$ *such that*

$$\{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-p} = n^p(\textstyle\sum_{i=r+1}^{r+k} Z_i)^{-p} f^p\{F^{-1}(1 - r/n)\}$$
$$\times (1 - p\{\textstyle\sum_{i=1}^r (Z_i - 1)/(n-i+1)\}H''[\log\{n/(n-r)\}]/H'[\log\{n/(n-r)\}]$$
$$+ O_p\{n^{-1}(n/r)^{\gamma_+ - \alpha_+}\log n\})$$

uniformly in $n^\varepsilon \leq r \leq n\Delta$, *as* $n \to \infty$.

PROOF. Under the condition $(\gamma - \alpha)(1 - \varepsilon) < 1$ the remainder in the expansion (3.14) equals $o_p(1)$ uniformly in $n^\varepsilon \leq r \leq n\Delta$, and since $2(\beta - \alpha) \leq \gamma - \alpha$ then the term preceding the remainder also equals $o_p(1)$ (see Lemma 4). Lemmas 2 and 3 may now be combined to give

$$\{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-p}$$
$$= \{\textstyle\sum_{i=1}^{r+k} Z_i/(n-i+1)\}^{-p}\exp\{p\sum_{i=1}^r (n-i+1)^{-1}\}f^p[H\{\sum_{i=1}^r (n-i+1)^{-1}\}]$$
$$\times [1 - p\{\textstyle\sum_{i=1}^r (Z_i - 1)/(n-i+1)\}H''\{\sum_{i=1}^r (n-i+1)^{-1}\}/H'\{\sum_{i=1}^r (n-i+1)^{-1}\}$$
$$+ O_p\{n^{-1}(n/r)^{2(\beta-\alpha)}\log n + n^{-1}(n/r)^{\gamma-\alpha}\log n\}].$$

The proof of Lemma 5 may be completed using some routine computations based on the estimates in (3.10).

LEMMA 6. *Under condition* (B_+) *there exists* $\Delta > 0$ *such that*

$$\rho X_{n,n-r+1} + (1-\rho)X_{n,n-(r+k)+1}$$
$$= F^{-1}(1 - r/n) + \{\textstyle\sum_{i=1}^r (Z_i - 1)/(n-i+1)\}H'[\log\{n/(n-r)\}] + O_p\{n^{-1}(n/r)^{\beta_+}\log n\}$$

uniformly in $1 \leq r \leq n\Delta$, *as* $n \to \infty$.

PROOF. Since f is ultimately nonincreasing then with probability tending to one as $n \to \infty$, and for some $0 < \phi < 1$,

$$X_{n,n-r+1} - X_{n,n-(r+k)+1} \le \{\textstyle\sum_{i=r+1}^{r+k} Z_i/(n-i+1)\}/f[H\{\sum_{i=1}^{r} Z_i/(n-i+1) + \phi \sum_{i=r+1}^{r+k} Z_i/(n-i+1)\}]$$
$$= O_p(n^{-1}\log n)/f(X_{n,n-r+1}) = O_p\{n^{-1}(n/r)^{\beta}\log n\},$$

using (3.13). Therefore

$$\rho X_{n,n-r+1} + (1-\rho)X_{n,n-(r+k)+1} + O_p\{n^{-1}(n/r)^{\beta}\log n\} = X_{n,n-r+1}$$
$$= H\{\textstyle\sum_{i=1}^{r}(n-i+1)^{-1}\} + \{\sum_{i=1}^{r}(Z_i-1)/(n-i+1)\}H'\{\sum_{i=1}^{r}(n-i+1)^{-1}\}$$
$$+ \tfrac{1}{2}\{\textstyle\sum_{i=1}^{r}(Z_i-1)/(n-i+1)\}^2 H''\{\sum_{i=1}^{r}(n-i+1)^{-1} + \psi\sum_{i=1}^{r}(Z_i-1)/(n-i+1)\}$$

where $0 < \psi < 1$. The last term in this expansion may be shown to equal $O_p\{n^{-1}(n/r)^{\beta}\}$ using an argument like that in the proof of Lemma 4, and simplified expressions for the first two terms may be derived using (3.10). This ultimately leads to Lemma 6.

We are now in a position to prove Theorem 3. We shall only derive the limit theorem for $\hat{b}_n$, since the other results may be obtained in a like manner. Given $0 < \varepsilon, \Delta < 1$, we break the series expression for $\hat{b}_n$ into four parts:

$$n^{p+1}\hat{b}_n = \{\textstyle\sum_{r=1}^{n^{\varepsilon}} + \sum_{r=n^{\varepsilon}+1}^{n\Delta} + \sum_{r=n\Delta+1}^{n(1-\Delta)} + \sum_{r=n(1-\Delta)+1}^{n}\}\{\rho X_{n,n-r+1}$$
$$+ (1-\rho)X_{n,n-(r+k)+1}\} \times \{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-p}$$
$$= S_1 + S_2 + S_3 + S_4,$$

say. (Here $\sum_{n\Delta+1}^{n(1-\Delta)}$ denotes $\sum_{[n\Delta]+1}^{[n(1-\Delta)]}$, etc.) If g is a differentiable function satisfying $\int_0^1 |g'(x)|\,dx < \infty$ then $\sum_{r=1}^{n} g(1-r/n) = \int_0^n g(1-x/n)\,dx + O(1)$ as $n \to \infty$, and so

$$n^{p+1}a = n^p\mu \int_0^n F^{-1}(1-x/n)f^p\{F^{-1}(1-x/n)\}\,dx$$
$$= n^p\mu\{\textstyle\sum_{r=1}^{n^{\varepsilon}} + \sum_{r=n^{\varepsilon}+1}^{n\Delta} + \sum_{r=n\Delta+1}^{n(1-\Delta)} + \sum_{r=n(1-\Delta)+1}^{n}\}$$
$$\cdot F^{-1}(1-r/n)f^p\{F^{-1}(1-r/n)\} + O(n^p)$$
$$= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + O(n^p),$$

say. It is readily proved using the techniques we shall use below that the variable $n^{-p-1/2}(S_1 + S_2 - \lambda_1 - \lambda_2)$ is asymptotically normally distributed with zero mean and a variance which converges to zero as $\Delta \to 0$. From considerations of symmetry we see that the same must be true of $n^{-p-1/2}(S_4 - \lambda_4)$. Therefore it will suffice to prove that for all sufficiently small $\Delta > 0$, $n^{-p-1/2}(S_1 + S_2 + S_3 - \lambda_1 - \lambda_2 - \lambda_3)$ is asymptotically normally distributed with zero mean, and its variance converges to σ_b^2 as $\Delta \to 0$. We shall consider the series S_1, S_2 and S_3 individually.

(i) Series S_1: If $\delta < \alpha p + 1$ then in view of Lemma 1, $S_1 = O_p\{n^{p+1-(1-\varepsilon)(\alpha p+1-\delta)}\}$. Moreover, $\lambda_1 \le Cn^p \sum_1^{n^{\varepsilon}} [1 - F\{F^{-1}(1-r/n)\}]^{-\delta} \times [1 - F\{F^{-1}(1-r/n)\}]^{\alpha p} = Cn^p \sum_1^{n^{\varepsilon}} (r/n)^{\alpha p-\delta}$, and combining these estimates we deduce that

(3.16) $$n^{-p}(S_1 - \lambda_1) = o_p(n^{1/2}) \quad \text{if} \quad (\alpha p + 1 - \delta)(1-\varepsilon) > \tfrac{1}{2}.$$

(ii) Series S_2: If $(\gamma - \alpha)(1-\varepsilon) < 1$ then in view of Lemmas 5 and 6, and the expressions (3.1) and (3.2),

$$\{\rho X_{n,n-r+1} + (1-\rho)X_{n,n-(r+k)+1}\}\{X_{n,n-r+1} - X_{n,n-(r+k)+1}\}^{-p}$$
$$= n^p Y_r f^p\{F^{-1}(1-r/n)\}(F^{-1}(1-r/n)$$
$$+ S_{nr}[pF^{-1}(1-r/n)\{1 - f'(F^{-1}(1-r/n))(1-r/n)/f^2(F^{-1}(1-r/n))\}$$
$$- (1-r/n)/f\{F^{-1}(1-r/n)\}]) + R_{3nr},$$

where $Y_r = (\sum_{i=r+1}^{r+k} Z_i)^{-p}$, $S_{nr} = \sum_{i=1}^{r} (Z_i - 1)/(n - i + 1)$ and

$$R_{3nr} = O_p(1)n^p Y_r f^p\{F^{-1}(1 - r/n)\}\{n^{-1}(n/r)^\beta \log n + n^{-1}(n/r)^{\gamma-\alpha+\delta}\log n + S_{nr}^2(n/r)^{2\beta-\alpha}\}$$

$$= O_p(1)n^{p-1}Y_r f^p\{F^{-1}(1 - r/n)\}(n/r)^{2\beta-\alpha}\log n.$$

(Note that (B) implies $2\beta - \alpha \geq \beta$, while (D) entails $2\beta - \alpha \geq \gamma - \alpha + \delta$.) Therefore

$$S_2 = n^p \sum_{r=n^\varepsilon+1}^{n\Delta} Y_r f^p\{F^{-1}(1 - r/n)\}(F^{-1}(1 - r/n)$$

$$+ S_{nr}[pF^{-1}(1 - r/n)\{1 - f'(F^{-1}(1 - r/n))(1 - r/n)/f^2(F^{-1}(1 - r/n))\} \tag{3.17}$$

$$- (1 - r/n)/f\{F^{-1}(1 - r/n)\}]) + R_n,$$

where $R_n = \sum_{r=n^\varepsilon+1}^{n\Delta} R_{3nr} = O_p(1)n^{p-1}(\log n) \sum_{r=n^\varepsilon+1}^{n\Delta} (r/n)^{\alpha p+\alpha-2\beta}Y_r$. Using Markov's inequality the series in this expression is easily shown to equal $O_p(n \log n)$ if $\alpha p + \alpha - 2\beta \geq -1$, while the series equals

$$O_p(1)n^{2\beta-\alpha-\alpha p} \sum_{r=n^\varepsilon+1}^{\infty} r^{\alpha p+\alpha-2\beta} = O_p\{n^{1+(1-\varepsilon)(2\beta-\alpha-\alpha p-1)}\}$$

if $\alpha p + \alpha - 2\beta < -1$. Consequently

$$R_n = o_p(n^{p+1/2}) \text{ if } (2\beta-\alpha-\alpha p-1)(1 - \varepsilon) < ½. \tag{3.18}$$

Combining (3.16) and (3.18) we see that ε must satisfy the inequalities $(\alpha p + 1 - \delta)(1 - \varepsilon) > ½$, $(2\beta - \alpha - \alpha p - 1)(1 - \varepsilon) < ½$ (and also $(\gamma - \alpha)(1 - \varepsilon) < 1$). These can all be satisfied if and only if $2(\alpha p + 1 - \delta) > 1$ and $\max\{2(2\beta - \alpha - \alpha p - 1), \gamma - \alpha\} < 2(\alpha p + 1 - \delta)$. When $2\beta - \gamma \geq \delta$ (see (D)) this last condition is equivalent to $\gamma - \alpha < 2(\alpha p + 1 - \delta)$, and so they are jointly equivalent to

$$\max(\gamma - \alpha, 1) < 2(\alpha p + 1 - \delta). \tag{3.19}$$

Note that this implies that $\delta < \alpha p + 1$, which was assumed in (i) above.

(iii) Series S_3: The density f is bounded away from zero on the interval $[F^{-1}(\Delta), F^{-1}(1 - \Delta)]$ whenever $0 < \Delta < ½$, and so a simpler version of the argument in the earlier lemmas may be used to prove an analogue of (3.17) in which S_2 is replaced by S_3, $\sum_{n^\varepsilon+1}^{n\Delta}$ by $\sum_{n\Delta+1}^{n(1-\Delta)}$ and R_n by $o_p(n^{1/2})$.

Combining this result with (3.16) and (3.18) we deduce that whenever (3.19) holds,

$$\begin{aligned} n^{-p} \sum_1^3(S_i - \lambda_i) &= \sum_{r=1}^{n(1-\Delta)} (Y_r - \mu)F^{-1}(1 - r/n)f^p\{F^{-1}(1 - r/n)\} \\ &+ \mu\sum_{r=1}^{n(1-\Delta)} S_{nr}f^p\{F^{-1}(1 - r/n)\}[pF^{-1}(1 - r/n) \\ &\cdot\{1 - f'(F^{-1}(1 - r/n))(1 - r/n)/f^2(F^{-1}(1 - r/n))\} \\ &- (1 - r/n)/f\{F^{-1}(1 - r/n)\}] \\ &+ \sum_{r=1}^{n(1-\Delta)} (Y_r - \mu)S_{nr}f^p\{F^{-1}(1 - r/n)\} \\ &\cdot[pF^{-1}(1 - r/n)\{1 - f'(F^{-1}(1 - r/n)) \\ &\cdot(1 - r/n)/f^2(F^{-1}(1 - r/n))\} - (1 - r/n)/f\{F^{-1}(1 - r/n)\}] + o_p(n^{1/2}). \end{aligned} \tag{3.20}$$

Any two summands in the last series distance k or more apart may be shown to be uncorrelated, and so the series can be written as a sum of k series each containing only uncorrelated terms. The variance (or in this case, mean square error) of the last series is therefore dominated by a constant multiple of the sum of the variances, which in turn is dominated by

$$C_1 n^{-1} \sum_{r=1}^{n(1-\Delta)} (r/n) f^{2p}\{F^{-1}(1-r/n)\}[\{F^{-1}(1-r/n)\}^2\{1 + |f'(F^{-1}(1-r/n))|^2/ f^4(F^{-1}(1-r/n))\} + 1/f^2\{F^{-1}(1-r/n)\}]$$
$$\le C_2 \int_{F^{-1}(\Delta)}^{F^{-1}(1-1/n)} \{1-F(x)\}\{x^2 f^{2p+1}(x) + x^2|f'(x)|^2 f^{2p-3}(x) + f^{2p-1}(x)\}\, dx.$$

The infinite integrals of the first and last terms converge, while for $y > -\infty$,

$$\int_y^{F^{-1}(1-1/n)} x^2|f'(x)|^2 f^{2p-3}(x)\, dx \le C \int_y^{F^{-1}(1-1/n)} \{1-F(x)\}^{\alpha(2p+1)-\gamma-2\delta} f(x)\, dx$$
$$= C\int_y^{F^{-1}(1-1/n)} \{1-F(x)\}^{-2+\eta} f(x)\, dx,$$

where $\eta > 0$ under (3.19). Combining these estimates we deduce that the third series in (3.20) equals $o_p(n^{1/2})$. The sum of the first two series may be written as $S = \sum_{r=1}^{n(1-\Delta)} \{(Y_r - \mu) c_{nr} + \mu(Z_r - 1)(n-r+1)^{-1} \sum_{i=r}^{n(1-\Delta)} d_{ni}\}$, where

$$c_{nr} = F^{-1}(1-r/n) f^p\{F^{-1}(1-r/n)\}$$

and

$$d_{nr} = f^p\{F^{-1}(1-r/n)\}[pF^{-1}(1-r/n)\{1 - f'(F^{-1}(1-r/n))(1-r/n)/f^2(F^{-1}(1-r/n))\} - (1-r/n)/f\{F^{-1}(1-r/n)\}].$$

The series S is a sum of k-dependent random variables, and the classical central limit theorem for bounded, m-dependent variables (see Chung (1974, page 214)) may be used to prove that S is asymptotically normally distributed. The argument involves two truncations, the first of the series (summing from $r = n\eta$ to $n(1-\Delta)$, where $\eta > 0$ is arbitrarily small) and the second of the random variables themselves. The asymptotic variance is that of S:

$$\operatorname{var}(S) = \Big(\sum_{r=1}^{n(1-\Delta)} c_{nr}^2\Big) \sum_{i=1}^{2k-1} \operatorname{cov}(Y_i, Y_k) + 2k\mu\Big\{\sum_{r=1}^{n(1-\Delta)} c_{nr}(n-r+1)^{-1}\Big(\sum_{i=r}^{n(1-\Delta)} d_{ni}\Big)\Big\}\operatorname{cov}(Y_1, Z_1) + \mu^2 \sum_{r=1}^{n(1-\Delta)} (n-r+1)^{-2}\Big(\sum_{i=r}^{n(1-\Delta)} d_{ni}\Big)^2 + o(n).$$

Routine calculations yield

$$\sum_{r=1}^{n(1-\Delta)} c_{nr}^2 \sim \int_0^{n(1-\Delta)} \{F^{-1}(1-x/n)\}^2 f^{2p}\{F^{-1}(1-x/n)\}\, dx = n\int_{F^{-1}(\Delta)}^{\infty} x^2 f^{2p+1}(x)\, dx$$

and

$$\sum_{i=r}^{n(1-\Delta)} d_{ni} \sim n\Big[p\int_{F^{-1}(\Delta)}^{F^{-1}(1-r/n)} x f^{p+1}(x)\, dx - \int_{F^{-1}(\Delta)}^{F^{-1}(1-r/n)} \{pxf'(x)f^{p-1}(x) + F(x) f^p(x)\}\, dx\Big]$$
$$= n\Big[(p+1)\int_{F^{-1}(\Delta)}^{F^{-1}(1-r/n)} x f^{p+1}(x)\, dx - F^{-1}(1-r/n) f^p\{F^{-1}(1-r/n)\}(1-r/n) + F^{-1}(\Delta) f^p\{F^{-1}(\Delta)\}\Delta\Big].$$

From these estimates we may obtain a formula for the limit of $n^{-1}\operatorname{var}(S)$, which leads in turn to the result

$$\lim_{\Delta\to 0}\{\lim_{n\to\infty} n^{-1}\text{var}(S)\} = \tau^2 \quad (\text{say}) \quad = \left\{\int_{-\infty}^{\infty} x^2 f^{2p+1}(x)\,dx\right\} \textstyle\sum_{i=1}^{2k-1}\text{cov}(Y_i,\, Y_k)$$

$$+ 2k\mu\left[(p+1)\int_{-\infty}^{\infty} xf^{p+1}(x)\{F(x)\}^{-1}\,dx \int_{-\infty}^{x} yf^{p+1}(y)\,dy - \int_{-\infty}^{\infty} x^2 f^{2p+1}(x)\,dx\right]\text{cov}(Y_1,\, Z_1)$$

$$+ \mu^2 \int_{-\infty}^{\infty}\left[(p+1)\{F(x)\}^{-1}\int_{-\infty}^{x} yf^{p+1}(y)\,dy - xf^{p}(x)\right]^2 f(x)\,dx.$$

It may be proved that $\text{cov}(Y_1, Z_1) = -p\mu/k$, and then it is a simple matter to show that $\tau^2 = \sigma_b^2$, completing the proof of Theorem 3.

Acknowledgment. The referee's comments have enabled me to clarify my notation and improve my presentation.

REFERENCES

ANDRIANO, K., GENTLE, J. E. and SPOSITO, V. A. (1977a). On the asymptotic bias of Grenander's mode estimator. *Comm. Statist.-Theor. Meth.* **A6** 773–776.

ANDRIANO, K., GENTLE, J. E. and SPOSITO, V. A. (1977b). Direct estimation of the mode. *Proc. Statist. Comput. Section of Amer. Statist. Assoc.* 118–122.

ANDRIANO, K., GENTLE, J. E. and SPOSITO, V. A. (1978). Comparison of some estimators of the mode. *Proc. Social Statist. Section of Amer. Statist. Assoc.* 760–764.

CHUNG, K. L. (1974). *A Course in Probability Theory.* Academic, New York.

DALENIUS, T. (1965). The mode—a neglected statistical parameter. *J. Royal Statist. Soc.* **A128** 110–117.

DANIELS, H. E. (1945). The statistical theory of the strength of bundles of threads. *Proc. Royal Soc.* **A183** 405–435.

DAVID, H. A. (1970). *Order Statistics.* Wiley, New York.

EKBLOM, H. (1972). A Monte Carlo investigation of mode estimators in small samples. *Applied Statistics* **21** 177–184.

GRENANDER, U. (1965). Some direct estimates of the mode. *Ann. Math. Statist.* **36** 131–138.

HALL, P. (1981). Asymptotic theory of Grenander's mode estimator. *Z. Wahrsch. verw. Gebiete.* To appear.

HOLST, L. (1979). Asymptotic normality of sum-functions of spacings. *Ann. Probability* **7** 1066–1072.

ROBBINS, H. (1954). A one-sided confidence interval for an unknown distribution function. (Abstract) *Ann. Math. Statist.* **25** 409.

ROSENTHAL, H. P. (1970). On the subspaces of $L^p(p > 2)$ spanned by sequences of independent random variables. *Israel J. Math.* **8** 273–303.

WELLNER, J. A. (1978). Limit theorems for the ratio of the empirical distribution function to the true distribution function. *Z. Wahrsch. verw. Gebiete.* **45** 73–88.

DEPARTMENT OF STATISTICS
THE FACULTIES
THE AUSTRALIAN NATIONAL UNIVERSITY
P.O. BOX 4
CANBERRA ACT 2600
AUSTRALIA