The Annals of Probability
1983, Vol. 11, No. 1, 168–177

DOMAINS OF ATTRACTION OF MULTIVARIATE EXTREME VALUE DISTRIBUTIONS

BY ALBERT W. MARSHALL[1] AND INGRAM OLKIN[2]

University of British Columbia and Stanford University

The univariate conditions of Gnedenko characterizing domains of attraction for univariate extreme value distributions are generalized to higher dimensions. In addition, it is shown that random variables with a multivariate extreme value distribution are associated. Applications are given to a number of parametric families of joint distributions with given marginal distributions.

1. Introduction. Since their introduction by Fisher and Tippett (1928), univariate extreme value distributions have been extensively studied, perhaps most notably by Gnedenko (1943). Results for the multivariate case, obtained by a number of authors, have recently been summarized by Galambos (1978). The purpose of this paper is to obtain some multivariate analogs of Gnedenko's characterizations of domains of attractions. In addition, some results concerning the nature of the limiting distributions are obtained.

For $\mathbf{a}, \mathbf{b}, \mathbf{x} \in \mathscr{R}^k$, write $\mathbf{ax} + \mathbf{b}$ to denote the vector

$$(a_1x_1 + b_1, \cdots, a_kx_k + b_k).$$

Let $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \cdots$ be a sequence of independent k-dimensional random vectors with common distribution F and let

$$Z_j^{(n)} = \max_{1\le i\le n} X_j^{(i)}, \qquad W_j^{(n)} = \min_{1\le i\le n} X_j^{(i)}, \qquad j = 1, \cdots, k.$$

If there exist sequences $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \cdots$ and $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \cdots$ in $\mathscr{R}^k$ such that $\mathbf{a}^{(n)}\mathbf{Z}^{(n)} + \mathbf{b}^{(n)}$ converges in distribution to a random vector $\mathbf{U}$ with nondegenerate distribution G, then F is said to be in the *max domain of attraction* of G and G is said to be a *max extreme value distribution*. The convergence in distribution is equivalent to the condition

$$\lim_{n\to\infty} F^n(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)}) = G(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \tag{1}$$

because the fundamental representation of Pickands (1980) [see Galambos (1978), page 265] shows that extreme value distributions are continuous (but not always absolutely continuous).

Similar definitions of *min extreme value distributions* and their domains of attraction are made. For minima, (1) is replaced by

$$\lim_{n\to\infty} \bar{F}^n(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)}) = \bar{G}(\mathbf{x}) \quad \text{for all} \quad \mathbf{x}, \tag{2}$$

where for any distribution H of random variables $Y_1, \cdots, Y_k$,

$$\bar{H}(y) = P\{Y_1 > y_1, \cdots, Y_k > y_k\}$$

is called the *survival function* of $Y_1, \cdots, Y_k$. When $k = 1$, $\bar{H}(y) = 1 - H(y)$, but in general, this simple relation fails to hold.

Notice that if $\mathbf{a}^{(n)}\mathbf{Z}^{(n)} + \mathbf{b}^{(n)}$ (or $\mathbf{a}^{(n)}\mathbf{W}^{(n)} + \mathbf{b}^{(n)}$) converges in distribution to $\mathbf{U}$, then the ith component of $\mathbf{a}^{(n)}\mathbf{Z}^{(n)} + \mathbf{b}^{(n)}$ (or $\mathbf{a}^{(n)}\mathbf{W}^{(n)} + \mathbf{b}^{(n)}$) must converge to the ith component

Received October 1981; revised April 1982.

[1] Supported in part by the National Science Foundation at Stanford University, and in part by the National Sciences and Engineering Research Council of Canada.

[2] Supported by the National Science Foundation.

AMS 1980 *subject classification numbers.* 60F99, 62H05.

Key words and phrases. Extreme value distributions, association of random variables, domains of attraction.

of $\mathbf{U}$ and thus appropriate sequences $\{a_i^{(n)}\}$, $\{b_i^{(n)}\}$ can be determined from well-known univariate considerations, $i = 1, \cdots, k$. Once that is done the limiting distribution can in principle be obtained from (1) or (2). But usually this is not easy to do directly.

Sibuya (1960) obtains a representation of bivariate max extreme value distributions G that asymmetrically involve the marginal distributions, and Berman (1961/1962) obtains necessary and sufficient conditions for F to be in the domain of attraction of such a G. The lack of symmetry between the marginals is not entirely satisfying when one thinks of generalizing these results to $k > 2$. For another approach to this problem, see de Haan and Resnick (1977); they make use of the theory of max infinite divisible distributions as developed by Balkema and Resnick (1977). Sibuya (1960) introduces the notion of a "dependence function" which is also used by Deheuvels (1978, 1980) and by Galambos (1978) to study multivariate extreme value distributions and their domains of attraction. The approach of this paper, which avoids the use of dependence functions, is to generalize to higher dimensions the results of Gnedenko (1943).

1.1 LEMMA. *Equation* (1) *is equivalent to*

$$\lim_{n\to\infty} n[1 - F(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)})] = -\log G(\mathbf{x}) \quad \textit{for all } \mathbf{x} \textit{ such that } \quad G(\mathbf{x}) > 0, \tag{3}$$

and (2) *is equivalent to*

$$\lim_{n\to\infty} n[1 - \bar{F}(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)})] = -\log \bar{G}(\mathbf{x}) \quad \textit{for all } \mathbf{x} \textit{ such that } \quad G(\mathbf{x}) > 0. \tag{4}$$

This lemma is essentially Lemma 4 of Gnedenko (1943, page 438) because the dimensionality of x is not critical to the proof. For an alternate proof that easily carries over to the multivariate case, see Barlow and Proschan (1975, page 241).

2. Summary of univariate results. The univariate max(min) extreme value distributions are of the same type as Φ_α, Ψ_α or $\Lambda(\Phi_\alpha^*, \Psi_\alpha^*$ or $\Lambda^*)$, where

$$\Phi_\alpha(x) = e^{-x^{-\alpha}} \quad x > 0; \qquad \Phi_\alpha^*(x) = 1 - e^{-(-x)^{-\alpha}}, \quad x < 0 \qquad (\alpha > 0),$$

$$\Psi_\alpha(x) = e^{-(-x)^{\alpha}}, \quad x \le 0; \qquad \Psi_\alpha^*(x) = 1 - e^{-x^{\alpha}}, \quad x \ge 0 \qquad (\alpha > 0),$$

$$\Lambda(x) = e^{-e^{-x}}, \quad -\infty < x < \infty; \qquad \Lambda^*(x) = 1 - e^{-e^{x}}, \quad -\infty < x < \infty.$$

Write $F \in D_{\max}(G)$ to mean F is the max domain of attraction of the extreme value distribution G, and analogously define $F \in D_{\min}(G)$. Also when $k = 1$, let

$$F^{-1}(p) = \inf\{x : F(x-) \le p \le F(x)\}, \qquad \bar{F}^{-1}(p) = \inf\{x : \bar{F}(x-) \ge p \ge \bar{F}(x)\},$$

$$x^0 = \sup\{x : F(x) < 1\} \le \infty \quad \text{and} \quad x_0 = \inf\{x : F(x) > 0\} \ge -\infty.$$

The following summarizes some univariate results.

2.1 (Gnedenko, 1943).

$$F \in D_{\max}(\Phi_\alpha) \Leftrightarrow \lim_{t\to\infty} \bar{F}(tx)/\bar{F}(t) = x^{-\alpha}, \qquad x > 0,$$

$$F \in D_{\min}(\Phi_\alpha^*) \Leftrightarrow \lim_{t\to-\infty} F(tx)/F(t) = x^{-\alpha}, \qquad x > 0.$$

2.2 (Gnedenko, 1943).

$$F \in D_{\max}(\Psi_\alpha) \Leftrightarrow x^0 < \infty \quad \text{and} \quad \lim_{t\downarrow 0} \bar{F}(x^0 + tx)/\bar{F}(x^0 - t) = (-x)^\alpha, \qquad x < 0,$$

$$F \in D_{\min}(\Psi_\alpha^*) \Leftrightarrow x_0 > -\infty \quad \text{and} \quad \lim_{t\downarrow 0} F(x_0 + tx)/F(x_0 + t) = x^\alpha, \qquad x > 0.$$

2.3 (von Mises, 1936).

$$\lim_{t\uparrow x^0} \frac{d}{dt}\frac{1}{r(t)} = 0 \Rightarrow F \in D_{\max}(\Lambda),$$

where $r(t) = F'(t)/\bar{F}(t)$ is the hazard rate of F,

$$\lim_{t\downarrow x_0} \frac{d}{dt}\frac{1}{r^*(t)} = 0 \Rightarrow F \in D_{\min}(\Lambda^*),$$

where $r^*(t) = F'(t)/F(t)$.

Necessary and sufficient conditions for $F \in D_{\max}(\Lambda)$ similar to the one dimensional version of Propositions 3.3 is given by de Haan (1971). Other conditions not particularly convenient to verify are obtained by Marcus and Pinsky (1969).

3. Max domains of attraction. If H is the joint distribution of $Y_1, \cdots, Y_k$, then H_i denotes the marginal distribution of Y_i, $i = 1, \cdots, k$.

The following result is a k-dimensional version of 2.1.

3.1 PROPOSITION. *Let G be a k-dimensional* max *extreme value distribution such that $G_i = \Phi_{\alpha_i}$, $i = 1, \cdots, k$, and let $\phi_i(t) = \bar{F}_i^{-1}\bar{F}_1(t)$ $i = 2, \cdots, k$, $-\infty < t < \infty$. F is in the* max *domain of attraction of G if and only if*

$$\lim_{t\to\infty} \frac{1 - F(tx_1, \phi_2(t)x_2, \cdots, \phi_k(t)x_k)}{1 - F_1(t)} = -\log G(\mathbf{x}) \quad \textit{for all} \quad \mathbf{x} \tag{5}$$

such that $G(\mathbf{x}) > 0$.

PROOF. Suppose that $F \in D_{\max}(G)$. It is known (Gnedenko, 1943) that one can take $b_i^{(n)} = 0$ for all i, $a_1^{(n)} = \bar{F}_1^{-1}(1/n)$ and $a_i^{(n)} = \bar{F}_i^{-1}(1/n) = \bar{F}_i^{-1}\bar{F}_1(a_1^{(n)}) = \phi_i(a_1^{(n)})$, $i = 2, \cdots, k$, $n = 1, 2, \cdots$. Use of this in (3) of Lemma 1.1 leads to the condition

$$\begin{aligned}
&\lim_{n\to\infty} n[1 - F(a_1^{(n)}x_1, \phi_2(a_1^{(n)})x_2, \cdots, \phi_k(a_1^{(n)})x_k)] \\
&\qquad = \lim_{n\to\infty} \frac{[1 - F(a_1^{(n)}x_1, \phi_2(a_1^{(n)})x_2, \cdots, \phi_k(a_1^{(n)})x_k)]}{1 - F_1(a_1^{(n)})} \\
&\qquad = \lim_{t\to\infty} \frac{1 - F(tx_1, \phi_2(t)x_2, \cdots, \phi_k(t)x_k)}{1 - F_1(t)} \\
&\qquad = -\log G(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ such that} \quad G(\mathbf{x}) > 0.
\end{aligned}$$

The above equality of limits on n and t is a straightforward consequence of the monotonicity of F and the relation $[1 - F_1(a_1^{(n)})]/[1 - F_1(a_1^{(n+1)})] = (n + 1)/n$.

Conversely suppose that (5) holds, where $G_i = \Phi_{\alpha_i}$, $i = 1, \cdots, k$. Let $a_i^{(n)}$ and $b_i^{(n)}$ be as defined above. Then by reversing the steps of the above argument it follows from (5) that

$$\lim_{n\to\infty} n[1 - F(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)})] = -\log G(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ such that} \quad G(\mathbf{x}) > 0,$$

and hence by Lemma 1.1, $F \in D_{\max}(G)$. □

3.2 PROPOSITION. *Let G be a k-dimensional* max *extreme value distribution such that $G_i = \Psi_{\alpha_i}$, $i = 1, \cdots, k$. Then $F \in D_{\max}(G)$ if and only if* (6a) *there exists $\mathbf{x}^0 \in \mathscr{R}^k$ such that $F(\mathbf{x}^0) = 1$ and $F(\mathbf{x}) < 1$ if $\mathbf{x} \neq \mathbf{x}^0$, $x_i \le x_i^0$, $i = 1, \cdots, k$,.*

$$\lim_{t\downarrow 0} \frac{1 - F((tx_1, \phi_2(t)x_2, \cdots, \phi_k(t)x_k) + \mathbf{x}^0)}{1 - F_1(x_1^0 - t)} = -\log G(\mathbf{x}) \tag{6b}$$

for all $\mathbf{x}$ such that $G(\mathbf{x}) > 0$, where

$$\phi_i(t) = x_i^0 - \bar{F}_i^{-1}(\bar{F}_1(x_1^0 - t)), \qquad i = 2, \cdots, k.$$

PROOF. Suppose that $F \in D_{\max}(G)$. Then by 2.2, $x_i^0 = \sup\{x : F_i(x) < 1\} < \infty$; take $\mathbf{x}^0 = (x_1^0, \cdots, x_k^0)$ to obtain (6a). Gnedenko (1943) shows that one can take $a_1^{(n)}$ to satisfy $\bar{F}_1(x_1^0 - a_1^{(n)}) = 1/n$, $a_i^{(n)} = \phi_i(a_1^{(n)})$, $i = 2, \cdots, k$, and $\mathbf{b}^{(n)} = \mathbf{x}^0$. Then from Lemma 1.1 and

$\lim_{n\to\infty} a_1^{(n)} = 0$, it follows that

$$\lim_{n\to\infty} n[1 - F(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)})]$$

$$= \lim_{n\to\infty} \frac{1 - F((a_1^{(n)}x_1, \phi_2(a_1^{(n)})x_2, \cdots, \phi_k(a_1^{(n)})x_k) + \mathbf{b}^{(n)})}{1 - F_1(x_1^0 - a_1^{(n)})}$$

$$= \lim_{t\downarrow 0} \frac{1 - F((tx_1, \phi_2(t)x_2, \cdots, \phi_k(t)x_k) + \mathbf{b}^{(n)})}{1 - F_1(x_1^0 - t)} = -\log G(\mathbf{x})$$

for all $\mathbf{x}$ such that $G(\mathbf{x}) > 0$.

Next, suppose that (6a) and (6b) are satisfied. Then by reversing the steps of the previous argument, one obtains $\lim n[1 - F(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)})] = -\log G(\mathbf{x})$ for all $\mathbf{x}$ such that $G(\mathbf{x}) > 0$. Thus by Lemma 1.1, $F \in D_{\max}(G)$. □

3.3 PROPOSITION. *Let G be a k-dimensional* max *extreme value distribution such that $G_i = \Lambda$, $i = 1, \cdots, k$. Then $F \in D_{\max}(G)$ if and only if*

$$\text{(7)} \qquad \lim_{t\uparrow x_1^0} \frac{1 - F(\mathbf{a}(t)\mathbf{x} + \mathbf{b}(t))}{1 - F_1(t)} = -\log G(\mathbf{x}) \quad \textit{for all } \mathbf{x} \textit{ such that} \quad G(\mathbf{x}) > 0,$$

where $x_1^0 = \sup\{t : F_1(t) < 1\}$, $a_i(t) = \bar{F}_i^{-1}(e^{-1}\bar{F}_i(t)) - \bar{F}_i^{-1}\bar{F}_1(t)$, $b_i(t) = \bar{F}_i^{-1}\bar{F}_1(t)$, $i = 1, \cdots, k$.

PROOF. The proof of this result is similar to the proof of the preceeding propositions. Gnedenko (1943) shows that in (3) one can take

$$b_i^{(n)} = \bar{F}_i^{-1}\bar{F}_1(b_1^{(n)}) = \bar{F}^{-1}\left(\frac{1}{n}\right), \qquad i = 1, \cdots, k,$$

and

$$a_i^{(n)} = \bar{F}_i^{-1}\left(\frac{1}{ne}\right) - \bar{F}_i^{-1}\bar{F}_1(b_1^{(n)}) = \bar{F}_i^{-1}(e^{-1}\bar{F}_1(b_1^{(n)})) - \bar{F}_i^{-1}\bar{F}_1(b_1^{(n)}).$$

This expresses all normalizing constants in terms of $b_1^{(n)}$. Since $\lim_{n\to\infty} b_1^{(n)} = x_1^0$ and $b_1^{(n)}$ is an increasing sequence, one can write t in place of $b_1^{(n)}$ and take the limit as $t \uparrow x_1^0$. □

In the following, each F_i is assumed to have a density f_i and $r_i = f_i/\bar{F}_i$ is the corresponding hazard rate.

In case

$$\lim_{t\uparrow x_i^0} \frac{d}{dt}\frac{1}{r_i(t)} = 0, \qquad i = 1, \cdots, k,$$

where $x_i^0 = \sup\{x : F_i(x) < 1\}$, $a_i^{(n)}$ can be replaced by the simpler norming constant

$$1/nF_i'(b_i^{(n)}) = \bar{F}_i(b_i^{(n)})/F_i'(b_i^{(n)}) = 1/r_i(b_i^{(n)}).$$

This leads to the following proposition.

3.4 PROPOSITION. *Let G be a k-dimensional* max *extreme value distribution such that $G_i = \Lambda$, $i = 1, \cdots, k$. Let $\phi_i(t) = \bar{F}_i^{-1}\bar{F}_1(t)$, $i = 2, \cdots, k$, and let $x_1^0 = \sup\{x : F_1(x) < 1\}$. Then $F \in D_{\max}(G)$ if*

$$\text{(8)} \qquad \lim_{t\uparrow x_1^0} \frac{1 - F\left(\dfrac{x_1}{r_1(t)} + t, \dfrac{x_2}{r_2(\phi_2(t))} + \phi_2(t), \cdots, \dfrac{x_k}{r_k(\phi_k(t))} + \phi_k(t)\right)}{1 - F_1(t)} = -\log G(\mathbf{x})$$

for all $\mathbf{x}$ such that $G(\mathbf{x}) > 0$.

4. Min domains of attraction. The results of this section are all analogous to those of Section 3, and are stated without proof.

4.1 PROPOSITION. *Let G be a k-dimensional* min *extreme value distribution such that $G_i = \Phi^*_{\alpha_i}$, $i = 1, \cdots, k$, and let $\phi_i(t) = F_i^{-1}F_1(t)$, $i = 2, \cdots, k$, $-\infty < t < \infty$. F is in the* min *domain of attraction of G if and only if*

$$\text{(9)}\quad \lim_{t\to-\infty} \frac{1 - \bar{F}(tx_1, \phi_2(t)x_2, \cdots, \phi_k(t)x_k)}{F_1(t)} = -\log \bar{G}(\mathbf{x}) \text{ for all } \mathbf{x} \text{ such that } G(\mathbf{x}) > 0.$$

4.2 PROPOSITION. *Let G be a k-dimensional* min *extreme value distribution such that $G_i = \Psi^*_{\alpha_i}$, $i = 1, \cdots, k$. Then $F \in D_{\min}(G)$ if and only if*

$$\text{(10a)}\quad \begin{array}{l} \text{there exists } \mathbf{x}_0 = (x_{01}, \cdots, x_{0k}) \text{ such that} \\ F(\mathbf{x}_0) = 0 \text{ and } F(\mathbf{x}) > 0 \text{ if } \mathbf{x} \neq \mathbf{x}_0, \quad x_i \geq x_{0i} \quad i = 1, \cdots, k, \end{array}$$

and

$$\text{(10b)}\quad \lim_{t\downarrow 0} \frac{1 - \bar{F}((tx_1, \phi_2(t)x_2, \cdots, \phi_k(t)x_k) + \mathbf{x}_0)}{F_1(t + x_{01})} = -\log \bar{G}(\mathbf{x})$$

for all $\mathbf{x}$ such that $\bar{G}(\mathbf{x}) > 0$, where

$$\phi_i(t) = F_i^{-1}F_1(x_{01} + t) - x_{0i}.$$

4.3 PROPOSITION. *Let G be a k-dimensional* min *extreme value distribution such that $G_i = \Lambda^*$, $i = 1, \cdots, k$. Then $F \in D_{\min}(G)$ if and only if*

$$\text{(11)}\quad \lim_{t\downarrow x_{01}} \frac{1 - \bar{F}(\mathbf{a}(t)\mathbf{x} + \mathbf{b}(t))}{F_1(t)} = -\log \bar{G}(\mathbf{x})$$

for all $\mathbf{x}$ such that $G(\mathbf{x}) > 0$, where $x_0 = \inf\{t : F_1(t) > 0\}$, $a_i(t) = F_i^{-1}(e^{-1}F_1(t)) - F_i^{-1}F_1(t)$, $b_i(t) = F_i^{-1}F_1(t)$, $i = 1, \cdots, k$.

Now, assume that each F_i has a density and let $r_i^* = f_i/F_i$.

4.4 PROPOSITION. *Let G be a k-dimensional* min *extreme value distribution such that $G_i = \Lambda^*$, $i = 1, \cdots, k$. Let $\phi_i(t) = F_i^{-1}F_1(t)$, $i = 2, \cdots, k$ and let $x_{01} = \inf\{t : F_1(t) > 0\}$. Then $F \in D_{\min}(G)$ if*

$$\text{(12)}\quad \lim_{t\downarrow x_{01}} \frac{1 - \bar{F}\left(\dfrac{x_1}{r_1^*(t)} + t, \dfrac{x_2}{r_2^*(\phi_2(t))} + \phi_2(t), \cdots, \dfrac{x_k}{r_k^*(\phi_k(t))} + \phi_k(t)\right)}{F_1(t)} = -\log \bar{G}(\mathbf{x})$$

for all $\mathbf{x}$ such that $\bar{G}(\mathbf{x}) > 0$.

5. Association and independence. As noted by Tiago de Oliveira (1962/63), all multivariate extreme value distributions G of k variables $X_1, \cdots, X_k$ satisfy the condition

$$\text{(13)}\quad G(x_1, \cdots, x_k) \geq \textstyle\prod_1^k G_i(x_i),$$

a property called "positive quadrant dependence" by Lehmann (1966) in case $k = 2$. Positive quadrant dependence implies that all covariances are nonnegative.

Random variables $X_1, \cdots, X_k$ are said to be *associated* if for every pair θ, ψ of nondecreasing functions defined on $\mathscr{R}^k$,

$$\operatorname{Cov}(\theta(\mathbf{X}), \psi(\mathbf{X})) \geq 0$$

whenever the relevant expectations exist. This concept of positive dependence was introduced by Esary, Proschan and Walkup (1967), who show that it is stronger than positive quadrant dependence.

5.1 PROPOSITION. *If $X_1, \cdots, X_k$ have a multivariate extreme value distribution, then $X_1, \cdots, X_k$ are associated.*

PROOF. Consider a distribution F with survival function

$$\bar{F}(x) = \exp\left[-\int_S (\max_{1\le i\le k} q_i x_i)\, d\mu(\mathbf{q})\right], \tag{14}$$

where μ is a finite measure with support $S = \{\mathbf{q} : q_i \ge 0,\, i = 1, \cdots, k,\, \sum_1^k q_i = 1\}$.

Let $Z_1, \cdots, Z_m$ be independent exponentially distributed random variables with unit expectation and let

$$Y_i = \min_{1\le j\le m} a_{ij} Z_j, \qquad i = 1, \cdots, k,$$

where $0 < a_{ij} \le \infty$ for all i, j. Then $Y_1, \cdots, Y_k$ has a joint survival function

$$\begin{aligned} P\{Y_1 > y_1, \cdots, Y_k > y_k\} &= P\{\min_{1\le j\le m} a_{ij} Z_j > y_i,\, i = 1, \cdots, k\} \\ &= P\left\{Z_j > \max_{1\le i\le k} \frac{y_i}{a_{ij}},\, j = 1, \cdots, m\right\} \\ &= \exp[-\textstyle\sum_1^m \theta_j \max_{1\le i\le k} q_{ij} y_i], \end{aligned}$$

where $\theta_j = \sum_{i=1}^k 1/a_{ij}$ and $q_{ij} = 1/(\theta_j a_{ij})$. If ν is the measure on S which puts mass θ_j at $(q_{1j}, \cdots, q_{kj})$, $j = 1, \cdots, m$, then

$$P\{Y_1 > y_1, \cdots, Y_k > y_k\} = \exp\left[-\int_S [\max_{1\le i\le k} q_i y_i]\, d\nu(\mathbf{q})\right]. \tag{15}$$

Because increasing functions of independent random variables are associated [Esary, Proschan and Walkup, 1967], $Y_1, \cdots, Y_k$ are associated.

Now, any distribution of the form (14) is the weak limit of distributions of the form (15), where ν has finite support. Because limits in distribution of associated random variables are associated, it follows that random variables with a distribution of the form (14) are associated.

According to the representation of Pickands (1980), $X_1, \cdots, X_k$ have a min extreme value distribution with marginals $G_1, \cdots, G_k$ if and only if there exist random variables $Y_1, \cdots, Y_k$ with a distribution of the form (14) such that

$$X_i = -\log \bar{G}_i(Y_i), \qquad i = 1, \cdots, k.$$

Since increasing functions of associated random variables are associated, it follows that $X_1, \cdots, X_k$ are associated.

The proof for maxima is similar, or can be obtained from the fact that when $X_1, \cdots, X_k$ are associated, then so are $-X_1, \cdots, -X_k$. □

The above theorem shows that random variables with a multivariate extreme value distribution are always associated; in practice the stronger property of independence is often encountered. The study of independence in multivariate extreme value distributions is greatly simplified by the fact (Berman, 1961/1962), that pairwise independent random variables $X_1, \cdots, X_k$ having a multivariate extreme value distribution are mutually independent; this fact follows from Proposition 5.1 and the fact (Newman and Wright, 1981) that uncorrelated associated random variables are jointly independent. This allows studies of asymptotic independence to be confined to the bivariate case.

Geffroy (1958/1959, Chapter VII) gives a necessary and sufficient condition for asymptotic independence of two maxima. Other conditions are obtained by Sibuya (1960), Berman (1961); see also Galambos (1978).

Versions of the following proposition have been obtained by Mikhailov (1974) and by Galambos (1975); it has a simple direct proof.

5.2 PROPOSITION. *Suppose* $k = 2$. *Under the conditions accompanying the equations* (3), (5), (6b) *or* (7), *asymptotic independence occurs if and only if the limit in the respective equation is* 0 *when* $1 - F$ *is replaced by* $\bar{F}$. *Under the conditions accompanying the equations* (4), (9), (10b) *or* (11), *asymptotic independence occurs if and only if the limit in the respective equation is* 0 *when* $1 - \bar{F}$ *is replaced by* F.

6. Examples. There are a number of parametric families of joint distributions with arbitrary marginal distributions. Some of these are:

6.1 EXAMPLE. Farlie (1960), Gumbel (1958), Morgenstern (1956):

$$F(x_1, x_2) = F_1(x_1)F_2(x_2)[1 + \alpha\bar{F}_1(x_1)\bar{F}_2(x_2)],$$

i.e.,

$$\bar{F}(x_1, x_2) = \bar{F}_1(x_1)\bar{F}_2(x_2)[1 + \alpha F_1(x_1)F_2(x_2)], \qquad -1 \le \alpha \le 1.$$

6.2 EXAMPLE. Ali, Mikhail, Haq (1978):

$$F(x_1, x_2) = \frac{F_1(x_1)F_2(x_2)}{1 - \alpha\bar{F}_1(x_1)\bar{F}_2(x_2)}, \qquad -1 \le \alpha \le 1.$$

6.3 EXAMPLE. Plackett (1965):

$$F(x_1, x_2)\bar{F}(x_1, x_2) = \theta[F_1(x_1) - F(x_1, x_2)][F_2(x_2) - F(x_1, x_2)], \qquad 0 \le \theta \le \infty.$$

6.4 EXAMPLE. Fréchet (1951):

$$F(x_1, x_2) = \theta F_L(x_1, x_2) + (1 - \theta)F_U(x_1, x_2), \qquad 0 \le \theta \le 1,$$

where

$$F_U(x_1, x_2) = \min[F_1(x_1), F_2(x_2)], \qquad F_L(x_1, x_2) = \max[0, F_1(x_1) + F_2(x_2) - 1].$$

6.5 EXAMPLE. Gumbel (1960):

$$[-\log F(x_1, x_2)]^m = [-\log F_1(x_1)]^m + [-\log F_2(x_2)]^m, \qquad m \ge 1.$$

All of these families can be used to obtain bivariate distributions with marginals that are extreme value distributions, and it is natural to ask if the distributions so obtained are in fact bivariate extreme value distributions.

By using Lemma 1.1 or Proposition 5.2, it is not difficult to show that for Examples 6.1 and 6.2, the limiting extreme value distributions both for maxima and minima are always the case of independence. Since a bivariate extreme value distribution belongs to its own domain of attraction, it follows that the bivariate distributions of Examples 6.1 and 6.2 are bivariate extreme value distributions only in the case $\alpha = 0$ of independence.

Similarly for Example 6.3, the limiting distribution for maxima and minima are the case of independence except when $\theta = \infty$. In case $\theta = \infty$, $F(x_1, x_2) = \min[F_1(x), F_2(x)]$ and both for maxima and minima, the limiting distribution is $\min[G_1(x_1), G_2(x_2)]$, $-\infty < x_1, x_2 < \infty$, whenever $F_i \in D(G_i)$, $i = 1, 2$.

Again using Lemma 1.1, it is easy to see that whenever $F_i \in D_{\max}(G_i)$ [respectively, $D_{\min}(G_i)$], $i = 1, 2$, then the bivariate distributions of Example 6.4 are in $D_{\max}(G)$

[respectively $D_{\min}(G)$] where

$$G(x_1, x_2) = \{\min[G_1(x_1), G_2(x_2)]\}^{\theta}\{G_1(x_1)G_2(x_2)\}^{\theta}, \qquad -\infty < x_1, x_2 < \infty.$$

Unless $\theta = 1$, this has a different form than the distributions of Example 6.4 so those distributions are not extreme value distributions, $\theta \neq 1$.

As noted by Berman (1961/1962)[see also Tiago de Oliveira, 1975], distributions of Example 6.5 are in $D_{\max}(G)$ where

$$[-\log G(x_1, x_2)]^m = [-\log G_1(x_1)]^m + [-\log G_2(x_2)]^m \quad \text{and} \quad F_i \in D_{\max}(G_i),$$

$i = 1, 2$. Of course this means that if F_1 and F_2 are max extreme value distributions, then so are the bivariate distributions F of Example 6.5. On the other hand, if $F_i \in D_{\min}(G_i)$, $i = 1, 2$, then such distributions F are in $D_{\min}(G)$ where $G(x_1, x_2) = G_1(x_1)G_2(x_2)$, $-\infty < x_1$, $x_2 < \infty$. With $m = 1$, this fact is easily verified using Lemma 1.1, but for $m > 1$ the demonstration is somewhat more involved.

The dependency model

$$X_1 = \min(U_1, W), \qquad X_2 = \min(U_2, W)$$

has been studied in various contexts [see, e.g. Marshall and Olkin, 1967]. If U_i has distribution H_i, $i = 1, 2$, and W has distribution H_3, then it is easy to see that X_1, X_2 have joint distribution F given by

$$\bar{F}(\mathbf{x}) = \bar{H}_1(x_1)\bar{H}_2(x_2)\bar{H}_3(\max(x_1, x_2)), \qquad -\infty < x_1, x_2 < \infty. \tag{17}$$

6.6 Example. Suppose that $\bar{F}$ is given by (17) with

$$H_1(z) = H_2(z) = z^{\theta} \quad \text{and} \quad H_3(z) = z^{\nu}, \qquad 0 \leq z \leq 1$$

for some $\theta > 0$, $\nu > 0$. Here, the marginals $F_1, F_2 \in D_{\max}(\Psi_2)$, and it is not difficult to obtain from Proposition 3.2 that $F \in D_{\max}(G)$ where

$$G(\mathbf{x}) = e^{-[(-x_1)^2 + (-x_2)^2]}, \qquad x_1, x_2 \leq 0,$$

In this example, F_1, $F_2 \in D_{\min}(\Psi^*_{\min(\theta,\nu)})$ and from Proposition 4.2 it follows that $F \in D_{\min}(G)$, where for $x_1, x_2 \geq 0$,

$$\begin{aligned} G(\mathbf{x}) &= \exp -[\max(x_1, x_2)]^{\nu}, && \text{if } \nu < \theta, \\ &= \exp - \tfrac{1}{2}[x_1^{\theta} + x_2^{\theta} + \max(x_1^{\theta}, x_2^{\theta})], && \text{if } \nu = \theta, \\ &= \exp -[x_1^{\theta} + x_2^{\theta}], && \text{if } \nu > \theta. \end{aligned}$$

6.7 Example. Suppose that $\bar{F}$ is given by (17) with

$$\bar{H}_1(z) = e^{-\lambda_1 z}, \qquad \bar{H}_2(z) = e^{-\lambda_2 z}, \qquad \bar{H}_3(z) = e^{-\lambda_3 z},$$

$$z \geq 0 \quad \text{for some } \lambda_1, \lambda_2, \lambda_3 \geq 0, \lambda_1 + \lambda_3 > 0, \lambda_2 + \lambda_3 > 0.$$

Then F is a bivariate exponential distribution (Marshall and Olkin, 1967). Here F_1, $F_2 \in D_{\max}(\Lambda)$. Using Proposition 3.4, it can be shown that $F \in D_{\max}(G)$ where

$$\begin{aligned} G(x_1, x_2) &= \exp[-e^{-x_1} - e^{-x_2}], && \max(\lambda_1, \lambda_2) > 0 \\ &= \exp[-e^{-x_1} - e^{-x_2} + e^{-\max(x_1, x_2)}], && \lambda_1 = \lambda_2 = 0. \end{aligned}$$

(cf. Galambos, 1978, Example 5.2.2). In this example, F_1, $F_2 \in D_{\min}(\Psi^*)$ and from Proposition 4.2, it can be shown that $F \in D_{\min}(G)$ where

$$\bar{G}(x_1, x_2) = \exp -\left[\lambda_1 x_1 + \lambda_2 \frac{\lambda_1 + \lambda_3}{\lambda_2 + \lambda_3} x_2 + \lambda_3 \max\left(x_1, \frac{\lambda_1 + \lambda_3}{\lambda_2 + \lambda_3} x_2\right)\right], \quad x_1, x_2 \geq 0.$$

6.8 EXAMPLE. Another bivariate exponential distribution, given by Mardia (1970) is

$$\bar{F}(x_1, x_2) = (e^{x_1} + e^{x_2} - 1)^{-1}, \qquad x_1, x_2 \geq 0.$$

Using Proposition 3.4, it is easy to verify that $F \in D_{\max}(G)$ where

$$G(x_1, x_2) = \exp\text{-}[e^{-x_1} + e^{-x_2} - (e^{x_1} + e^{x_2})^{-1}].$$

This result was also obtained by Galambos (1978, Example 5.2.2) using different methods.
From Proposition 4.1, it follows also that $F \in D_{\min}(G)$ where

$$G(x_1, x_2) = e^{-x_1 - x_2}, \qquad x_1, x_2 \geq 0.$$

6.9 EXAMPLE. Bivariate Logistic Distribution. The logistic distribution given by

$$F_1(x_1) = (1 + e^{-x_1})^{-1}, \qquad -\infty < x_1 < \infty$$

belongs to $D_{\max}(\Lambda)$ since the von Mises condition (7) is satisfied. Here it is easily verified using Proposition 3.4 that the bivariate version

$$F(x_1, x_2) = (1 + e^{-x_1} + e^{-x_2})^{-1}, \qquad -\infty < x_1, x_2 < \infty$$

belongs to $D_{\max}(G)$ where

$$G(x_1, x_2) = \exp[-e^{-x_1} - e^{-x_2}], \qquad -\infty < x_1, x_2 < \infty$$

is the case of independence. Because F_1 has a symmetric density it is easy to see that $F_1 \in D_{\min}(\Lambda^*)$ and that $F \in D_{\min}(G^*)$ where

$$\bar{G}^*(x_1, x_2) = \exp[-e^{x_1} - e^{x_2}], \qquad -\infty < x_1, x_2 < \infty.$$

REFERENCES

[1] ALI, M. M., MIKHAIL, N. N., and HAQ, M. S. (1978). A class of bivariate distributions including the bivariate logistic. *J. Multivariate Anal.* **8** 405–412.

[2] BALKEMA, A. and RESNICK, S. I. (1977). Max-infinite divisibility. *J. Appl. Probab.* **14** 309–319.

[3] BARLOW, R. E. and PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability models.* Holt, Rinehart and Winston, New York.

[4] BERMAN, S. M. (1961/1962). Convergence to bivariate limiting extreme value distributions. *Ann. Inst. Statist. Math.* **13** 217–223.

[5] DEHEUVELS, P. (1978). Caractérisation complète des lois extrêmes multivariées et de la convergence des types extrêmes. *Publ. Inst. Statist. Univ. Paris* **23** 1–36.

[6] DEHEUVELS, P. (1980). The decomposition of infinite order and extreme multivariate distributions, 259–286 in *Asymptotic Theory of Statistical Tests and Estimation* (ed. by I. M. Chakravarti) Academic, New York.

[7] ESARY, J. D., PROSCHAN, F., and WALKUP, D. W. (1967). Association of random variables, with applications. *Ann. Math. Statist.* **38** 1466–1474.

[8] FARLIE, D. J. G. (1960). The performance of some correlation coefficients for a general bivariate distribution. *Biometrika* **47** 307–323.

[9] FISHER, R. A. and TIPPETT, L. H. C. (1928). Limiting forms of the frequency distributions of the largest or smallest member of a sample. *Proc. Cambridge Philos. Soc.* **24** 180–190.

[10] FRÉCHET, M. (1951). Sur les tableaux de corrélation dont les marges sont données. *Annales de l'Université de Lyon,* Sec. A, Ser. 3, **14** 53–77.

[11] GALAMBOS, J. (1975). Order statistics of samples from multivariate distributions. *J. Amer. Statist. Assoc.* **70** 674–680.

[12] GALAMBOS, J. (1978). *The asymptotic theory of extreme order statistics.* Wiley, New York.

[13] GEFFROY, J. (1958/1959). Contribution à la théorie des valeurs extrêmes. *Publ. Instit. Statist. Univ. Paris,* Chapters I-VII, **7-8** 37–184.

[14] GNEDENKO, B. V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.* **44** 423–453.

[15] GUMBEL, E. J. (1958). Distributions à plusieurs variables dont les marges sont données (with remarks by M. Fréchet). *Comptes Rendus de l'Académie des Sciences,* Paris, **246** 2717–2720.

[16] GUMBEL, E. J. (1960). Distributions des valeurs extrêmes en plusieurs dimensions. *Publ. l'Institut Statist. Univ. Paris* **9** 171–173.

[17] DE HAAN, L. (1971). A form of regular variation and its application to the domain of attraction of the double exponential. *Z. Wahrsch. verw. Gebiete* **17** 241–258.

[18] DE HAAN, L. and RESNICK, S. I. (1977). Limit theory for multivariate sample extremes. *Z. Wahrsch. verw. Gebiete* **40** 317–337.
[19] LEHMANN, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–1153.
[20] MARCUS, M. B. and PINSKY, M. (1969). On the domain of attraction of $\exp(-e^{-x})$. *J. Math. Anal. Appl.* **28** 440–449.
[21] MARDIA, K. V. (1970). *Families of bivariate distributions.* Griffin's Statistical Monographs, Griffin, London.
[22] MARSHALL, A. W. and OLKIN, I. (1967). A multivariate exponential distribution. *J. Amer. Statist. Assoc.* **62** 30–44.
[23] MIKHAILOV, V. G. (1974). Asymptotic independence of vector components of multivariate extreme order statistics. *Theory Prob. Appl.* **19** 817–821.
[24] VON MISES, R. (1936). La distribution de la plus grande de n valeurs. *Revue Mathématique de l'Union Interbalkanique* **1** 141–160. [Also pages 271–294 in vol. 2, Selected Papers of Richard von Mises, 1964, American Mathematical Society, Providence, R.I.]
[25] MORGENSTERN, D. (1956). Einfache Beispiele zweidimensionaler Verteilungen, *Mitteilungsblatt für Mathematische Statistik* **8** 234–235.
[26] NEWMAN, C. M. and WRIGHT, A. L. (1981). An invariance principle for certain dependent sequences. *Ann. Probab.* **9** 671–675.
[27] TIAGO DE OLIVEIRA, J. (1962/1963). Structure theory of bivariate extremes; extensions. *Estudos de Matemática, Estatistica e Econometria,* **7** 165–195.
[28] TIAGO DE OLIVEIRA, J. (1975). Bivariate extremes: extensions. *Bulletin International Statist. Inst., Proceedings of the 40th Session,* Warsaw, **46,** Book 2.
[29] PICKANDS, J. III (1980). Multivariate negative exponential and extreme value distributions. Unpublished manuscript, University of Pennsylvania.
[30] PLACKETT, R. L. (1965). A class of bivariate distributions. *J. Amer. Statist. Assoc.* **60** 516–522.
[31] SIBUYA, M. (1960). Bivariate extreme statistics, I. *Ann. Inst. Statist. Math.* **11** 195–210.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, B.C., CANADA V6T 1Y4

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305