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MOMENTS AND ERROR RATES OF TWO-SIDED STOPPING RULES[1]

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For $X_1, X_2, \cdots$ i.i.d., $EX_1 = \mu \neq 0$, $S_n = X_1 + \cdots + X_n$, the asymptotic behavior of moments and error rates of the two-sided stopping rules

$$\inf\{n \geq 1: |S_n| > cn^\alpha\}, \; c > 0, \; 0 \leq \alpha < 1,$$

is considered. Convergence of (normalized) moments of all orders as $c \to \infty$ is obtained, without the higher moment assumptions needed in the one-sided case of extended renewal theory (Gut, 1974), and in a more general setting than just the i.i.d. case. Necessary and sufficient conditions are given for convergence of series involving the error rates, in terms of the moments of X_1.

1. Introduction. Let $X, X_1, X_2, \cdots$ be i.i.d. with mean μ, and let $S_n = X_1 + \cdots + X_n$. If $\mu > 0$, the stopping time N_c of extended renewal theory is defined by

$$N_c = \inf\{n \geq 1: S_n > cn^\alpha\}, \; c > 0, \; 0 \leq \alpha < 1. \tag{1.1}$$

Since $N_c \to \infty$ a.s. as $c \to \infty$, by the Strong Law of Large Numbers

$$(c/\mu)^{-1/(1-\alpha)} N_c \to 1 \text{ a.s.} \quad \text{as} \quad c \to \infty. \tag{1.2}$$

As for the moment convergence in (1.2), Gut (1974) has shown that for $p > 1$,

$$\begin{aligned} E(X^-)^p < \infty &\Leftrightarrow EN_c^p < \infty, \quad \text{all} \quad c > 0 \\ &\Leftrightarrow EN_c^p \sim (c/\mu)^{p/(1-\alpha)} \quad \text{as} \quad c \to \infty. \end{aligned} \tag{1.3}$$

Assume now that $\mu \neq 0$, and define the two-sided stopping rule $\bar{N}_c$ by

$$\bar{N}_c = \inf\{n \geq 1: |S_n| > cn^\alpha\}, \; c > 0, \quad 0 \leq \alpha < 1. \tag{1.4}$$

It is clear that

$$(c/|\mu|)^{-1/(1-\alpha)} \bar{N}_c \to 1 \text{ a.s.} \quad \text{as} \quad c \to \infty. \tag{1.5}$$

Because $\bar{N}_c \leq N_c$, from (1.3) and (1.5) it follows that if $\mu > 0$ and $E(X^-)^p < \infty$, $p > 1$, then

$$\{(c^{-1/(1-\alpha)} \bar{N}_c)^p: c \geq 1\} \text{ is uniformly integrable}$$

and

$$E\bar{N}_c^p \sim (c/|\mu|)^{p/(1-\alpha)} \quad \text{as} \quad c \to \infty.$$

It is natural to ask whether the sufficient condition $E(X^-)^p < \infty$ is also necessary, as it is in the one-sided case. Theorem 1 of Section 2 and its corollaries assert that this condition is not necessary, that is,

$$\{(c^{-1/(1-\alpha)} \bar{N}_c)^p: c \geq 1\} \text{ is uniformly integrable for all } p > 0,$$

provided only that $\mu \neq 0$. In fact, this result is shown to be true not only for i.i.d. sequences,

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but for more general sequences of independent random variables under conditions to be discussed below.

Another issue of importance is the asymptotic behavior of $P(S_{\bar{N}_c} < 0)$ as $c \to \infty$ for $\mu > 0$ (and of $P(S_{\bar{N}_c} > 0)$, $c \to \infty$, for $\mu < 0$), which corresponds to the error probability in the case of a sequential test. This asymptotic behavior when $|X|$ has a finite moment-generating function has been investigated by Berk (1978), for very general stopping boundaries. In Section 3, necessary and sufficient conditions for the convergence of series involving $P(S_{\bar{N}_j} < 0)$, $j = 1, 2, \cdots$ are obtained in terms of the moments of X^-. These results are similar in nature to the random walk results of Hsu and Robbins (1947), Baum and Katz (1965), and Chow and Lai (1975).

2. Asymptotic behavior of moments of $\bar{N}$. The requirement imposed on the sequence of independent random variables in Theorem 1 involves a corresponding sequence of centering constants a_n, and is twofold. First, the delayed averages of the a_n should converge uniformly to $\mu \neq 0$ and finite (Condition (2.1)). Second, the centered random variables $X_n - a_n$ must obey the Weak Law of Large Numbers uniformly in their delayed sums (Condition (2.2)). When these conditions are satisfied, Theorem 1 gives the uniform integrability of $(c^{-1/(1-\alpha)}\bar{N}_c)^p$, all $p > 0$, $0 \le \alpha < 1$.

THEOREM 1. *Let $X_1, X_2, \cdots$ be independent random variables, and assume there exists a sequence of real numbers a_n such that if $A_{k,n} = \sum_1^n a_{k+j}$, then*

$$n^{-1}A_{k,n} \to \mu \neq 0 \text{ (finite) as } n \to \infty, \text{ uniformly in } k, \tag{2.1}$$

and for all $\varepsilon > 0$,

$$P(|S_{k,n} - A_{k,n}| \ge \varepsilon n) \to 0 \text{ as } n \to \infty, \text{ uniformly in } k, \tag{2.2}$$

where $S_{k,n} = \sum_1^n X_{k+j}$. Define

$$\bar{N}_c = \inf\{n \ge 1 : |S_n| > cn^\alpha\}, \quad c > 0, 0 \le \alpha < 1. \tag{2.3}$$

Then

$$\{(c^{-1/(1-\alpha)}\bar{N}_c)^p : c \ge 1\} \text{ is uniformly integrable, all } p > 0. \tag{2.4}$$

PROOF. Without loss of generality, $\mu = 1$. Choose δ so that $0 \le \alpha < \delta < 1$, and assume $c \ge 1$, $K \ge \max(K_0^{1/\delta}, 4^{1/(\delta-\alpha)})$, where $K_0 \ge 1$ is chosen so that

$$n \ge K_0 \Rightarrow P(|S_{j,n}| \le (½)n) < ½ \text{ for all } j \tag{2.5}$$

(this can be done by (2.1) and (2.2)). It follows from (2.5) and independence that

$$n \ge K_0 \Rightarrow P(\max_{j \le m} |S_{(j-1)n,n}| \le (½)n) \le (½)^m,$$

for all m. In order to simplify notation, assume that $Kc^{1/(1-\alpha)}$ and $K^\delta c^{1/(1-\alpha)}$ are integers. Then by the triangle inequality,

$$\begin{aligned} P(\bar{N}_c > Kc^{1/(1-\alpha)}) &= P(\max_{j \le Kc^{1/(1-\alpha)}} j^{-\alpha}|S_j| \le c) \\ &\le P(\max_{j \le Kc^{1/(1-\alpha)}} |S_j| \le K^\alpha c^{\alpha/(1-\alpha)}c) \\ &= P(\max_{j \le Kc^{1/(1-\alpha)}} |S_j| \le K^\alpha c^{1/(1-\alpha)}) \\ &\le P(\max_{j \le [K^{1-\delta}]} |S_{(j-1)K^\delta c^{1/(1-\alpha)}, K^\delta c^{1/(1-\alpha)}}| \le 2K^\alpha c^{1/(1-\alpha)}) \\ &\le P(\max_{(j \le [K^{1-\delta}]} | S_{(j-1)K^\delta c^{1/(1-\alpha)}, K^\delta c^{1/(1-\alpha)}} \le (½)K^\delta c^{1/(1-\alpha)}) \\ &< (½)^{[K^{1-\delta}]} \end{aligned} \tag{2.6}$$

Hence for $L \geq \max(K_0^{1/\delta}, 4^{1/(\delta-\alpha)})$,

$$\int_L^\infty x^{p-1} P(\bar{N}_c > xc^{1/(1-\alpha)})dx \leq \int_L^\infty x^{p-1} (½)^{[x^{1-\delta}]} dx \to 0 \tag{2.7}$$

as $L \to \infty$, uniformly in c. That is,

$$\{(c^{-1/(1-\alpha)}\bar{N}_c)^p: c \geq 1\} \text{ is uniformly integrable for all } p > 0.$$

In the following two corollaries, results are given only for the case $\mu > 0$; the analogous results for $\mu < 0$ can be obtained easily from these by replacing X with $-X$, S_n with $-S_n$.

COROLLARY 1. *Let* $X, X_1, X_2, \cdots$ *be* i.i.d., $EX = \mu > 0$. *Define*

$$\bar{N} = \bar{N}_{c_1,c_2} = \inf\{n \geq 1: S_n \notin [-c_1 n^\alpha, c_2 n^\alpha]\}, \; c_1, c_2 > 0, \; 0 \leq \alpha < 1. \tag{2.8}$$

If $c_1 = O(c_2)$ *as* $c = \min(c_1, c_2) \to \infty$, *then*

$$\{(c_2^{-1/(1-\alpha)}\bar{N})^p: c \geq 1\} \textit{ is uniformly integrable} \tag{2.9}$$

and

$$E\bar{N}^p \sim (c_2/\mu)^{p/(1-\alpha)} \textit{ as } c \to \infty, \textit{ all } p > 0. \tag{2.10}$$

PROOF. Consider first the case $c_1 = c_2 = c$. If $\mu \in (0, \infty)$, (2.9) is immediate from Theorem 1, putting $a_n = \mu$ for all n. When $\mu = \infty$, $n^{-1}S_n \to \infty$ a.s., and therefore

$$\sup_j P(|S_{j,n}| \leq (½)n) < ½ \tag{2.11}$$

if n is sufficiently large. By (2.11) and the proof of Theorem 1,

$$\{(c^{-1/(1-\alpha)}\bar{N})^p: c \geq 1\} \text{ is uniformly integrable} \tag{2.12}$$

for all $p > 0$, proving (2.9) when $\mu = \infty$. In the general case, assuming that $c_1 = O(c_2)$ as $c = \min(c_1, c_2) \to \infty$, define

$$\bar{N}_{c_1+c_2} = \inf\{n \geq 1: |S_n| > (c_1 + c_2)n^\alpha\}. \tag{2.13}$$

From the proof above,

$$\{[(c_1 + c_2)^{-1/(1-\alpha)}\bar{N}_{c_1+c_2}]^p: c \geq 1\} \text{ is uniformly integrable for all } p > 0. \tag{2.14}$$

Because $\bar{N} \leq \bar{N}_{c_1+c_2}$ and $c_1 + c_2 = O(c_2)$ as $c \to \infty$, (2.9) follows from (2.14). (2.10) now follows from (2.9), since $c_2^{-1/(1-\alpha)}\bar{N} \to \mu^{-1/(1-\alpha)}$ a.s. as $c \to \infty$.

COROLLARY 2. *Let* $X_1, X_2, \cdots$ *be independent random variables with* $EX_n = a_n$, *and assume* (2.1) *holds for some* $\mu \in (0, \infty)$, *and that*

$$\sup_n E|X_n - EX_n|^r < \infty \textit{ for some } r > 1. \tag{2.15}$$

Define $\bar{N}$ *by* (2.8). *If* $c_1 = O(c_2)$ *as* $c = \min(c_1, c_2) \to \infty$, *then*

$$\{(c_2^{-1/(1-\alpha)}\bar{N})^p: c \geq 1\} \textit{ is uniformly integrable,} \tag{2.16}$$

and

$$E\bar{N}^p \sim (c_2/\mu)^{p/(1-\alpha)} \textit{ as } c \to \infty, \textit{ all } p > 0. \tag{2.17}$$

PROOF. We may take $r \leq 2$. By the Tchebychev and Marcinkiewicz-Zygmund inequalities (see Chow and Teicher, 1978, page 356), for $n, k = 1, 2, \cdots$, $A_{k,n} = \sum_1^n a_{k+j}$, and some $B_r \in (0, \infty)$ (depending only on r),

$$
\begin{aligned}
P(|S_{k,n} - A_{k,n}| \geq \varepsilon n) &\leq \varepsilon^{-r} n^{-r} E|S_{k,n} - A_{k,n}|^r \\
&\leq B_r \varepsilon^{-r} n^{-r} E(\textstyle\sum_{k+1}^{k+n} (X_j - a_j)^2)^{r/2} \\
&\leq B_r \varepsilon^{-r} n^{-r} E(\textstyle\sum_{k+1}^{k+n} |X_j - a_j|^r) \qquad (2.18)\\
&\leq B_r \varepsilon^{-r} n^{-r}(nM), \text{ where } M = \sup_n E|X_n - a_n|^r, \\
&= B_r M \varepsilon^{-r} n^{1-r} \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

uniformly in k.

Thus (2.2) holds, so by Theorem 1

$$\{[(c_1 + c_2)^{-1/(1-\alpha)} \bar{N}_{c_1+c_2}]^p : c \geq 1\}$$

is uniformly integrable, all $p > 0$, and (2.16) follows as in the proof of Corollary 1. By a theorem of Loève (see Chow and Teicher, 1978, page 121),

$$n^{-1} S_n \to \mu \quad \text{a.s. as} \quad n \to \infty,$$

so

$$c_2^{-1/(1-\alpha)} \bar{N} \to \mu^{-1/(1-\alpha)} \quad \text{a.s.,} \quad c \to \infty, \tag{2.19}$$

and thus (2.17) follows immediately from (2.16) and (2.19), finishing the proof.

3. Asymptotic behavior of the error rates associated with $\bar{N}_c$. For $X, X_1, X_2, \cdots$ i.i.d., $EX = \mu$, $S_n = X_1 + \cdots X_n$, Berk (1978) has proved (as a special case of a much more general theorem) that if $E(\exp(t|X|)) < \infty$ for some $t > 0$,

$$\log[P(S_{\bar{N}_c} < 0)] \sim -\lambda c^{1/(1-\alpha)} \quad \text{as} \quad c \to \infty$$

for $\mu > 0$, and similarly

$$\log[P(S_{\bar{N}_c} > 0)] \sim -\lambda c^{1/(1-\alpha)} \quad \text{as} \quad c \to \infty$$

for $\mu < 0$, where λ is a positive constant which depends on the moment-generating function of X and on α. It is natural to ask about the asymptotic behavior of these probabilities when X does not necessarily have finite moment-generating function, but does have finite pth moment for some $p > 1$. Theorem 2 gives necessary and sufficient moment conditions on X^- for convergence of series involving the probabilities $P(S_{\bar{N}_c} < 0)$, $\mu > 0$ (the analogous results about $P(S_{\bar{N}_c} > 0)$, $\mu < 0$, follow immediately upon replacing X by $-X$ throughout). Using the random walk results of Chow and Lai (1975), upper bounds for such series in terms of the moments of X are also obtained.

THEOREM 2. *Let $X, X_1, X_2, \cdots$ be* i.i.d., *$EX = \mu > 0$, and define*

$$\bar{N}_c = \inf\{n \geq 1 : |S_n| > cn^\alpha\},\ c > 0,\ 0 \leq \alpha < 1.$$

Assume $p > 1$.

(i) *If $\alpha = 0$, then for every $\gamma \in (1, 2]$,*

$$\textstyle\sum_1^\infty j^{p-2} P(\inf_{k \geq j} S_{\bar{N}_K} < 0) \leq A\mu^{p-1}\{[\mu^{-1}E(X-\mu)^-]^p + (\mu^{-\gamma}E|X-\mu|^\gamma)^{(p-1)/(\gamma-1)}\}, \tag{3.1}$$

where $A = A_{p,\gamma} \in (0, \infty)$ depends only on p and γ. Furthermore,

$$E(X^-)^p < \infty \Leftrightarrow \textstyle\sum_1^\infty j^{p-2} P(\inf_{k \geq j} S_{\bar{N}_k} < 0) < \infty \Leftrightarrow \sum_1^\infty j^{p-2} P(S_{\bar{N}_j} < 0) < \infty. \tag{3.2}$$

(ii) *If $\alpha > 0$, then* (3.1) *holds for every $\gamma \in (1, 2]$. Furthermore,*

$$E(X^-)^p < \infty \Rightarrow \textstyle\sum_1^\infty j^{p-2} P(\inf_{k \geq j} S_{\bar{N}_k} < 0) < \infty, \tag{3.3}$$

and

$$\sum_1^\infty j^{p-2} P(S_{\bar{N}_j} < 0) < \infty \Rightarrow E(X^-)^s < \infty, \tag{3.4}$$

where $s = \max(p-1, (p-1)(1-\alpha)+1)$.

PROOF. (i) Assume $E(X^-)^p < \infty$. Then by Kiefer and Wolfowitz (1956),

$$E(\sup_{n\ge 0}(-S_n))^{p-1} < \infty, \tag{3.5}$$

so that

$$\begin{aligned}\sum_1^\infty j^{p-2} P(\inf_{k\ge j} S_{\bar{N}_k} < 0) &\le \sum_1^\infty j^{p-2} P(\sup_{n\ge 0}(-S_n) > j)\\ &\le K_p E(\sup_{n\ge 0}(-S_n))^{p-1} < \infty,\end{aligned} \tag{3.6}$$

where $K_p \in (0, \infty)$ depends only on p. By Theorem 1 and Lemma 2 of Chow and Lai (1975),

$$\begin{aligned}E(\sup_{n\ge 0}(-S_n))^{p-1} &= E(\sup_{n\ge 0}(\mu n - S_n - \mu n))^{p-1}\\ &\le C\mu^{p-1}\{[\mu^{-1}E(X-\mu)^-]^p + (\mu^{-\gamma}E|X-\mu|^\gamma)^{(p-1)/(\gamma-1)}\}\end{aligned} \tag{3.7}$$

for every $\gamma \in (1, 2]$, where $C = C_{p,\gamma}$ depends only on p and γ. Combining (3.6) and (3.7) proves (3.1), with $A = K_p C_{p,\gamma}$. Finally,

$$\begin{aligned}\sum_1^\infty j^{p-2} P(S_{\bar{N}_j} < 0) &\ge \sum_{j=1}^\infty j^{p-2} \sum_{k=1}^\infty P(|X_1| \le j, \cdots, |S_{k-1}| \le j, X_k^- > 2j)\\ &= \sum_{j=1}^\infty j^{p-2} \sum_{k=1}^\infty P(|X_1| \le j, \cdots, |S_{k-1}| \le j) P(X^- > 2j)\\ &= \sum_{j=1}^\infty j^{p-2} P(X^- > 2j) \sum_{k=1}^\infty P(\bar{N}_j \ge k)\\ &= \sum_{j=1}^\infty j^{p-2} P(X^- > 2j) E(\bar{N}_j).\end{aligned} \tag{3.8}$$

Suppose that

$$\sum_1^\infty j^{p-2} P(S_{\bar{N}_j} < 0) < \infty.$$

Since $E\bar{N}_j \sim j/\mu$ as $j \to \infty$, it follows from (3.8) that

$$\sum_1^\infty j^{p-1} P(X^- > 2j) < \infty,$$

and hence $E(X^-)^p < \infty$; combining this result with (3.6) finishes the proof of (3.2).

(ii) The proofs of (3.1) and (3.3) are similar to those in Part (i). To show (3.4), assume

$$\sum_1^\infty j^{p-2} P(S_{\bar{N}_j} < 0) < \infty.$$

We have

$$\begin{aligned}\sum_1^\infty j^{p-2} P(S_{\bar{N}_j} < 0) &\ge \sum_{j=1}^\infty j^{p-2} \sum_{k=1}^\infty P(|X_1| \le j, \cdots, |S_{k-1}| \le j(k-1)^\alpha, X_k^- > 2jk^\alpha)\\ &= \sum_{j=1}^\infty j^{p-2} \sum_{k=1}^\infty P(\bar{N}_j \ge k) P(X^- > 2jk^\alpha)\\ &\ge \sum_{j=1}^\infty j^{p-2} \sum_{k=1}^{(j/\mu)^{1/(1-\alpha)}} P(\bar{N}_j \ge k) P(X^- > 2jk^\alpha)\\ &\ge \sum_{j=1}^\infty j^{p-2} P(X^- > 2j^{1/(1-\alpha)}\mu^{-\alpha/(1-\alpha)}) \sum_{k=1}^{(j/\mu)^{1/(1-\alpha)}} P(\bar{N}_j \ge k).\end{aligned} \tag{3.9}$$

Now

$$j^{-1/(1-\alpha)} \min(\bar{N}_j, (j/\mu)^{1/(1-\alpha)}) \to \mu^{-1/(1-\alpha)} \text{ a.s.} \tag{3.10}$$

as $j \to \infty$, and

$$j^{-1/(1-\alpha)} \min(\bar{N}_j, (j/\mu)^{1/(1-\alpha)}) \le \mu^{-1/(1-\alpha)}, \tag{3.11}$$

hence by dominated convergence

$$E[\min(\bar{N}_j, (j/\mu)^{1/(1-\alpha)})] \sim (j/\mu)^{1/(1-\alpha)} \quad \text{as} \quad j \to \infty. \tag{3.12}$$

Therefore, from (3.9) and (3.12),

$$\sum_1^\infty j^{[(p-2)(1-\alpha)+1]/(1-\alpha)} P(X^- > 2\mu^{-\alpha/(1-\alpha)} j^{1/(1-\alpha)}) < \infty \tag{3.13}$$

and it follows that

$$E(X^-)^{(p-1)(1-\alpha)+1} < \infty. \tag{3.14}$$

Since

$$\sum_1^\infty j^{p-2} P(S_{\bar{N}_j} < 0) \geq \sum_1^\infty j^{p-2} P(X^- > j), \tag{3.15}$$

we have also

$$E(X^-)^{p-1} < \infty, \tag{3.16}$$

which together with (3.14) completes the proof of (3.4).

REMARK. The implications

$$E(X^-)^p < \infty \Rightarrow \sum_1^\infty j^{p-2} P(\inf_{k \geq j} S_{\bar{N}_k} < 0) < \infty,$$

as well as the inequality (3.1), remain true for

$$\bar{N}_c = \inf\{n \geq 1 : S_n \notin [-cn^\alpha, f_c(n)]\},$$

where f_c is any positive function. The implications

$$\sum_1^\infty j^{p-2} P(S_{\bar{N}_j} < 0) < \infty \Rightarrow E(X^-)^p < \infty, \alpha = 0$$

and

$$\sum_1^\infty j^{p-2} P(S_{\bar{N}_j} < 0) < \infty \Rightarrow E(X^-)^{\max(p-1,(p-1)(1-\alpha)+1)} < \infty, \quad \alpha > 0,$$

hold if

$$\bar{N}_c = \inf\{n \geq 1 : S_n \notin [-cn^\alpha, dn^\alpha]\},$$

provided that $d = O(c)$ as $\min(c, d) \to \infty$ (in both cases the proofs given for Theorem 2 apply with minimal change).

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