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THE DOMAIN OF NORMAL ATTRACTION OF AN OPERATOR-STABLE LAW

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The idea of the domain of normal attraction was earlier extended to probabilities on a finite-dimensional inner-product space. We obtain a necessary and sufficient condition that a probability be in the domain of normal attraction of a given probability in terms of their covariance operators and of a limit involving the Lévy measure. This condition appears to be the natural generalization of the corresponding univariate condition. We also show that the domains of normal attraction of two probabilities are either the same or disjoint, with a condition that is necessary and sufficient for them to be the same.

1. Introduction, notation, and summary. In this paper are three basic results concerning the "domain of normal attraction" of an operator-stable law. A Borel probability measure μ on a real finite-dimensional inner-product space $\mathscr{V}$ is *operator-stable* if there exist independent identically distributed random vectors $\{X_k\}$ in $\mathscr{V}$, linear operators $\{A_n\}$ on $\mathscr{V}$, and vectors $\{a_n\}$ in $\mathscr{V}$ such that $\{A_n \sum_1^n X_k + a_n\}$ converges in law to μ as $n \to \infty$. These distributions are the natural multivariate analogue of the familar univariate stable laws. This work is limited to *full* measures, namely, those measures which are not concentrated in some hyperplane. Full operator-stable measures were first investigated by Sharpe. In his fundamental paper [10], he proved a number of basic results. In particular he proved that full operator-stable measures are infinitely divisible. Thus if μ is full and operator-stable on $\mathscr{V}$, there exists a $\mathscr{V}$-valued stochastic process $\{X(t): t \geq 0\}$ which is continuous in probability, which has stationary independent increments, and is such that $X(1)$ is distributed acording to μ. Sharpe showed that there is a nonsingular linear operator B on $\mathscr{V}$ such that $X(t)$ has the same distribution as does $t^B X(1) + b(t)$ where $t^B = e^{(\ln t)B} = \sum_{k=0}^{\infty} (\ln t)^k B^k/k!$ and b is some function from $(0, \infty)$ to $\mathscr{V}$. The linear operator B is called an exponent for μ. The distribution of $X(t)$ is denoted by μ^t and if A is a linear operator on $\mathscr{V}$ the distribution of $AX(1)$ is denoted by $A\mu$. Sharpe's result may then be written $\mu^t = t^B \mu * \delta(b(t))$, $t > 0$, where $\delta(x)$ denotes the probability measure concentrated at $\{x\}$. A distribution λ on $\mathscr{V}$ belongs to the *domain of normal attraction* $\mathscr{D}_{\mathscr{N}}(\mu)$ of a full operator-stable measure μ on $\mathscr{V}$ if there exist vectors $\{a_n\}$ in $\mathscr{V}$ such that the sequence $\{n^{-B}\lambda^n * \delta(a_n)\}_{n=1}^{\infty}$ converges weakly to μ where B is some exponent for μ. As noted by Jurek in [3], this domain does not depend on the choice of an exponent for μ. The theorems below give necessary and sufficient conditions for a distribution λ to belong to $\mathscr{D}_{\mathscr{N}}(\mu)$. We also show that if μ_1 and μ_2 are two full operator-stable laws on $\mathscr{V}$, then their domains of normal attraction are either disjoint or identical.

Two basic facts concerning full operator-stable measures will be needed below and it is convenient to simultaneously establish some notation.

ASSUMPTION. μ is a full operator-stable measure on $\mathscr{V}$. All measures are Borel.

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The Levy measure of μ will be denoted throughout by M so the characteristic function $\hat{\mu}(y)$ of μ is given by

$$\hat{\mu}(y) = \exp\{i(a, y) - \tfrac{1}{2}(y, Cy) + \int_{\mathscr{V}} \Psi(x, y)M(dx)\}$$

where $y, a \in \mathscr{V}$, (a, y) is the inner product of a and y, C is the covariance operator for the Gaussian component of μ, $\Psi(x, y) = \exp[i(x, y)] - 1 - i(x, y)/(1 + |x|)^2$, and $|x| = (x, x)^{1/2}$ is the norm determined by the inner product of $\mathscr{V}$. In particular $M(\{0\}) = 0$, $\int_{|x|<1} |x|^2 M(dx) < \infty$, and for all $\varepsilon > 0$, $\int_{|x|>\varepsilon} M(dx) < \infty$. Sharpe showed that M is a mixture of operator-stable Lévy measures concentrated on single orbits of $\{t^B : t > 0\}$. This result was extended by Kucharczak [7], Jurek [6], and Hudson and Mason [2]. We will use the following version.

THEOREM A. *Let B be an exponent for μ. Define $L = \{x : |x| = 1$ and for all $t > 1$, $|t^B x| > 1\}$. If A is a Borel subset of $\mathscr{V}$, then there exists a unique finite Borel measure ν on L such that*

$$M(A) = \int_L \left(\int_0^\infty I_A(t^B x)t^{-2}\, dt\right) \nu(dx).$$

If D is a Borel subset of L, then

$$\nu(D) = M(\{t^B x : t > 1 \quad \text{and} \quad x \in D\}).$$

The second fact is a refinement of a theorem of Sharpe and may be found in [2]. Let B be an exponent for μ and let $f = gh$ denote the minimal polynomial of B (i.e. the polynomial of smallest degree which annihilates B) where the roots of g are simple and have real parts equal to ½ and the roots of h have real parts greater than ½. Put $\mathscr{V}_g$ = kernel $g(B)$ and $\mathscr{V}_p$ = kernel $h(B)$; then $\mathscr{V} = \mathscr{V}_p \oplus \mathscr{V}_g$.

THEOREM B. *There exist Borel probability measures μ_g and μ_p on $\mathscr{V}$ such that*
(a) $\mu = \mu_p * \mu_g$
(b) *μ_g is Gaussian and concentrated in $\mathscr{V}_g$,*
(c) *μ_p has no Gaussian component and is concentrated in $\mathscr{V}_p$.*
(d) *$\mu_g | \mathscr{V}_g$ is full and operator-stable on $\mathscr{V}_g$ and $B | \mathscr{V}_g$ is an exponent for $\mu_g | \mathscr{V}_g$.*
(e) *$\mu_p | \mathscr{V}_p$ is full and operator-stable on $\mathscr{V}_p$ and $B | \mathscr{V}_p$ is an exponent for $\mu_p | \mathscr{V}_p$.*

This decomposition, $\mathscr{V} = \mathscr{V}_p \oplus \mathscr{V}_g$, determines a pair of projections, F_p and F_g. The projection F_g maps $\mathscr{V}$ onto $\mathscr{V}_g$ and has kernel $\mathscr{V}_p$. Similarly, the projection $F_p = I - F_g$ maps $\mathscr{V}$ onto $\mathscr{V}_p$ along $\mathscr{V}_g$. Although the last theorem puts $\mathscr{V}_p$ and $\mathscr{V}_g$ in terms of an exponent, they do not depend on the choice of exponent. In fact, $\mathscr{V}_g$ is the range of the covariance operator of μ and $\mathscr{V}_p$ is the subspace generated by the support of the Lévy measure of μ. Since F_p and F_g are determined by $\mathscr{V}_p$ and $\mathscr{V}_g$, they, too, do not depend on the particular choice of an exponent.

Our first result is

THEOREM 1. *A probability distribution $\lambda \in \mathscr{D}_{\mathscr{N}}(\mu)$ if and only if*
(a) *$F_g\lambda \in \mathscr{D}_{\mathscr{N}}(\mu_g)$ and*
(b) *$F_p\lambda \in \mathscr{D}_{\mathscr{N}}(\mu_p)$.*

Our second theorem will use the notation $\partial' A$ which denotes the boundary of $A \cap \mathscr{V}_p$ as a subset of $L \cap \mathscr{V}_p$. The symbol ∂A will denote the boundary of A in $\mathscr{V}$.

THEOREM 2. *A probability distribution λ on $\mathscr{V}$ is in $\mathscr{D}_{\mathscr{N}}(\mu)$ if there is an exponent B such that*

(a) *for every Borel subset A of $L \cap \mathscr{V}_p$ such that $\nu(\partial' A) = 0$,*

$$\lim_{t\to\infty} t\lambda\{s^B x : x \in A,\ s > t\} = \nu(A)$$

and

(b) *$F_g\lambda$ has a covariance operator which is also the covariance operator of μ_g.*
Conversely, if $\lambda \in \mathscr{D}_{\mathscr{N}}(\mu)$, then (a) *and* (b) *hold for every exponent B of μ.*

The proof of this theorem follows immediately from our Theorem 1 and a theorem of Jurek stated below, so the details are omitted.

To state our last theorem, we denote by (A, a) the affine transformation $\sigma x = Ax + a$ taking $\mathscr{V}$ onto $\mathscr{V}$ where A is a linear operator on $\mathscr{V}$ and $a \in \mathscr{V}$. Let $\mathscr{S}_a(\mu)$ be the set of all affine transformations (A, b) such that $\mu = A\mu * \delta(b)$. Theorem 2 of Urbanik [11] says that $\mathscr{S}_a(\mu)$ is a compact group of affine transformations if and only if μ is full. (The topology of pointwise convergence is used for the space of affine transformations on $\mathscr{V}$. This topology is the same as the topology determined by the norm, $|(A, a)| = |A| + |a|$ where $|A|$ denotes the operator norm of A.)

THEOREM 3. *Let γ be full and operator-stable on $\mathscr{V}$ and let B be any exponent for μ. Then either $\mathscr{D}_{\mathscr{N}}(\mu) = \mathscr{D}_{\mathscr{N}}(\gamma)$ or $\mathscr{D}_{\mathscr{N}}(\mu) \cap \mathscr{D}_{\mathscr{N}}(\gamma) = \phi$. Furthermore, $\mathscr{D}_{\mathscr{N}}(\mu) = \mathscr{D}_{\mathscr{N}}(\gamma)$ if and only if there is a nonsingular linear operator A taking $\mathscr{V}$ onto $\mathscr{V}$ such that*

(i) *for some a in $\mathscr{V}$, $\mu = A\gamma * \delta(a)$,*

(ii) *for some sequence $\{v_n\}$ in $\mathscr{V}$, the sequence $\{(n^{-B}An^B, v_n)\}$ is relatively compact in* Aff *$\mathscr{V}$, the space of affine transformations on $\mathscr{V}$, and*

(iii) *every limit point of $\{(n^{-B}An^B, v_n)\}$ is contained in $\mathscr{S}_a(\mu)$.*

To illustrate Theorem 3 suppose μ is full and operator-stable on R^2 with exponent $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and let $\mu = A\gamma * \delta(a)$. Then

$$n^{-B}An^B = \begin{pmatrix} a_{11} & na_{12} \\ \frac{1}{n}\, a_{21} & a_{22} \end{pmatrix}$$

so $\mathscr{D}_{\mathscr{N}}(\mu) = \mathscr{D}_{\mathscr{N}}(\gamma)$ if and only if $a_{12} = 0$ and $\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \in \mathscr{S}(\mu)$. E.g., if $\mathscr{S}(\mu) = \left\{I, -I, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$, then $\mathscr{D}_{\mathscr{N}}(\mu) = \mathscr{D}_{\mathscr{N}}(\gamma)$ if and only if $a_{12} = 0$ and $a_{11}, a_{22} \in \{-1, 1\}$.

2. Related work. Domains of normal attraction were also considered by Jurek and by Salter. In [3] and [4], Jurek investigated the domain of normal attraction of full operator-stable laws in two important special cases, namely for purely Gaussian distributions and for operator-stable distributions without a Gaussian component. He combined these cases (see [4]) in the following.

THEOREM (Jurek). *Let $\lambda = \lambda_g * \lambda_p$ be a Borel probability measure on $\mathscr{V}$ where λ_g is concentrated on $\mathscr{V}_q$ and λ_p is concentrated on $\mathscr{V}_p$. Then $\lambda \in \mathscr{D}_{\mathscr{N}}(\mu)$ if and only if*

(i) *λ_g on $\mathscr{V}_g$ has the same covariance operator as μ_g, and*

(ii) *if A is a Borel subset of $L \cap \mathscr{V}_p$, and if $\nu(\partial' A) = 0$, then $\lim_{t\to\infty} t\lambda_p(s^B x : x \in A,\ s \geq t) = \nu(A)$.*

This theorem is basically a combination of the two special cases mentioned above and is extended by our Theorems 1 and 2 above. It is easy using Theorem 1 to give an example of an operator-stable distribution μ and a $\lambda \in \mathscr{D}_{\mathscr{N}}(\mu)$ such that λ is not a convolution of the form $\lambda_g * \lambda_p$. Indeed, put $\mathscr{V} = R^2$ and define μ by $\hat{\mu}(x, y) = \exp\{-y^2/2 - |x|\}$ so $\mathscr{V}_g$ is the y-axis, $\mathscr{V}_p$ is the x-axis, μ_g is the standard normal distribution and μ_p is the symmetric Cauchy distribution. Then μ has the exponent $B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $\mu^t = t^B\mu$ for $t > 0$, so μ is clearly operator-stable. Let λ_1 be any symmetric probability distribution with variance 2 concentrated on $\mathscr{V}_g$ and let λ_2 be the symmetric probability distribution concentrated on $\mathscr{V}_p$ defined by

$$\lambda_2\{(x, 0) : x > t\} = \begin{cases} 2(\pi t)^{-1} & \text{if} \quad t \geq 4\pi^{-1} \\ \frac{1}{2} & \text{if} \quad 0 \leq t < 4\pi^{-1}. \end{cases}$$

The mixing measure ν for μ_p is given by $\nu\{(1, 0)\} = \nu\{(-1, 0)\} = \pi^{-1}$. Put $\lambda = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2$. Then the support of λ is contained in $\mathscr{V}_p \cup \mathscr{V}_g$ so λ cannot be the convolution of two measures concentrated on $\mathscr{V}_p$ and $\mathscr{V}_g$ respectively. Also $F_p\lambda = \frac{1}{2}\lambda_2 + \frac{1}{2}\delta(0)$, and $F_g\lambda = \frac{1}{2}\lambda_1 + \frac{1}{2}\delta(0)$. It follows that $F_g\lambda$ has mean zero and variance 1 so $F_g\lambda \in \mathscr{D}_{\mathscr{N}}(\mu_g)$. Since for $t > 4\pi^{-1}tF_p\lambda\{(x, 0) : x > t\} = \nu\{(1, 0)\}$, $F_p\lambda \in \mathscr{D}_{\mathscr{N}}(\mu_p)$. So by Theorem 1, $\lambda \in \mathscr{D}_{\mathscr{N}}(\mu)$.

Salter in an unpublished part of his thesis [8] also characterized the domain of normal attraction for the same two cases. (Actually he defined a slightly more general domain of attraction and characterized it.) Salter attempted the general case. However, he assumed that the Gaussian component μ_g and the Poisson component μ_p were concentrated on orthogonal subspaces. This is not true in general; a counterexample is given in [2]. Also, one of his conditions for $\lambda \in \mathscr{D}_{\mathscr{N}}(\mu)$ is that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} n\left(\int_{K(n\varepsilon)} (\zeta, n^{-B}y)^2 \lambda(dy) - \left(\int_{K(n\varepsilon)} (\zeta, n^{-B}y)\lambda(dy) \right)^2 \right) = (C\zeta, \zeta)$$

where C is the covariance operator of μ_g and $K(n\varepsilon) = \{t^B x : x \in L \backslash \mathscr{V}_p, \ t \leq n\varepsilon\}$. The corresponding condition of Theorems 1 and 2 is the natural extension of the univariate domain of normal attraction to a Gaussian law, in which λ is required to have the same variance as μ.

Domains of normal attraction have also been defined for G-stable probability measures (see Schmidt [9] for the definition of G-stable). These domains have been characterized for the same special cases (i.e. Gaussian case and Poisson case) as have the domains for full operator-stable laws. But any G-stable distribution may be centered to be full and operator-stable on the space spanned by the support of its symmetrization. Furthermore, such a distribution is stable with respect to some one-parameter group and the infinitesimal generator of any such group is a scalar multiple of some operator-stable exponent of the distribution. The domain of normal attraction for a G-stable distribution as defined in [5] is the same as the domain of normal attraction for an operator-stable distribution and so is covered by the results above.

3. Proof of Theorem 1.

For the proof of Theorem 1, two lemmas will be needed.

LEMMA 1. *Let B be an exponent for μ. Then B commutes with F_p and F_g.*

PROOF. Since $\mathscr{V}_p$ and $\mathscr{V}_g$ are invariant under B and since $\mathscr{V} = \mathscr{V}_p \oplus \mathscr{V}_g$, $Bx = BF_px + BF_gx = F_pBX + F_gBx$ where $BF_px \in \mathscr{V}_p$ and $BF_gx \in \mathscr{V}_g$. Hence $BF_px = F_pBx$ and $BF_gx = F_gBx$. □

COROLLARY. *For all $t > 0$, $t^BF_p = F_pt^B$ and $t^BF_g = F_gt^B$.*

PROOF. Since t^B is a power series in B, this is immediate from Lemma 1. □

In [3], Jurek shows that the domain of normal attraction of μ does not depend on the choice of exponent. For the convenience of the reader and for the sake of completeness we provide a proof here.

Let $S(\mu)$ be the set of all linear operators A on $\mathscr{V}$ such that for some $a \in \mathscr{V}$, $\mu = A\mu * \delta(a)$. Then $S(\mu)$ is a compact group of linear operators in the operator-norm topology (see for example Billingsley [1] or Urbanik [11]).

PROPOSITION 1. *If for a sequence $\{A_n\}$ in $S(\mu)$ and for a sequence $\{a_n\}$ in $\mathscr{V}$, $A_n\mu_n * \delta(a_n) \to \mu$ where $\{\mu_n\}$ is a sequence of Borel probability measures on $\mathscr{V}$, then there exists a sequence $\{c_n\}$ of vectors in $\mathscr{V}$ such that*

$$\mu_n * \delta(c_n) \to \mu.$$

PROOF. Since $\{A_n\} \subset S(\mu)$, there exist vectors b_n such that $\mu = A_n\mu * \delta(b_n)$. For $x \in \mathscr{V}$, define an affine transformation σ_n by $\sigma_n x = A_n x + b_n$. Then $\sigma_n\mu \equiv A_n\mu * \delta(b_n) = \mu$ so $\sigma_n \in S_a(\mu)$. Let $\lambda_n = A_n\mu_n * \delta(a_n)$ so that $\lambda_n \to \mu$. We will show that every subsequence of $\{\sigma_n^{-1}\lambda_n\}$ contains a further subsequence which converges to μ. Indeed, let $\{\sigma_{n'}^{-1}\lambda_{n'}\}$ be any subsequence. Since $S_a(\mu)$ is a compact group, the sequence $\{\sigma_{n'}\}$ contains a convergent subsequence $\{\sigma_{n''}\}$. Let σ denote the limit of $\{\sigma_{n''}\}$; then $\sigma \in S_a(\mu)$. Since $\lambda_{n''} \to \mu$, $\sigma_{n''}^{-1}\lambda_{n''} \to \sigma^{-1}\mu = \mu$. It follows that $\sigma_n^{-1}\lambda_n \to \mu$. But $\sigma_n^{-1}\lambda_n = \mu_n * \delta(C_n)$ for a suitable C_n in $\mathscr{V}$. □

COROLLARY. (Jurek). *The domain of normal attraction of μ does not depend on the choice of exponent. That is, if B_1 and B_2 are any two exponents for μ and if $n^{-B_1}\lambda^n * \delta(a_n) \to \mu$ for some $\{a_n\} \subset \mathscr{V}$ and Borel probability measure λ on $\mathscr{V}$, then there exist $\{a_n'\}$ in $\mathscr{V}$ such that $n^{-B_2}\lambda^n * \delta(a_n') \to \mu$.*

PROOF. Since $\mu^n = n^{B_1}\mu * \delta(b_n) = n^{B_2}\mu * \delta(b_n')$, it follows that $\{n^{-B_1}n^{B_2}\}_{n=1}^{\infty} \subset S(\mu)$. Now $n^{-B_1}\lambda^n * \delta(a_n) = (n^{-B_1}n^{B_2})(n^{-B_2}\lambda)^n * \delta(a_n) \to \mu$ so by Proposition 1, there exist $\{a_n'\} \subset \mathscr{V}$ such that $n^{-B_2}\lambda^n * \delta(a_n') \to \mu$. □

LEMMA 2. *Let α be an infinitely divisible probability measure on $\mathscr{V}$ and let μ be as above. If $F_p\alpha = \mu_p$ and $F_g\alpha = \mu_g$, then $\alpha = \mu$.*

PROOF. The measure α is uniquely determined by the triple $(a_\alpha, C_\alpha, M_\alpha)$ where a_α is in $\mathscr{V}$, C_α is the covariance operator of the Gaussian component α_g of α, and M_α is the Lévy measure of α. Since $F_p\alpha = \mu_p$ and μ_p has no Gaussian component, the covariance operator of $F_p\alpha$ must be zero. Also, the Lévy measure of $F_p\alpha$ is the same as that of μ_p. Thus

$$F_pC_\alpha F_p^* = 0 \quad \text{and} \quad F_pM_\alpha = M. \tag{1}$$

Similarly, since $F_g\alpha = \mu_g$,

$$F_gC_\alpha F_g^* = C \quad \text{and} \quad F_gM_\alpha = 0. \tag{2}$$

From (2) we see that the support of M_α is contained in $\mathscr{V}_p$, hence, $F_pM_\alpha = M_\alpha$. Thus, $M_\alpha = M$.

Since C_α is nonnegative definite and self-adjoint, it has a self-adjoint square root $C_\alpha^{1/2}$. From (1) we obtain $(F_pC_\alpha^{1/2})(F_pC_\alpha^{1/2})^* = 0$, which implies that $F_pC_\alpha^{1/2} = 0$, so $F_pC_\alpha = 0$. But, $C_\alpha = (F_p + F_g)C_\alpha(F_p + F_g)^* = F_gC_\alpha F_g^* = C$, by (2). Therefore, $C_\alpha = C$.

Finally, the centering terms, F_pa and F_ga, of μ_p and μ_g, respectively, must equal the corresponding centering terms, F_pa_α and F_ga_α, of $F_p\alpha$ and $F_g\alpha$. It easily follows that $a = a_\alpha$. □

Now for the proof of Theorem 1, first we assume that $F_p\lambda \in \mathscr{D}_{\mathscr{N}}(\mu_p)$ and $F_g\lambda \in \mathscr{D}_{\mathscr{N}}(\mu_g)$ and we show that $\lambda \in \mathscr{D}_{\mathscr{N}}(\mu)$. Let $\{X_n\}$ be a sequence of i.i.d. random vectors with common probability distribution λ and set $S_n = \sum_1^n X_j$. Let B be any exponent for μ. Then

$\lambda \in \mathcal{D}_{\mathcal{N}}(\mu)$ if and only if there exist a sequence $\{b_n\}$ of vectors in $\mathcal{V}$ such that $n^{-B}S_n + b_n \to \mu$ in law as $n \to \infty$. The triangular system $\{n^{-B}X_j: j = 1, \cdots, n; n = 1, 2, \cdots\}$ is infinitesimal since $\lim n^{-B} = 0$. Since $F_p\lambda \in \mathcal{D}_{\mathcal{N}}(\mu_p)$ and $F_g\lambda \in \mathcal{D}_{\mathcal{N}}(\mu_g)$, there exist sequences $\{p_n\}$ and $\{g_n\}$ of vectors in $\mathcal{V}_p$ and $\mathcal{V}_g$, respectively, such that $n^{-B}F_pS_n + p_n \to \mu_p$ and $n^{-B}F_gS_n + g_n \to \mu_g$ in law. Since n^{-B} commutes with F_p and F_g, $F_p(n^{-B}S_n + p_n) \to \mu_p$ and $F_g(n^{-B}S_n + g_n) \to \mu_g$ in law. Since $F_p + F_g = I$, it follows that the sequence of distributions of $\{n^{-B}S_n + p_n + g_n\}$ is tight. Let α be any limiting distribution of this sequence and assume that $n_k^{-B}S_{n_k} + p_{n_k} + g_{n_k} \to \alpha$ in law as $k \to \infty$. Then $F_p(n_k^{-B}S_{n_k} + p_{n_k} + g_{n_k}) = n_k^{-B}F_pS_{n_k} + p_{n_k} \to F_p\alpha$ in law, so $F_p\alpha = \mu_p$. Similarly, $F_g\alpha = \mu_g$. By Lemma 2, $\alpha = \mu$. Thus, $n^{-B}S_n + p_n + g_n \to \mu$ in law, so $\lambda \in \mathcal{D}_{\mathcal{N}}(\mu)$.

We now assume that $\lambda \in \mathcal{D}_{\mathcal{N}}(\mu)$. Then there is a sequence $\{a_n\}$ of vectors in $\mathcal{V}$ such that

$$\lim_{n\to\infty} n^{-B}\lambda^n * \delta(a_n) = \mu. \tag{3}$$

The Corollary of Lemma 1 shows that (1) implies $n^{-B}(F_p\mu)^n * \delta(F_pa_n) \to F_p\mu$, so $F_p\lambda \in \mathcal{D}_{\mathcal{N}}(\mu_p)$. Similarly, $F_g\lambda \in \mathcal{D}_{\mathcal{N}}(\mu_g)$.

4. The Proof of Theorem 3. First, assume that $\mathcal{D}_{\mathcal{N}}(\mu) \cap \mathcal{D}_{\mathcal{N}}(\gamma)$ is not empty. Let $\lambda \in \mathcal{D}_{\mathcal{N}}(\mu) \cap \mathcal{D}_{\mathcal{N}}(\gamma)$ and let B and D be any exponents for μ and γ respectively. Then there exist vectors $\{a_n\}$ and $\{a'_n\}$ in $\mathcal{V}$ such that as $n \to \infty$

$$n^{-B}\lambda^n * \delta(a_n) \to \mu, \qquad \text{and} \tag{1}$$

$$n^{-D}\lambda^n * \delta(a'_n) \to \gamma. \tag{2}$$

Rewrite (1) to get

$$n^{-B}n^D(n^{-D}\lambda^n * \delta(a'_n)) * \delta(a_n - n^{-B}n^Da'_n) \to \mu. \tag{3}$$

By a theorem on convergence of types, (e.g. [12], Theorem 2.3) the set $\{(n^{-B}n^D, a_n - n^{-B}n^Da'_n)\}_{n=1}^{\infty}$ is relatively compact, and if (A, a) is any limit point, then

$$\mu = A\gamma * \delta(a) \tag{4}$$

so (i) of Theorem 3 holds. Note that A is nonsingular since μ is full. From (4) and the fact that B is an exponent for μ, it follows that $A^{-1}BA$ is an exponent for γ. Thus, for a suitable choice of vectors $\{a'_n\}$, (2) holds with $D = A^{-1}BA$ and it may be written in the form

$$n^{-A^{-1}BA}n^Bn^{-B}\lambda^n * \delta(a'_n) \to \gamma. \tag{5}$$

Apply the affine transformation (A, a) to both sides of (5) and use (4) to see that

$$(n^{-B}An^B)(n^{-B}\lambda^n * \delta(a_n)) * \delta(Aa'_n + a - n^{-B}An^Ba_n) \to \mu. \tag{6}$$

Put $v_n = Aa'_n + a - n^{-B}An^Ba_n$. By (1), (6), and the theorem on convergence of types, $\{(n^{-B}An^B, v_n)\}_{n=1}^{\infty}$ is relatively compact, and if (L, ℓ) is any limit point, then

$$\mu = L\mu * \delta(\ell) \tag{7}$$

so (L, ℓ) belongs to $S_a(\mu)$, and conditions (ii) and (iii) of Theorem 3 follow.

Next, assume (i), (ii), and (iii) hold. It suffices to show that $\mathcal{D}_{\mathcal{N}}(\mu) = \mathcal{D}_{\mathcal{N}}(\gamma)$. Let $\lambda \in \mathcal{D}_{\mathcal{N}}(\mu)$. Then (1) holds for a suitable choice of vectors in $\mathcal{V}$. It follows from (ii) and (1) that $\{n^{-B}An^B(n^{-B}\lambda^n * \delta(a_n)) * \delta(v_n)\}_{n=1}^{\infty}$ is tight. From (1) and (iii) it follows that every convergent subsequence converges to μ and hence

$$n^{-B}A\lambda^n * \delta(v_n + n^{-B}An^Ba_n) \to \mu. \tag{8}$$

Apply $(A^{-1}, -A^{-1}a)$ to both sides of (8) to see that

$$n^{-A^{-1}BA}\lambda^n * \delta(A^{-1}v_n + A^{-1}n^{-B}An^Ba_n - A^{-1}a) \to \gamma. \tag{9}$$

Since $A^{-1}BA$ is an exponent of γ, this proves that $\lambda \in \mathcal{D}_{\mathcal{N}}(\gamma)$. Finally assume $\lambda \in \mathcal{D}_{\mathcal{N}}(\gamma)$.

Then for suitable vectors $\{a'_n\}$,

(10) $$n^{-A^{-1}BA}\lambda^n * \delta(a'_n) \to \gamma.$$

Apply (A, a) to both sides of (10) to obtain

(11) $$n^{-B}A\lambda^n * \delta(Aa'_n + a) \to \mu;$$

that is,

(12) $$(n^{-B}An^B)(n^{-B}\lambda^n * \delta(n^{-B}A^{-1}n^B(Aa'_n + a - v_n))) * \delta(v_n) \to \mu.$$

Since $\{(n^{-B}An^B, v_n)\}$ is relatively compact and since every limit point of $\{(n^{-B}An^B, v_n)\}$ is in $\mathscr{S}_a(\mu)$, a group, the set of inverse transformations $\{(n^{-B}A^{-1}n^B, -n^{-B}A^{-1}n^Bv_n)\}$ is relatively compact and every limit point lies in $\mathscr{S}_a(\mu)$. Apply $(n^{-A}A^{-1}n^B, -n^{-B}A^{-1}n^Bv_n)$ to the left side of (12). The resulting sequence is tight and every convergent subsequence converges to μ so

(13) $$n^{-B}\lambda^n * \delta(d_n) \to \mu$$

for a suitable sequence of vectors d_n in $\mathscr{V}$. Thus $\lambda \in \mathscr{D}_{\mathscr{N}}(\mu)$ so $\mathscr{D}_{\mathscr{N}}(\mu) = \mathscr{D}_{\mathscr{N}}(\gamma)$. □

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