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A REMARK ON STOCHASTIC FUNDAMENTAL MATRICES

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The determinant of the fundamental matrix of a linear system of stochastic differential equations is computed.

Consider the stochastic system

$$d\Phi(\tau, t) = A(t)\Phi(\tau, t)\, dt + \textstyle\sum_{i=1}^{m} B_i(t)\Phi(\tau, t)\, dW_i(t), \quad t \geq \tau, \tag{1}$$

$$\Phi(\tau, \tau) = I. \tag{2}$$

Here $(W_1, W_2, \cdots, W_m)$ is an m-dimensional Brownian motion, and $A(t)$ and $B_i(t)$ (i=1, $\cdots$, m) are bounded measurable $d \times d$ matrices. It is known (e.g. Arnold [1, Theorem 8.1.5]) that there is a unique solution $\Phi(\tau, t)$ to (1), (2) for $0 \leq \tau \leq t < \infty$, which is continuous in t. However, only when the matrices $A(t)$, $B_i(t)$ are mutually commutative over various times t (i.e. $A(t)B_i(t') = B_i(t')A(t)$, $B_i(t)B_j(t') = B_j(t')B_i(t)$ for all i, j, t, t') can we write

$$\Phi(\tau, t) = \exp\left\{\int_\tau^t \left[A(s) - \frac{1}{2}\textstyle\sum_{i=1}^{m} B_i^2(s)\right] ds + \textstyle\sum_{i=1}^{m} \displaystyle\int_\tau^t B_i(s)\, dW_i(s)\right\}. \tag{3}$$

It would then follow that

$$\det \Phi(\tau, t) = \exp\left\{\int_\tau^t \operatorname{tr}\left[A(s) - \frac{1}{2}\textstyle\sum_{i=1}^{m} B_i^2(s)\right] ds + \textstyle\sum_{i=1}^{m} \displaystyle\int_\tau^t \operatorname{tr} B_i(s)\, dW_i(s)\right\}. \tag{4}$$

If this commutativity condition does not hold, then in general one does not expect an explicit closed form expression for $\Phi(\tau, t)$ to exist. We show below that in any event (4) always remains valid. Note first that this is completely analogous to the deterministic situation where $B_i(t) = 0$ (i=1, $\cdots$, m) (e.g. Coddington and Levinson [3, Theorem 7.3, Chapter 1]).

Lemma. *Let A, B be $d \times d$ matrices. For a subset of S $\{1, 2, \cdots, d\}$ denote by $A_S B$ the matrix obtained from B by substituting the* ith *row of AB for the* ith *row of B, for each $i \in S$. Then*

$$\textstyle\sum_{\#(S)=k} \det(A_S B) = s_k(\lambda_1, \lambda_2, \cdots, \lambda_d) \det B, \tag{5}$$

where s_k is the kth *symmetric function and $\lambda_1, \lambda_2, \cdots, \lambda_d$ are the eigenvalues of A.*

Proof. First note that

$$\textstyle\sum_S \det(A_S B) = \det(I + A)B, \tag{6}$$

where S ranges over all subsets of $\{1, 2, \cdots, d\}$. By substituting λA in (6) it follows that the left hand side of (5) is precisely the coefficient of λ^k in the polynomial $\det(I + \lambda A) \det B$. When B is nonsingular, the roots of this polynomial are $-\lambda_1, -\lambda_2, \cdots, -\lambda_d$.

Theorem. (1), (2) $\Rightarrow$ (4).

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PROOF. First note that by Itô's formula

(7) $$d(\det \Phi(\tau, t)) = \sum_{\#(S)=1,2} \det(d_S\Phi(\tau, t)),$$

where $d_S\Phi(\tau, t)$ denotes the matrix obtained from $\Phi(\tau, t)$ by substituting the ith row of $d\Phi(\tau, t)$ for the ith row of $\Phi(\tau, t)$, for each $i \in S$. Here we are adopting the Itô mnemonic

(8) $$dW_i(t)\, dW_j(t) = \delta_{ij}\, dt,$$

where δ_{ij} is the Kronecker delta. From (1)

(9) $$d_S\Phi(\tau, t) = [A(t)\, dt + \textstyle\sum_{i=1}^m B_i(t)\, dW_i(t)]_S\Phi(\tau, t).$$

For $\#(S) = 2$ it follows now from (8) that

(10) $$\det(d_S\Phi(\tau, t)) = \textstyle\sum_{i=1}^m \det(B_i(t)_S\Phi(\tau, t))\, dt.$$

Thus, back to (7), if we use the Lemma, and note

(11) $$2s_2(\lambda_1, \lambda_2, \cdots, \lambda_d) = (\textstyle\sum_{i=1}^d \lambda_i)^2 - \sum_{i=1}^d \lambda_i^2,$$

then we obtain the linear scalar equation

(12) $$\begin{aligned} d(\det \Phi(\tau, t)) &= \left\{\mathrm{tr}\left[A(t) - \frac{1}{2}\textstyle\sum_{i=1}^m B_i^2(t)\right] + \frac{1}{2}\sum_{i=1}^m [\mathrm{tr}\, B_i(t)]^2\right\} \\ &\quad \times \det \Phi(\tau, t)\, dt + \textstyle\sum_{i=1}^m \mathrm{tr}\, B_i(t) \det \Phi(\tau, t)\, dW_i(t). \end{aligned}$$

From (2) and (12), (4) follows at once.

For a generalization of this result involving the higher order processes of Hochberg [4] the reader is referred to Berger and Sloan [2, Section 8.1]. The methods above can be extended to include systems relative to more general martingales $(M_1, M_2, \cdots, M_m)$ than Brownian motion, where (8) is replaced by

(13) $$dM_i(t)\, dM_j(t) = d < M_i, M_j > (t)$$

(e.g. Kunita and Watanabe [5, Theorem 2.2]).

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