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REMAINDER TERM ESTIMATES OF THE RENEWAL FUNCTION

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Let μ be a probability measure and $H(x) = \sum_{n=0}^{\infty} \mu^{n*}(-\infty, x]$ its renewal function. It is well-known that $H(x) - x/\mu_1 - \mu_2/2\mu_1^2 \to 0$ as $x \to +\infty$ if $\mu_1 > 0$ and μ is a nonlattice measure. (μ_k is the kth moment of μ.) The rate of this convergence is studied under further conditions on μ.

1. Introduction. Let $X_1, X_2, \cdots$ be independent random variables with a common distribution μ that has finite moments or order α, that is

$$\int |x|^{\alpha}\, d\mu(x) < +\infty.$$

(Unspecified integrations are always taken over the whole real line.) The renewal measure υ is defined by

$$\upsilon = \textstyle\sum_{n=0}^{\infty} \mu^{n*}$$

and the renewal function H by

$$H(x) = \upsilon(-\infty, x].$$

Here μ^{n*} denotes n-fold convolution and μ^{0*} is the Dirac measure at 0. Let

$$\mu_k = \int x^k\, d\mu(x)$$

be the kth moment of μ and

$$f(t) = \int e^{-itx}\, d\mu(x)$$

its characteristic function. The measure μ is said to be nonlattice if

$$f(t) = 1 \Leftrightarrow t = 0,$$

otherwise μ is called lattice.

The following theorems, stated in the nonlattice case, are well known.

THEOREM A. (Blackwell [1]) *Assume that μ is a nonlattice measure with a positive first moment. Then*

$$\lim_{x\to+\infty} \upsilon(x, x + h) = h/\mu_1.$$

THEOREM B. (Smith [14]) *Assume that μ is a nonlattice measure with a positive first and a finite second moment. Then*

$$\lim_{x\to+\infty}\left(H(x) - \frac{x}{\mu_1} - \frac{\mu_2}{2\mu_1^2}\right) = 0.$$

Estimates of the rate of convergence to zero in these theorems, when further moments exist, have been extensively studied, see for instance Stone [16], Stone-Wainger [18], Essén [4] and Lindvall [9]. In these papers sharp estimates are given in either the lattice case or in the nonlattice case under some further assumption on μ. The weakest such assumption

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is that μ is strongly nonlattice, that is

$$\liminf_{|t|\to\infty} |1 - f(t)| > 0.$$

However, this does not for instance cover the simple case where the measure μ has pointmass ½ at 1 and α, for some irrational $\alpha > 1$. Moreover, for different types of irrationalities $|1 - f(t)|$ may have much different types of behavior at infinity. This motivates defining classes of measures that satisfy

$$\liminf_{|t|\to\infty} |t^p(1 - f(t))| > 0$$

for some $p > 0$. We call such a measure nonlattice of type p.

Before stating our first result we need to introduce some more notation. Let $F(x) = \mu(-\infty, x]$ be the distribution function of X_i and put

$$R(x) = \begin{cases} \displaystyle\int_x^{+\infty} (1 - F(y))\, dy, & x \geq 0 \\ \displaystyle\int_{-\infty}^{x} F(y)\, dy, & x < 0 \end{cases}$$

and

$$S(x) = \begin{cases} \displaystyle -\int_x^{+\infty} R(y) dy, & x \geq 0 \\ \displaystyle\int_{-\infty}^{x} R(y)\, dy, & x < 0. \end{cases}$$

It is easily seen that S is well defined and that $S(x) = o(x^{2-\alpha})$, $|x| \to \infty$, if μ has finite moments of order $\alpha \geq 2$. Also $R * R$ exist and $R * R(x) = o(x^{1-\alpha})$, $|x| \to +\infty$.

THEOREM 1. *Assume that μ is nonlattice of type p with finite moments of integer order $m \geq 2$ and $\mu_1 > 0$. Then*

$$H(x) = \begin{cases} \dfrac{x}{\mu_1} + \dfrac{\mu_2}{2\mu_1^2} + \dfrac{S(x)}{\mu_1^2} + \dfrac{R * R(x)}{\mu_1^3} + o(x^{-m/1+p(m+1)}), & x \to +\infty \\ \dfrac{S(x)}{\mu_1^2} + \dfrac{R * R(x)}{\mu_1^2} + o(x^{-m/1+p(m+1)}), & x \to -\infty. \end{cases}$$

Consider again the example with supp $\mu = \{1, \alpha\}$. In Section 5 we will show that for almost all α, and in particular if α is algebraic, μ is nonlattice of type p for all $p > 2$. Thus Theorem 1 implies

$$H(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + o(x_1^{-p}), \qquad x \to +\infty,$$

for all $p < ½$. (It is not possible to have a better estimate than $O(x^{-1/2})$.) However, for certain α, μ is not nonlattice of type p for any p, and in fact α can be chosen so that the decrease to zero in Theorem B is slower than any prescribed ρ with $\rho(x) \to 0$, $x \to +\infty$, although μ has finite moments of all orders.

The proof of Theorem 1 is by Fourier analysis. Earlier estimates of the renewal function have been obtained by integrating estimates of the renewal measure. We will use a Bohr-type inequality to estimate H directly and this technique will enable us to prove sharper estimates of the renewal function in the strongly nonlattice case.

THEOREM 2. *Assume that μ is a strongly nonlattice measure with finite moments of* *integer order $m \geq 2$ and $\mu_1 > 0$. Then*

$$H(x) = \begin{cases} \dfrac{x}{\mu_1} + \dfrac{\mu_1}{2\mu_1^2} + \dfrac{S(x)}{\mu_1^2} + \dfrac{R * R(x)}{\mu_1^3} + o(x^{-m} \log x), & x \to +\infty, \\ \dfrac{S(x)}{\mu_1^2} + \dfrac{R * R(x)}{\mu_1^3} + o(x^{-m} \log |x|), & x \to -\infty. \end{cases}$$

The term $\mu_1^{-3} R * R$ is new. Since $R * R(x) = o(x^{1-\alpha})$, $|x| \to \infty$, we have the following corollary.

COROLLARY 1. *Under the conditions in Theorem 2 we have*

$$H(x) = \begin{cases} \dfrac{x}{\mu_1} + \dfrac{\mu_2}{2\mu_1^2} + \dfrac{S(x)}{\mu_1^2} + o(x^{1-m}), & x \to +\infty, \\ \dfrac{S(x)}{\mu_1^2} + o(x^{1-m}), & x \to -\infty. \end{cases}$$

In [16] Stone obtained the estimate $o(x^{1-m}\log x)$, and in [17], assuming μ to have a small singular part he proved the estimate in Corollary 1. That μ has a small singular part means that μ^{n*} is not purely singular with respect to Lebesgue measure if n is sufficiently large.

REMARK 1. In [10] Makarov gave an example of a singular continuous measure μ with $f(t) \to 0$, $|t| \to \infty$, such that μ and $\mu * \mu$ are absolutely continuous with respect to each other. By induction we get $\mu^{n*} \ll \mu$ and in particular μ^{n*} is singular for all n. Thus the condition that μ has a small singular part is strictly stronger than the strongly nonlattice condition. This confirms a conjecture by Smith [15].

If $m \geq 3$, we can simplify Theorem 2 somewhat, since then $R * R(x) = \mu_2 R(x) + o(x^{-m})$, $|x| \to \infty$.

COROLLARY 2. *Let μ be a strongly nonlattice measure with finite moments of integer order $m \geq 3$ and $\mu_1 > 0$. Then*

$$H(x) = \begin{cases} \dfrac{x}{\mu_1} + \dfrac{\mu_2}{2\mu_1^2} + \dfrac{S(x)}{\mu_1^2} + \dfrac{\mu_2}{\mu_1^3} R(x) + o(x^{-m} \log x), & x \to +\infty, \\ \dfrac{S(x)}{\mu_1^2} + \dfrac{\mu_2}{\mu_1^3} R(x) + o(x^{-m} \log |x|), & x \to -\infty. \end{cases}$$

Corollary 2 is close to an integrated version of Corollary 3 in Stone-Wainger [18] (given in the lattice case). However, by integrating their result we only obtain the estimate $o(x^{1-m})$, $|x| \to \infty$.

REMARK 2. Theorems 1 and 2 are true also when μ has finite moments of order α, α not necessarily an integer. See the end of Section 4.

REMARK 3. There is a corresponding statement of Theorem 2 in the lattice case with the remainder $o(k^{-m})$, $|k| \to \infty$. Its proof is somewhat simpler as the dual group of $\mathbb{Z}$ is compact.

2. A Bohr-type inequality. Our idea to prove estimates of the renewal function is to form $G(x) = H(x) - h(x)$, where h is chosen such that $\hat{G}$ and sufficiently many of its derivatives are locally integrable. We then put

$$G_T(x) = G(x) - \frac{1}{2\pi} \int_{-T}^{T} e^{itx} \hat{G}(t)\, dt$$

(for large T) and estimate G_T with a Bohr-type inequality. In this section we prepare the way by giving a suitable form of Bohr's inequality.

Let $\mathcal{M}_\Omega$ be the family of those nondecreasing functions $m:\mathbb{R} \to [1, +\infty)$ that satisfy $m(2x) \le \Omega m(x)$ and $m(x) = 1$ if $x \le 1$.

LEMMA 1. *Let g be a tempered function such that*

$$\sup_{x\le y\le x+1/T}(g(y) - g(x)) \le K/m_T(x),\ m_T \in M_\Omega.$$

Also assume that $\hat{g}$ has its support outside $[-T, T]$. Then

$$|g(x)| \le MK/m_T(x),$$

where M depends on Ω but not T for $T \ge 1$.

This lemma is very close to Theorem 7.3 in Ganelius [5]. Ganelius requires g to be bounded, but this is implied by the condition in Lemma 1; compare Hörmander [7], Theorem 2.6. We may also have different functions m_T for different T, but by checking the proof by Ganelius, it is clear that M is only depending on Ω. Note that we require $m_T(x) = 1$ if $x \le 1$, instead of $x \le 0$ as in Ganelius [5], to ensure that there is a constant M_1, only depending on Ω, such that $|h(x)| \le M_1/m_T(x)$, where h is a rapidly decreasing function.

3. Calculations of Fourier transforms. In this section we will compute the Fourier transform of

$$G(x) = H(x) - \left(\frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2}\right) E(x) - \frac{S(x)}{\mu_1^2},$$

where E is the Heaviside function,

$$E(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0. \end{cases}$$

The Fourier transforms will be computed in the sense of distributions. For the theory of distributions and its standard notation we refer to Schwartz [13] and Gelfand-Shilov [6].

We have $E' = \delta$ and since $\hat{\delta} = 1$, it $\hat{E}(t) = 1$ and $\hat{E}(t) = 1/it + C\delta(t)$. As $E - ½$ is odd, $(E - ½)\hat{}(t) = 1/it + (C - \pi)\delta(t)$ is also odd and we must have $C = \pi$. (Compare Schwartz [13, page 259].) Hence

$$\hat{E}(t) = 1/it + \pi\delta(t) \quad \text{and} \quad it\hat{E}(t) = 1. \tag{1}$$

Furthermore $(xE(x))' = E(x)$ and we get

$$it(xE(x))\hat{}(t) = 1/it + \pi\delta(t). \tag{2}$$

By an integration by parts we get

$$\int R(x)dx = \frac{1}{2}\mu_2 \quad \text{and} \quad \hat{R}(t) = \frac{1}{(it)^2}(f(t) - 1 + it\mu_1). \tag{3}$$

Now $S' = R - ½\mu_2\delta$ since

$$\langle S', \varphi\rangle = -\langle S, \varphi'\rangle = \int R(x)\varphi(x)dx - \varphi(0)\int R(x)\,dx = \langle R - ½\mu_2\delta, \varphi\rangle$$

and thus

$$it\,\hat{S}(t) = \frac{1}{(it)^2}\left(f(t) - 1 + it\mu_1 + \frac{1}{2}\mu_2 t^2\right). \tag{4}$$

If μ is nonlattice with a positive first and a finite second moment, the Fourier transform of v is given by the following assertion:

$$\hat{v} = \frac{1}{1 - f(t)} + \frac{\pi}{\mu_1}\,\delta(t). \tag{5}$$

PROOF. Put

$$v_N = \textstyle\sum_{n=0}^{N-1} \mu^{n*}.$$

Then v_N is an increasing sequence of positive measures and

$$\begin{aligned}\hat{v}_N(t) &= \textstyle\sum_{n=0}^{N-1} f^n(t) = \frac{1 - f^N(t)}{1 - f(t)}\\ &= \frac{1}{1 - f(t)} - f^N(t)\left(\frac{1}{1 - f(t)} - \frac{1}{i\mu t}\right) - \frac{f^N(t)}{\mu_1}\left(\frac{1}{it} + \pi\delta(t)\right) + \frac{\pi}{\mu_1}\,\delta(t).\end{aligned}$$

As μ has a finite second moment,

$$f(t) = \int e^{-itx}\,d\mu(x) = 1 - it\mu_1 + O(t^2), \qquad t \to 0.$$

Thus $(1 - f(t))^{-1} - (i\mu_1 t)^{-1}$ is locally bounded and by dominated convergence we get

$$f^N(t)\left(\frac{1}{1 - f(t)} - \frac{1}{i\mu_1 t}\right) \to 0 \quad \text{in} \quad \mathscr{D}', \qquad N \to \infty.$$

As $f^N(t)(1/it + \pi\delta(t))$ is the Fourier transform of

$$\begin{aligned}F^{N*}(x) &= (\mu^{N*} * E)(x) = \mathbb{P}(X_1 + \cdots + X_N \le x)\\ &= \mathbb{P}\left(\frac{1}{N}\left(X_1 + \cdots X_N\right) \le \frac{x}{N}\right)\end{aligned}$$

and $\mu_1 > 0$, the weak law of large numbers implies $F^{N*}(x) \to 0$, $N \to \infty$. As $0 \le F^{N*}(x) \le 1$, this pointwise convergence implies $F^{N*} \to 0$, $N \to \infty$, also in $\mathscr{S}'$ and consequently

$$f^N(t)\left(\frac{1}{it} + \pi\delta(t)\right) \to 0 \quad \text{in} \quad \mathscr{S}', \qquad N \to \infty,$$

and

$$\hat{\nu}_N \to \frac{1}{1 - f(t)} + \frac{\pi}{\mu_1}\,\delta \quad \text{in} \quad \mathscr{D}',\ N \to \infty.$$

To see that this is true also in $\mathscr{S}'$, we need an a priori estimate of ν. To get this, fix a nonnegative $\phi \in \hat{\mathscr{D}} = \{\hat{\varphi};\ \varphi \in \mathscr{D}\}$ such that $\phi(x) \ge 1$ if $|x| \le 1$. Then

$$\left(\phi * \left(\nu_N - \frac{E}{\mu_1} + \frac{F^{N*}}{\mu_1}\right)\right)^{\wedge}(t) = \hat{\phi}(t)\left(\frac{1}{1 - f(t)} - \frac{1}{i\mu_1 t}\right)(1 - f^N(t)).$$

As

$$\left|\int e^{itx}\hat{\phi}(t)\left(\frac{1}{1 - f(t)} - \frac{1}{i\mu_1 t}\right)(1 - f^N(t))\right| dt \le 2\int \left|\hat{\phi}(t)\left(\frac{1}{1 - f(t)} - \frac{1}{i\mu_1 t}\right)\right| dt < +\infty$$

and $\| \phi * (-E + F^{N*})/\mu_1 \|_\infty$ is bounded uniformly in N, this implies $\| \phi * \nu_N \|_\infty < C < +\infty$. Hence

$$C > \int \phi(x - y)\,d\nu_N(y) \ge \int_{x-1}^{x+1} \phi(x - y)\,d\nu_N(y) \ge \int_{x-1}^{x+1} d\nu_N(y).$$

From this uniform bound we see that $\nu_N \to \nu$ in $\mathscr{S}'$, where ν is a positive measure with

$$\hat{\nu} = \frac{1}{1 - f(t)} + \frac{\pi}{\mu_1}\delta$$

as desired.

If we put

$$G_0(x) = H_0(x) - \left(\frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2}\right)E(x) - \frac{S(x)}{\mu_1^2},$$

where $H_0(x) = \int_0^x d\nu(y)$, G_0 is linearly bounded and (1), (2), (4) and (5) imply

$$it\hat{G}_0(t) = \frac{1}{1 - f(t)} - \frac{1}{it\mu_1} - \frac{\mu_2}{2\mu_1^2} - \frac{f(t) - 1 + it\mu_1 + \tfrac{1}{2}t^2\mu_2}{(it\mu_1)^2}$$

$$= \frac{(f(t) - 1 + it\mu_1)^2}{(1 - f(t))(it\mu_1)^2} = g(t) = O(t), \qquad t \to 0,$$

as μ has a finite second moment. Hence

$$\hat{G}_0(t) = \frac{g(t)}{it} + C\delta(t)$$

for some constant C or, if we put $G_1 = G_0 - C/2\pi$,

$$G_1(t) = \frac{g(t)}{it} \in L^1_{\text{loc}}(\mathbb{R}).$$

We will now use Lemma 1 to show that $G = G_1$ and at the same time we obtain a simple proof of Theorem B. Put

$$G_T(x) = G_1(x) - \frac{1}{2\pi}\int_{-T}^{T} e^{itx}\hat{G}_1(t)\,dt = G_1(x) - G_T^*(x).$$

Then $\hat{G}_T(t) = 0$ if $|t| \le T$. Since

$$\sup_{x \le y \le x+1/T}(S(x) - S(y)) \le S(x) - S\left(x + \frac{1}{T}\right) \le R(x)/T \le K/T$$

and $H_1 = H_0 - C/2\pi$ is nondecreasing, we have

$$\sup_{x \le y \le x+1/T}(G_1(x) - G_1(y)) \le K/T \quad \text{if} \quad x \ge 1.$$

Furthermore, G_T^* is bounded and $G_T^*(x) \to 0$, $x \to +\infty$, by the Riemann-Lebesgue lemma. Thus

$$\sup_{x \le y \le x+1/T}(G_T(x) - G_T(y)) \le \begin{cases} K_T, & x \le x_T \\ K/T, & x > x_T. \end{cases}$$

If we take x_T^* large enough, there is an $m_T \in \mathscr{M}_2$ with $m_T(x) = TK_T/K$ if $x > x_T^*$ and

$$\sup_{x \le y \le x+1/T}(G_T(x) - G_T(y)) \le K_T/m_T(x).$$

Hence Lemma 1 implies

$$|G_T(x)| \le MK_T/m_T(x).$$

By letting $x \to +\infty$, we get

$$\limsup_{x \to +\infty} |G_1(x)| \le MK/T$$

and since T is arbitrary

$$\lim_{x \to +\infty} G_1(x) = 0.$$

In the same way we obtain $\lim_{x\to-\infty} G_1(x) = 0$ and since $S(x) \to 0$, $|x| \to \infty$, we have $0 = \lim_{x\to-\infty} G_1(x) = \lim_{x\to-\infty} H_1(x)$. Hence $H(x) = \int_{-\infty}^{x} d\nu(y)$ exists and $\lim_{x\to-\infty} H(x) = 0$. As H and H_1 are both primitives of ν and have the same limit at $-\infty$, they must be equal. Thus $G = G_1$ and Theorem B follows as $S(x) \to 0$, $x \to +\infty$.

4. The proofs. To prove Theorems 1 and 2 we must have a better estimate of G_T^* than above. This will be obtained by integrations by parts. To be able to integrate by parts sufficiently many times we have to subtract further terms from G.

We recall from Section 3 that

$$\hat{G}(t) = \mu_1 \frac{(f(t) - 1 + it\mu_1)^2}{(1 - f(t))(it\mu_1)^3}.$$

Define T by

$$\hat{T}(t) = \mu_1 \frac{(f(t) - 1 + it\mu_1)^2}{(it\mu_1)^4}$$

and U by

$$\hat{U}(t) = \mu_1 \frac{(f(t) - 1 + it\mu_1)^3}{(it\mu_1)^5}.$$

Then we obtain, with $\Gamma = G - T - U$,

$$\hat{\Gamma}(t) = \hat{G}(t)\left(\frac{f(t) - 1 + it\mu_1}{it\mu_1}\right)^2 = \mu_1 \frac{(f(t) - 1 + it\mu_1)^4}{(1 - f(t))(it\mu_1)^5}. \tag{1}$$

We will prove the following estimate of Γ.

PROPOSITION 1. *Assume that μ has finite moments of integer order $m \geq 2$ and $\mu_1 > 0$.*
(a). *If μ is strongly nonlattice then*

$$\Gamma(x) = o(x^{-m} \log |x|), \qquad |x| \to \infty.$$

(b). *If μ is nonlattice of type p then*

$$\Gamma(x) = o(x^{-m/1+p(m+1)}), \qquad |x| \to \infty.$$

From (3.3) we have $\hat{T} = \mu_1^{-3}\hat{R}^2$ and thus $T = \mu_1^{-3} R*R$. Theorems 1, 2 and the corollaries will follow from this observation together with Proposition 1 and the following lemma.

LEMMA 2. (a). *If μ has finite moments of order $\alpha \geq 2$, then*

$$U(x) = o(x^{-\alpha}), \qquad |x| \to \infty.$$

(b). *If μ has finite moments of order $\alpha \geq 3$, then*

$$R*R(x) = \mu_2 R(x) + o(x^{-\alpha}), \qquad |x| \to \infty.$$

PROOF OF LEMMA 2a. We have

$$\hat{U}(t) = \frac{1}{\mu_1^5}\left(\frac{f(t) - 1 + it\mu_1}{(it)^2}\right)^2 \left(\frac{f(t) - 1 + it\mu_1}{it}\right).$$

Put $Q = F - E$ and $Q_0 = \mu_1\delta + Q$. By an integration by parts we obtain $\hat{Q}(t) = (f(t) - 1)/it$ and hence $\hat{Q}_0(t) = (f(t) - 1 + it\mu_1)/it$. Together with (3.3) this implies $U = \mu_1^{-5} R*R*Q_0$. Put $M = R*R$. As $R' = Q_0$, we have $M' = R*R' = R*Q_0 = \mu_1 R + R*Q$ and since $\int Q(x)\,dx = -\mu_1$, we get

$$M'(x) = \int (R(x-y) - R(x))Q(y)\,dy = \int_{y\leq x/2} + \int_{y>x/2} (R(x-y) - R(x))Q(y)\,dy. \tag{2}$$

For definiteness we consider the case $x \to +\infty$. Then $R'(x) = Q(x) = o(x^{-\alpha})$, $x \to +\infty$, and therefore

$$\left| \int_{y \le x/2} (R(x-y) - R(x))Q(y)\, dy \right| \le \int_{y \le x/2} \left| yQ\left(\frac{x}{2}\right)Q(y)\right| dy$$

$$\le \| yQ(y) \|_1 \left| Q\left(\frac{x}{2}\right) \right| = o(x^{-\alpha}), \qquad x \to +\infty,$$

since $yQ(y) \in L^1(\mathbb{R})$. Furthermore,

$$(4) \qquad \left| \int_{y>x/2} R(x-y)Q(y)\, dy \right| \le \left| Q\left(\frac{x}{2}\right)\right| \left| \int_{y>x/2} R(x-y)\, dy \right.$$

$$\le \left| Q\left(\frac{x}{2}\right)\right| \| R \|_1 = o(x^{-\alpha}), \qquad x \to +\infty,$$

and

$$(5) \qquad \int_{y>x/2} R(x)Q(y)\, dy = R(x)R\left(\frac{x}{2}\right) = o(x^{1-\alpha})o(x^{1-\alpha}) = o(x^{-\alpha}), \qquad x \to +\infty,$$

as $\alpha \ge 2$. (2)–(5) implies

$$(6) \qquad M'(x) = o(x^{-\alpha}), \qquad |x| \to \infty.$$

We also observe that $M' \in L^1(\mathbb{R})$ and $M(x) = o(x^{1-\alpha})$, $|x| \to \infty$. Write

$$(7) \qquad U(x) = \mu_1^{-5} M * Q_0(x) = \mu_1^{-5} \int (M(x-y) - M(x))Q(y)\, dy.$$

In the same way as we obtained (6) from (2), we get from (7) that $U(x) = o(x^{-\alpha})$, $|x| \to \infty$, as desired.

Proof of Lemma 2b. We recall (3.3), $\int R(x)\, dx = \frac{1}{2}\mu_2$, and observe that $xR(x) \in L^1(\mathbb{R})$ if $\alpha \ge 3$. Write

$$R*R(x) = \int_{-\infty}^{x/2} + \int_{x/2}^{3x/2} + \int_{3x/2}^{+\infty} R(x-y)R(y)\, dy = A_1(x) + A_2(x) + A_3(x).$$

To estimate A_1, we observe that (again we only consider $x \to +\infty$)

$$\int_{-\infty}^{x/2} R(x)R(y)\, dy = R(x)\left(\int_{-\infty}^{+\infty} - \int_{x/2}^{+\infty} R(y)\, dy \right)$$

$$= \frac{1}{2}\mu_2 R(x) - o(x^{1-\alpha})o(x^{2-\alpha})$$

$$= \frac{1}{2}\mu_2 R(x) + o(x^{-\alpha}), \qquad x \to +\infty,$$

as $\alpha \ge 3$. Thus

$$A_1(x) = \frac{1}{2}\mu_2 R(x) + o(x^{-\alpha}) + \int_{-\infty}^{x/2} (R(x-y) - R(x))R(y)\, dy, \qquad x \to +\infty.$$

As

$$\left|\int_{-\infty}^{x/2} (R(x-y)-R(x))R(y)\,dy\right| \le \int_{-\infty}^{x/2} \left|yQ\left(\frac{x}{2}\right)R(y)\right|\,dy$$

$$\le Q\left(\frac{x}{2}\right)\|yR(y)\|_1 = o(x^{-\alpha}), \qquad x\to+\infty,$$

we obtain

$$A_1(x) = \tfrac{1}{2}\mu_2 R(x) + o(x^{-\alpha}), \qquad x\to+\infty. \tag{8}$$

To estimate A_2, we observe that

$$\int_{x/2}^{3x/2} R(x-y)R(x)\,dx = R(x)\int_{-x/2}^{x/2} R(y)\,dy = R(x)\left(\int_{-\infty}^{+\infty} - \int_{-\infty}^{-x/2} - \int_{x/2}^{+\infty} R(y)\,dy\right)$$

$$= \frac{1}{2}\mu_2 R(x) + o(x^{1-\alpha})o(x^{2-\alpha})$$

$$= \frac{1}{2}\mu_2 R(x) + o(x^{-\alpha}), \qquad x\to+\infty.$$

Hence

$$A_2(x) = \frac{1}{2}\mu_2 R(x) + o(x^{-\alpha}) + \int_{x/2}^{3x/2} (R(y)-R(x))R(x-y)\,dy, \qquad x\to+\infty.$$

For the integral we have

$$\left|\int_{x/2}^{3x/2} (R(y)-R(x))R(x-y)\,dy\right| \le \int_{x/2}^{3x/2} \left|(x-y)Q\left(\frac{x}{2}\right)R(x-y)\right|\,dy$$

$$\le \left|Q\left(\frac{x}{2}\right)\right| \|yR(y)\|_1 = o(x^{-\alpha}), \qquad x\to+\infty.$$

This renders

$$A_2(x) = \tfrac{1}{2}\mu_2 R(x) + o(x^{-\alpha}), \qquad x\to+\infty. \tag{9}$$

Finally,

$$|A_3(x)| \le R\left(-\frac{x}{2}\right)\int_{3x/2}^{+\infty} R(y)\,dy = o(x^{1-\alpha})o(x^{2-\alpha}) = o(x^{-\alpha}), \qquad x\to+\infty \tag{10}$$

as $\alpha \ge 3$. Lemma 2b now follows from (8), (9) and (10).

Proof of Proposition 1. The proof is patterned after the proof of Theorem B. Put

$$\Gamma_T(x) = \Gamma(x) - \frac{1}{2\pi}\int e^{itx}\hat{\Gamma}(t)\psi_T(t)\,dt = \Gamma(x) - \Gamma_T^*(x),$$

where $\psi_T(t) = \psi(t/T)$ for some $\psi \in C^\infty(\mathbb{R})$ with supp $\psi \subset [-2,2]$ and $\psi = 1$ on $[-1,1]$. Then $\hat{\Gamma}(t) = 0$ for $|t| \le T$. From our estimates of S, T, U and the fact that H is nondecreasing, it is easy to see that

$$\sup_{x\le y\le x+1/T}(\Gamma(x) - \Gamma(y)) \le K/T.$$

Next we want an estimate of Γ_T^*. From the Taylor expansion of $f(t)$ we get as $t\to 0$, for details compare Stone [16, pages 333–334],

$$\left(\frac{1}{t}(f(t)-1)\right)^{(k)} = O(1) \quad \text{for} \quad k \le m-1,$$

$$\left(\frac{1}{t}(f(t)-1)\right)^{(m)} = o\left(\frac{1}{t}\right),$$

$$\left(\frac{1}{t^2}(f(t)-1+it\mu_1)\right)^{(k)} = O(1) \quad \text{for} \quad k \le m-2$$

and

$$\left(\frac{1}{t^2}(f(t)-1+it\mu_1)\right)^{(k)} = o(t^{m-k-2}) \quad \text{for} \quad k = m-1, m.$$

From (1) we have

$$\hat{\Gamma}(t) = ct^2 \frac{\left(\frac{1}{t^2}(f(t)-1+it\mu_1)\right)^4}{\frac{1}{t}(f(t)-1)}.$$

Thus the Leibnitz formula implies that the mth derivative of $\hat{\Gamma}$ is locally integrable (in fact even bounded). Hence we can integrate by parts m times in the integral defining Γ_T^* to obtain

$$\text{(11)} \qquad \Gamma_T^*(x) = \frac{1}{2\pi}\left(\frac{i}{x}\right)^m \int e^{itx}(\hat{\Gamma}(t)\psi_T(t))^{(m)}\, dt.$$

If μ is strongly nonlattice, the integrand above is bounded by a constant times $1/|t|$, $|t| \to \infty$. Hence a uniform version of the Riemann-Lebesque lemma implies $\Gamma_T^*(x) = \log T o(x^{-m})$, $x \to +\infty$. Consequently,

$$\sup_{x\le y\le x+1/T}(\Gamma_T(x)-\Gamma_T(y)) \le K\left(\frac{1}{T}+\log To(x^{-m})\right), \qquad x\to+\infty,$$

and Lemma 1 implies

$$|\Gamma_T(x)| \le MK\left(\frac{1}{T}+\log To(x^{-m})\right), \qquad x\to+\infty,$$

where M is independent of T. If we put $T = x^{m+1}$ and use the estimate of Γ_T^* once more, we obtain

$$\Gamma(x) = o(x^{-m}\log x), \qquad x\to+\infty.$$

The estimate at $-\infty$ is proved in the same way and the proposition is proved in the strongly nonlattice case.

If μ is nonlattice of type p the integrand in (11) is bounded by a constant times

$$\frac{1}{|t(1-f(t))^{m+1}|} = \frac{|t|^{p(m+1)-1}}{|t^p(1-f(t))|^{m+1}} \le C|t|^{p(m+1)-1}, \qquad |t|\to\infty.$$

Thus

$$\Gamma_T^*(x) = T^{p(m+1)}o(x^{-m}), \qquad x\to+\infty$$

and

$$\sup_{x\le y\le x+1/T}(\Gamma_T(x)-\Gamma_T(y)) \le K\left(\frac{1}{T}+T^{p(m+1)}o(x^{-m})\right), \qquad x\to+\infty.$$

As above we get

$$|\Gamma(x)| \le MK\left(\frac{1}{T} + T^{p(m+1)}o(x^{-m})\right), \qquad x \to +\infty.$$

If we put $T = \beta^{-1}(x)x^{m/1+p(m+1)}$, where $\beta(x) \to 0$ slowly enough as $x \to +\infty$, we get

$$\Gamma(x) = o(x^{-m/1+p(m+1)}), \qquad x \to +\infty,$$

and Proposition 1 is proved.

Proposition 1 and its consequences are true also when μ has finite moments or order α, α not necessarily an integer. We conclude this section by indicating a proof of this fact. Put $m = [\alpha]$ and $\alpha = m + \beta$. As $(\Gamma_T^*)\hat{} = \hat{\Gamma}\psi_T$ we have,

$$(x^m\Gamma_T^*(x))\hat{}(t) = i^m D^m(\hat{\Gamma}\psi_T) \in L^1(\mathbb{R}).$$

We also want to assert that

$$(|x|^\beta x^m \Gamma_T^*(x))\hat{} \in L^1(\mathbb{R}).$$

For this we need a generalization of the derivative to nonintegral numbers. From Gelfand-Shilov [6, page 173], we have that the Fourier transform of $|x|^\beta$ is $C_\beta|t|^{-(1+\beta)}$ for some constant C_β. Thus we want to examine

$$D^\beta D^m(\hat{\Gamma}\psi_T) = |t|^{-(1+\beta)} * D^m(\hat{\Gamma}\psi_T).$$

Put $\Delta_s g(t) = g(t-s) - g(t)$. Then, if g is a measurable function with compact support and

$$\int \left| \frac{\Delta_s g(t)}{s^{1+\beta}} \right| ds \in L^1_{\text{loc}}(\mathbb{R}^d),$$

we have

$$D^\beta g(t) = \int \frac{\Delta_s g(t)}{|s|^{1+\beta}}\, ds.$$

For a proof of this fact, we refer to Carlsson [2, Lemma 1].

From the moment condition imposed on μ, it can be shown that

$$\int \left| \frac{\Delta_s D^m(\hat{T}\psi_T)(t)}{s^{1+\beta}} \right| ds \in L^1(\mathbb{R})$$

and that the behavior at infinity is such that

$$|x|^\beta x^m\Gamma_T^*(x) = \frac{C_\beta}{2\pi}\int e^{itx}D^\beta D^m(\hat{\Gamma}\psi_T)\, dt = \begin{cases} o(1)\log T \\ o(1)T^{p(m+1)}, \end{cases} \qquad x \to +\infty, \tag{11}$$

if μ is strongly nonlattice or nonlattice of type p, respectively. We omit the somewhat tedious details and refer the reader to [2, Section 4] for a similar derivation. The proof now follows from (11) as in the integer case.

5. Examples in the weakly nonlattice case. In this section we consider remainder term estimates of the renewal function when μ is not strongly nonlattice.

We first assume that μ consists of pointmasses a_i at α_i, $i = 1, \cdots, n$, $n \ge 2$, $a_i > 0$ and $\sum_{i=1}^n a_i = 1$. Since μ is assumed to be nonlattice, at least one of $\gamma_i = \alpha_i/\alpha_n$ is irrational. We observe that μ^{k^*} has pointmass at least equal to (equal to if $\{\alpha_i\}$ is independent over $\mathbb{Z}$)

$$\frac{k!}{k_1! \cdots k_n!} a_1^{k_1} \cdots a_n^{k_n}$$

at $k_1\alpha_1 + \cdots + k_n\alpha_n$, $k_1 + \cdots + k_n = k$. In particular, if we take $k_i \approx a_iN$, we get by Stirling's approximation that μ^{N^*}, and consequently also H, has a discontinuity of order at least $N^{(1-n)/2}$ close to $N(a_1\alpha_1 + \cdots + a_n\alpha_n)$ and since the approximand is continuous for $x \neq 0$, we can not have a better estimate than

$$H(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + O(x^{(1-n)/2}), \qquad x \to +\infty.$$

For certain choices of α_i, the discontinuities that arise from the pointmasses of μ^{n^*} for different n, may be so close together that the remainder is arbitrarily weak.

EXAMPLE 1. Let ρ be an arbitrary function with $\rho(x) \to 0$, $x \to +\infty$. Then there exists a nonlattice measure μ with compact support such that

$$\limsup_{x\to+\infty} \frac{1}{\rho(x)}\left(H(x) - \frac{x}{\mu_1} - \frac{\mu_2}{2\mu_1^2}\right) > 0.$$

PROOF. Let μ have pointmass ½ at 1 and α, where α is of the form

$$\alpha = 1,\underbrace{0001\,00001\,0\cdots\,001}_{n \text{ digits}}\underbrace{0\cdots 001}_{m \text{ digits}}0\cdots,$$

where m is a rapidly increasing function of n. If we put

$$\alpha_n = 1,\underbrace{0001\,000010\,\cdots\,01}_{n \text{ digits}},$$

$10^n\alpha_n$ is an integer and if $k_1 + k_2 = N$,

$$|k_1 10^n\alpha_n \cdot 1 + k_2 10^n \cdot \alpha - N10^n\alpha| = k_1 10^n(\alpha - \alpha_n) \leq 2k_1 10^n 10^{-(n+m)} \leq 2N10^{-m}.$$

Thus the points $k_1 10^n\alpha_n \cdot 1 + k_2 10^n \cdot \alpha$, $k_1 + k_2 = N$, all belong to an interval I_N of length $4N10^{-m}$ centered at $N10^n\alpha$. In this interval ν has mass at least equal to

$$\sum_{k_1=0}^{N} \frac{1}{2^{10^n(k_1\alpha_n + k_2)}} \binom{10^n(k_1\alpha_n + k_2)}{k_1 10^n\alpha_n} \geq c_n > 0,$$

according to Lemma 3 below. We now choose m such that $c_n \geq 32\ 10^{-m}$ and take N such that

$$\frac{c_n}{8} \leq |I_N| = 4N10^{-m} \leq \frac{c_n}{4}.$$

Since $\mu_1 > 1$, the variation in I_N of $x_1/\mu_1 + \mu_2/2\mu_1^2$ is bounded by ½ c_n. Thus the approximation is not better than ½ c_n at a point x_n with

$$x_n \geq N10^n - 4N10^{-m} \geq \frac{c_n}{32}\,10^{n+m} - \frac{c_n}{4} \geq c_n 10^m$$

if $n \geq 2$. Hence if we choose m as a very rapidly increasing function of n, we see that we can have arbitrarily weak remainders.

It remains to prove the lower estimate for the binomial coefficients used above.

LEMMA 3. *Let $1 \leq q < p$. Then*

$$\sum_{k=0}^{N} \frac{1}{2^{pk+q(N-k)}} \binom{pk + q(N-k)}{pk} \geq c(p, q) > 0$$

for all N.

PROOF. Put $B(n, k) = 2^{-n}\binom{n}{k}$ and $\beta_\ell(N) = \sum_{k=0}^{N} B(pk + q(N-k), pk + \ell)$. We first observe that

(1) $$\sum_{k=0}^{m} B(n, k) \geq \sum_{k=0}^{m} B(n+1, k).$$

If $m \le n/2$ this is obvious, since then $B(n, k) \ge B(n+1, k)$, $0 \le k \le m$.
If $m > n/2$ then $B(n, k) \le B(n+1, k)$ for $m+1 \le k \le n$. Hence,

$$\sum_{k=0}^{m} B(n, k) = 1 - \sum_{k=m+1}^{n} B(n, k) \ge 1 - \sum_{k=m+1}^{n} B(n+1, k)$$
$$\ge 1 - \sum_{k=m+1}^{n+1} B(n+1, k) = \sum_{k=0}^{m} B(n+1, k)$$

as desired. By repeated use of (1), we obtain

$$\sum_{\ell=0}^{p-1} \beta_\ell(N) \ge \sum_{k=0}^{pN} B(pN, k) = 1,$$

which implies

(2) $$\max_{0 \le \ell < p-1} \beta_\ell(N) \ge 1/p.$$

Let $0 < \ell < p-1$. Then

(3) $$\beta_0(N) \ge \beta_\ell(N) - cN^{-1/2}.$$

To see this, we observe that if $k \le qN(p+q)^{-1}$, then $pk \le ½(pk + q(N-k))$ and thus $B(pk + q(N-k), pk) \ge B(pk + q(N-k), pk - \ell)$. If $k > qN(p+q)^{-1}$, we have $B(pk + q(N-k), pk) \ge B(pk + q(N-k), p(k+1) - \ell)$. If furthermore $k > (qN + p - q - 2)(p+q)^{-1}$, then $p(k+1) - \ell < (1/2)(p(k+1) + q(N-(k+1))$ and we get $B(pk + q(N-k), pk) \ge B(p(k+1) + q(N-(k+1)), p(k+1) - \ell)$. Since there are only a finite number of k : s that do not satisfy the above inequalities and since each term $B(n, k)$ is smaller than a constant times $n^{-1/2}$, we obtain (3). Lemma 1 now follows from (2) and (3).

EXAMPLE 2. Let $\mu = \sum_{i=1}^{n} a_i \delta_{\alpha_i}$, $a_i > 0$, $\sum_{i=1}^{n} a_i = 1$ and put $\gamma_i = \alpha_i/\alpha_n$. Then for almost all $(\gamma_1, \cdots, \gamma_{n-1})$ (with respect to $(n-1)$-dimensional Lebesgue measure) we have

(4) $$H(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + o(x^{-p}), \qquad x \to +\infty,$$

for all $p < ½(n-1)$.
To prove this we will use Theorem 1 and the following result on simultaneous diophantine approximation, contained in (the easy half of) a theorem by Khintchine [8].

LEMMA 4. *For almost all* $(\gamma_1, \cdots, \gamma_{n-1})$ *we have*

$$\left| \gamma_i - \frac{p_i}{p_n} \right| < \frac{1}{p_n^{(n/n-1)+\sigma}}, p_i \in \mathbb{Z}, \quad i = 1, \cdots, n-1,$$

only for a finite number of p_n : *s if* $\sigma > 0$.

We have $f(t) = \sum_{i=1}^{n} a_i e^{-i\alpha_i t}$, which is close to 1 if $\alpha_i t/2\pi$ are close to some integers, $i = 1, \cdots, n$. For $t \ne 0$ put $\alpha_i t/2\pi = p_i + \varepsilon_i$, where $p_i \in \mathbb{Z}$ and $0 \le \varepsilon_i < 1$. Then

$$|1 - f(t)| = |\sum_{i=1}^{n} a_i(1 - e^{-i\alpha_i t})| = |\sum_{i=1}^{n} a_i(1 - e^{i2\pi\varepsilon_i})|$$
$$\ge a_i Re(1 - e^{-i2\pi\varepsilon_i}) \ge c\varepsilon_i^2, \quad i = 1, \cdots, n.$$

If $\varepsilon_i < t^{-\sigma}$, $i = 1, \cdots, n$, then, since p_i and t are of the same order of magnitude,

$$\left| \gamma_i - \frac{p_i}{p_n} \right| = \left| \frac{p_i + \varepsilon_i}{p_n + \varepsilon_n} - \frac{p_i}{p_n} \right| = \left| \frac{p_n\varepsilon_i - p_i\varepsilon_n}{p_n(p_n + \varepsilon_n)} \right| \le C\frac{p_n^{1-\sigma}}{p_n^2} = \frac{C}{p_n^{1+\sigma}}, \quad i = 1, \cdots, n-1.$$

By Lemma 4, for almost all $(\gamma_1, \cdots, \gamma_{n-1})$, this can only happen for a finite number of p_n : s if $\sigma > (n-1)^{-1}$. Thus, for almost all $(\gamma_1, \cdots, \gamma_{n-1})$ and all $\sigma > (n-1)^{-1}$,

$$|t|^{2\sigma}|1 - f(t)| \ge c|t^{2\sigma}|\varepsilon_i^2 \ge c > 0$$

if $|t|$ is sufficiently large. Hence μ is nonlattice of type p for all $p > 2(n-1)^{-1}$. Since μ has finite moments of all orders, (4) follows from Theorem 1 as desired.

REMARK. If $n = 2$, the famous result of Roth [11] on approximation of algebraic numbers by rational numbers implies that no μ with α_1/α_2 algebraic belongs to the exceptional set in Lemma 4. Thus

$$H(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + o(x^{-p}), \qquad x \to +\infty,$$

for all $p < 1/2$ if α_1/α_2 is algebraic. A similar remark is true also for arbitrary n, due to the generalization of Roth's theorem by Schmidt, see [12, pages 151–153].

We conclude with an example which shows that we can have a strong remainder term even if $\liminf_{|t|\to\infty} |1 - f(t)| = 0$.

EXAMPLE 3. There exist weakly nonlattice measures with finite moments of order m such that

$$(5) \qquad H(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + \frac{S(x)}{\mu_1^2} + \frac{R*R(x)}{\mu_1^3} + o(x^{-p}), \qquad x \to +\infty,$$

for all $p < m$.

In [3] Esseen gave the following example of a singular continuous measure that is weakly nonlattice.

Let $(\lambda_n)_1^\infty$ be an increasing sequence, $\lambda_1 > 2$, $\lim_{n\to\infty}\lambda_n = +\infty$ and $\prod_{n=1}^\infty (1 - \lambda_n^{-1}) = 0$. Let σ_n be the measure with pointmass $1 - \lambda_n^{-1}$ at $x = 0$ and λ_n^{-1} at $x = 2^{-n}$ and put $\sigma_n^* = \sigma_1 * \sigma_2 * \cdots * \sigma_n$. Then $\sigma_n^* \to \sigma$, where σ is a continuous singular probability measure with support in $[0, 1]$ and

$$\hat\sigma(t) = \prod_{n=1}^\infty (1 - \lambda_n^{-1}(1 - e^{-it2^{-n}})).$$

Let τ_k be the measure with support in $[k, k + 1]$ defined by $\tau_k(E) = \sigma(E - k)$. Take $a_k \geq 0$, $\sum_{k=0}^\infty a_k = 1$ and put $\mu = \sum_{k=0}^\infty a_k\tau_k$. We also choose a_k such that μ has finite moments of order m, but not $m + \varepsilon$ for any $\varepsilon > 0$.

As $\hat\mu(t) = \sum_{k=0}^\infty a_k e^{-itk}\hat\sigma(t)$, we have

$$\liminf_{|t|\to\infty} |1 - \hat\mu(t)| \leq \lim_{m\to\infty} |1 - \hat\mu(2\pi 2^m)| = \lim_{m\to\infty} \left|\sum_{k=0}^\infty a_k(1 - \hat\sigma(2\pi 2^m))\right| = 0$$

and hence μ is weakly nonlattice.

Now $Re(1 - e^{itk}\hat\sigma(t)) \geq 1 - |\hat\sigma(t)| \geq 1 - |1 - \lambda_n^{-1}(1 - e^{-t2^{-n}})|$ for all n. If $|t| \in [2^m, 2^{m+1}]$, we get with $n = m + 1$, since $1/2 \leq |t| 2^{-(m+1)} \leq 1$, that $Re(1 - e^{-itk}\hat\sigma(t)) \geq c\lambda_{m+1}^{-1}$ for some $c > 0$. If we choose λ_n such that $a^n\lambda_n^{-1} \to +\infty$, $n \to +\infty$, for all $a > 1$ we get

$$|t^p(1 - e^{-itk}\hat\sigma(t))| \geq ct^p\lambda_{m+1}^{-1} \geq c2^{pm}\lambda_{m+1}^{-1} \to +\infty, \qquad |t| \to +\infty,$$

for all $p > 0$. Hence

$$|t^p(1 - \mu(t))| \geq |t|^p Re(1 - \hat\mu(t)) = \sum_{k=0}^\infty a_k |t|^p Re(1 - e^{-itk}\hat\sigma(t)) \to +\infty, \qquad |t| \to \infty,$$

for any $p > 0$. Thus μ is weakly nonlattice of type p for any $p > 0$ and Theorem 1 implies (5) as desired.

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