The Annals of Probability
1983, Vol. 11, No. 1, 16–33

ATTRACTIVE NEAREST PARTICLE SYSTEMS

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We consider certain Markov processes with state space $\{0, 1\}^Z$ which were introduced and first studied by Spitzer. In these systems, deaths occur at rate one independently of the configuration, and births occur at rate $\beta(\ell, r)$ where ℓ and r are the distances to the nearest particles to the left and right, respectively. In his paper, Spitzer gave a necessary and sufficient condition for this process to have a reversible invariant measure, and showed that such a measure must be a stationary renewal process. It was that fact which motivated the study of these systems. Assuming that the process is attractive in the sense of Holley, we give conditions under which (a) the pointmass on the configuration "all zeros" is invariant, and (b) the reversible renewal process is the only nontrivial invariant measure which is translation invariant. As an application, these results allow us to determine exactly the values of $\lambda > 0$ and $p > 0$ for which the process with $\beta(\ell, r) = \lambda(1/\ell + 1/r)^p$ is ergodic.

1. Introduction. In [13], Spitzer introduced and first studied an important and interesting class of spin systems which are called nearest particle systems. A spin system on Z, the set of integers, is a continuous time Markov process on $\Omega = \{0, 1\}^Z$ in which a "flip" occurs at the coordinate $x \in Z$ at rate $c(x, \eta)$, where $c(x, \eta)$ is an appropriate nonnegative function on $Z \times \Omega$. For a more formal description of spin systems and a survey of results and techniques, see [2] or [11]. In the nearest particle systems, the dependence of the rates on the configuration η takes a particular form. Let $\beta(\ell, r)$ and $\delta(\ell, r)$ be nonnegative functions defined for positive integers ℓ and r, and let

$$\Omega' = \{\eta \in \Omega: \textstyle\sum_{x<0}\eta(x) = \sum_{x>0}\eta(x) = \infty\}.$$

For $x \in Z$ and $\eta \in \Omega'$ put

$$\ell_x(\eta) = x - \max\{y < x : \eta(y) = 1\},$$

$$r_x(\eta) = \min\{y > x : \eta(y) = 1\} - x.$$

Then $c(x, \eta)$ is taken to be

$$c(x, \eta) = \beta[\ell_x(\eta), r_x(\eta)] \quad \text{if} \quad \eta(x) = 0$$

and

$$c(x, \eta) = \delta[\ell_x(\eta), r_x(\eta)] \quad \text{if} \quad \eta(x) = 1.$$

This defines $c(x, \eta)$ for $\eta \in \Omega'$, and as we shall see later, it can be defined in a natural way for $\eta \in \Omega\backslash\Omega'$, at least in the context in which we will be working.

In addition to [13], papers which treat various aspects of nearest particle systems are [1], [3] and [5]. The widely studied basic contact process which was introduced by Harris in [6] and is the subject of a recent survey paper by Griffeath [4] is of course the special case of a nearest particle system in which $\delta(\ell, r) = 1$ for $\ell, r \geq 1$, $\beta(1, 1) = 2\lambda$, and $\beta(1, r) = \beta(\ell, 1) = \lambda$ and $\beta(\ell, r) = 0$ for $\ell, r \geq 2$.

Received December 1981; revised May 1982.

[1] Research supported in part by NSF Grant 80-02732.

AMS 1970 *subject classification.* Primary 60K35.

Key words and phrases. Interacting particle systems, ergodicity, reversible Markov processes, free energy.

Spitzer's theorem [13], which is the starting point and motivation for much of the work on nearest particle systems, gives a necessary and sufficient condition for the process to have a nontrivial reversible invariant measure. Recall that an invariant measure for a Markov process is said to be reversible if the stationary process obtained by using that invariant measure as the initial distribution is symmetric in time.

THEOREM 1.1 (Spitzer). *Suppose that $\beta(\ell, r) > 0$ and $\delta(\ell, r) > 0$ for $\ell, r \geq 1$ and that δ is uniformly bounded. Then the nearest particle system has a reversible invariant measure ν which concentrates on Ω' if and only if*

$$\frac{\beta(\ell, r)}{\delta(\ell, r)} = \frac{f(\ell)f(r)}{f(\ell + r)} \quad \text{for} \quad \ell, r \geq 1 \tag{1.2}$$

where

$$f(\ell) > 0, \quad \textstyle\sum_{\ell=1}^{\infty} f(\ell) = 1 \quad \text{and} \quad \textstyle\sum_{\ell=1}^{\infty} \ell f(\ell) < \infty. \tag{1.3}$$

In this case, the unique reversible invariant measure which concentrates on Ω' is the stationary renewal measure ν_f which is determined by the density f.

REMARK. Note that if $f(\ell)$ satisfies (1.2), then so does $f_\theta(\ell) = \theta^\ell f(\ell)$ for $\theta > 0$. Thus the statement in (1.2) and (1.3) should be thought of as asserting the existence of a $\theta > 0$ so that $f_\theta(\ell)$ satisfies (1.3).

An important question which is suggested by this theorem (and is raised in Spitzer's paper) in the case in which there is a reversible renewal measure is whether there are other invariant measures concentrating on Ω'. In this paper, this question will be resolved under some additional assumptions. Most of what we will prove is either easy or well known when $\inf_{\ell,r}\beta(\ell, r) > 0$. Our interest centers on cases in which $\beta(\ell, r)$ is not bounded below.

From this point on, we will assume that

(a) $\delta(\ell, r) = 1$ for $\ell, r \geq 1$.

(b) $\beta(\ell, r) = \beta(r, \ell)$ for $\ell, r \geq 1$.

(c) $\beta(\ell, r)$ is nonincreasing in ℓ and r.

Assumption (a) is made for simplicity only. There would be no essential difference in our results or proofs if (a) were replaced by $0 < \inf_{\ell,r}\delta(\ell, r) \leq \sup_{\ell,r}\delta(\ell, r) < \infty$. Assumption (c) is important, and guarantees that the process is attractive in the sense of Holley [9]. For background on attractiveness and its implications, see [2] or [11]. Note that if β is of the form (1.2), then assumption (c) is equivalent to the statement that $f(n)/f(n + 1)$ is nonincreasing, or equivalently that $\log f(n)$ is convex.

In [3], Gray constructed nearest particle systems on Ω' under the assumption that $\delta(\ell,r)$ is uniformly bounded above by proving existence and uniqueness for an appropriate martingale problem. For our purposes, it will be important to define the process on all of Ω. This is relatively easy to do in the attractive case, as will be seen in Section 2. In many cases, the solution to the martingale problem is in fact not unique on $\Omega \backslash \Omega'$. While we will not formulate the construction in the context of the martingale problem, what we will do has the effect of choosing the "largest" solution when the solution is not unique. The attractiveness assumption makes it possible to compare various solutions so that one can talk about the "largest" one. The use of the largest solution has the advantage of guaranteeing that the resulting process has the Feller property on Ω, and in fact this is the only choice which leads to the Feller property.

The following theorem is due to Holley, but no proof of it has appeared in print, so in Section 2 we will show how to deduce it rather easily from Theorem 1.1. Recall that a spin system is said to be ergodic if (a) it has a unique invariant measure ν and (b) the distribution of the process at time t converges to ν for any initial distribution. One of the major problems in the area of interacting particle systems is to determine which spin systems are ergodic. In the presence of the attractiveness assumption, (a) implies (b) ([9]). Let η_t denote the nearest particle system.

THEOREM 1.4 (Holley). *Suppose that*

$$\beta(\ell, r) = \frac{f(\ell)f(r)}{f(\ell+r)}$$

for some positive function $f(\ell)$, but that for no $\theta > 0$ is it the case that $f_\theta(\ell)$ satisfies (1.3). *Then*

$$\lim_{t\to\infty} P^\eta[\eta_t(x) = 1] = 0$$

for all $x \in Z$ and all $\eta \in \Omega$. Therefore the process is ergodic with invariant measure δ_0, the pointmass on $\eta \equiv 0$.

For our first result, suppose that $\beta(\ell, r)$ is given by (1.2) where f satisfies (1.3). Recall that by attractiveness, $f(n)/f(n+1)$ is nonincreasing, so

$$\gamma = \lim_{n\to\infty} \frac{f(n)}{f(n+1)} \tag{1.5}$$

exists. Since $f(n)$ is a density, $\gamma \geq 1$.

THEOREM 1.6. *If $\gamma > 1$, or if $\gamma = 1$ and*

$$\sum_{n=1}^{\infty} \frac{f^2(n)}{f(2n)} < \infty, \tag{1.7}$$

then ν_f is the only invariant measure for the system which is translation invariant and concentrates on Ω'.

The proof of Theorem 1.6 is given in Section 3. The proof is based on the free energy technique which was developed by Holley in [7] and [8] and by Holley and Stroock in [10]. In our context, new difficulties arise from the fact that the interaction is long range, and especially from the fact that in most cases of interest, $\beta(\ell, r)$ is not bounded away from zero. Assumption (1.7), which is only needed when $\gamma = 1$, is almost always satisfied, as will be seen in the remarks at the end of Section 3.

The second major result of this paper (see Theorems 4.9 and 4.14) deals with the following problem: when is it the case that δ_0 is invariant? Of course, if $\beta(\ell, r)$ is bounded away from zero, then δ_0 is never invariant. In general, however, it may or may not be invariant. Our result gives a type of "integral test" for these alternatives. Put $\alpha(n) = \sum_{\ell+r=n+1}(\ell \wedge r)\beta(\ell, r)$, and let $\tilde{\alpha}(n) = \max_{k\leq n}\alpha(k)$. An interpretation of $\alpha(n)$ is that it gives the rate at which the length of a maximal interval of zeros in η_t decreases. In the following theorem, we do not assume that $\beta(\ell, r)$ has the form (1.2).

THEOREM 1.8. (a) *If $\sum_n 1/\alpha(n) < \infty$, then δ_0 is not invariant.* (b) *If $\sum_n 1/\tilde{\alpha}(n) = \infty$, then δ_0 is invariant.*

Our interest in Theorems 1.6 and 1.8 comes largely from the fact that they can often be used together to prove that reversible attractive nearest particle systems are ergodic. In particular, we obtain the following result.

COROLLARY 1.9. *Suppose $\beta(\ell, r)$ is of the form* (1.2) *where $f(\ell)$ satisfies* (1.3). *Define γ as in* (1.5) *and let $g(n) = \gamma^n f(n)$, which then decreases at a rate which is slower than exponential.*

(a) *If $\sum_{n=1}^{\infty} [n^2 g(n)]^{-1} < \infty$, then the process is ergodic.*
(b) *If $\lim_{n\to\infty} g(n)/g(2n)$ exists and $\sum_{n=1}^{\infty} [n^2 g(n)]^{-1} = \infty$, then the process is not ergodic.*

PROOF. Since $g(n)$ is nonincreasing,

$$\beta(\ell, r) = \frac{g(\ell)g(r)}{g(\ell+r)} \geq g(\ell+r),$$

so

$$\alpha(n) \geq \sum_{1\leq \ell \leq (n/2)} \ell\beta(\ell, n+1-\ell) \geq g(n+1) \sum_{1\leq \ell\leq (n/2)} \ell.$$

Therefore $[\alpha(n)]^{-1}$ is bounded above by a constant multiple of $[n^2 g(n)]^{-1}$. To complete the proof of part (a), use (a) of Theorem 1.8 to conclude that δ_0 is not invariant. Then use Theorem 1.6 and attractiveness as in I.2.2 of [11] to show that starting from δ_0 (or δ_1) the limit of the distribution of the process as $t \to \infty$ is ν_f, so the process is ergodic. In order to verify (1.7) when $\gamma = 1$, note that $\sum_{n=1}^{\infty} [n^2 f(n)]^{-1} < \infty$ implies that $n^2 f(n) \geq \varepsilon$ for some $\varepsilon > 0$, so that

$$\sum_{n=1}^{\infty} \frac{f^2(n)}{f(2n)} \leq 4\varepsilon^{-1} \sum_{n=1}^{\infty} n^2 f^2(n) < \infty$$

since $\sum_{n=1}^{\infty} nf(n) < \infty$, by (1.3). To prove part (b), it suffices to show that δ_0 is invariant, since ν_f is also invariant. Using Theorem 1.8, it is enough to show that $\sum_{n=1}^{\infty} [\tilde{\alpha}(n)]^{-1} = \infty$. But since g is nonincreasing,

$$\alpha(n) = \sum_{\ell+r=n+1} (\ell \wedge r) \frac{g(\ell)g(r)}{g(\ell+r)} \leq 2c \sum_{1\leq \ell\leq (n/2)} \ell g(\ell)$$

where c is an upper bound for $g(n)/g(2n)$. Since the right side above is increasing in n, it follows that

$$\tilde{\alpha}(n) \leq 2c \sum_{1\leq \ell \leq (n/2)} \ell g(\ell).$$

If $\lim_{n\to\infty} g(n)/g(2n) > 2$, it is easy to check, using the monotonicity of $g(n)$, that $ng(n)$ is bounded, so that $\tilde{\alpha}(n)$ is bounded above by a constant multiple of n. Therefore $\sum_{n=1}^{\infty} [\tilde{\alpha}(n)]^{-1} = \infty$ in this case. On the other hand, if $\lim_{n\to\infty} g(n)/g(2n) < 4$, one can check that $\sum_{\ell=1}^{n} \ell g(\ell)$ is bounded above by a constant multiple of $n^2 g(n)$, so that $\sum_{n=1}^{\infty} [n^2 g(n)]^{-1} = \infty$ implies that $\sum_{n=1}^{\infty} [\tilde{\alpha}(n)]^{-1} = \infty$.

We conclude this section by indicating the results which Theorems 1.1 and 1.4 and Corollary 1.9 give for the two parameter family of birth rates

$$\beta(\ell, r) = \lambda\left(\frac{1}{\ell} + \frac{1}{r}\right)^p, \quad \lambda > 0, p > 0$$

which are mentioned in [13]. Note that this $\beta(\ell, r)$ is of the form (1.2) with $f(\ell) = \lambda \ell^{-p}$. Let $\mathscr{I}$ be the set of extreme points of the class of invariant measures for the process which are also translation invariant. For $\lambda > 0$ and $p > 0$, $\nu_{\lambda,p}$ will be the stationary renewal process with density $f_\theta(\ell) = \lambda\theta^\ell \ell^{-p}$ where θ is chosen so that $\sum_{\ell=1}^{\infty} f_\theta(\ell) = 1$ and $\sum_{\ell=1}^{\infty} \ell f_\theta(\ell) < \infty$, when such a θ exists.

COROLLARY 1.10. (a) *If $p < 1$, then the process is ergodic for all $\lambda > 0$ and $\mathscr{I} = \{\nu_{\lambda,p}\}$.* (b) *If $p = 1$, then the process is not ergodic for all $\lambda > 0$ and $\mathscr{I} = \{\delta_0, \nu_{\lambda,1}\}$.* (c) *If $1 < p \leq 2$, then the process is ergodic with $\mathscr{I} = \{\delta_0\}$ if $\lambda \leq (\sum_{\ell=1}^{\infty} \ell^{-p})^{-1}$ and is nonergodic with $\mathscr{I} = \{\delta_0, \nu_{\lambda,p}\}$ if $\lambda > (\sum_{\ell=1}^{\infty} \ell^{-p})^{-1}$.* (d) *If $p > 2$, then the process is ergodic with $\mathscr{I} = \{\delta_0\}$ if $\lambda < (\sum_{\ell=1}^{\infty} \ell^{-p})^{-1}$ and is nonergodic with $\mathscr{I} = \{\delta_0, \nu_{\lambda,p}\}$ if $\lambda \geq (\sum_{\ell=1}^{\infty} \ell^{-p})^{-1}$.*

2. Preliminaries. We begin this section by giving a direct construction of the attractive nearest particle system corresponding to $\beta(\ell, r)$ as a Feller process on all of Ω. For $m \leq n$, let $Z_{m,n} = \{m, m+1, \cdots, n\}$ and $\Omega_{m,n} = \{0, 1\}^{Z_{m,n}}$. Let $X_{m,n}$ be the set of all functions on $\Omega_{m,n}$. Of course $X_{m,n}$ is naturally embedded in $C(\Omega)$, the set of all continuous functions on Ω. Consider the continuous time Markov chain on $\Omega_{m,n}$ in which flips occur on $Z_{m,n}$ according to the nearest particle mechanism with the convention that there are fixed ones at $m-1$ and $n+1$ which are used in determining the nearest one to an $x \in Z_{m,n}$ at times when there are no ones on $Z_{m,n}$ to the right or left of x. Let $S_{m,n}(t)$ be the transition semigroup for this Markov chain, and let $\mathscr{L}_{m,n}$ be its infinitesimal generator, both defined on $X_{m,n}$. Let

$$\mathcal{M}_{m,n} = \{g \in X_{m,n} : \eta \le \zeta \Rightarrow g(\eta) \le g(\zeta)\}$$

be the increasing functions in $X_{m,n}$. A simple and standard coupling argument using the attractiveness of the rates implies that for $g \in \mathcal{M}_{k,\ell}$, $S_{m,n}(t)g(\eta)$ is nonincreasing in n and nondecreasing in m for $m \le k \le \ell \le n$, $t > 0$ and $\eta \in \Omega$. Therefore we can define

$$S(t)g(\eta) = \lim_{n\to\infty, m\to-\infty} S_{m,n}(t)g(\eta) \tag{2.1}$$

for $g \in \cup_{k\le\ell}\mathcal{M}_{k,\ell}$, $t > 0$ and $\eta \in \Omega$. Finally, for $g \in \cup_{k\le\ell}\mathcal{M}_{k,\ell}$, define

$$\mathcal{L}g(\eta) = \textstyle\sum_x c(x,\eta)[g(\eta_x) - g(\eta)]$$

where $\eta_x(y) = \eta(y)$ for $y \ne x$ and $\eta_x(x) = 1 - \eta(x)$. Here $c(x,\eta)$ is defined in terms of $\beta(\ell, r)$ (with $\delta(\ell, r) \equiv 1$) as in the introduction, with the convention that if $\ell_x(\eta)$ or $r_x(\eta)$ is infinite, $\beta(\infty, r)$, $\beta(\ell, \infty)$ and $\beta(\infty,\infty)$ are defined by continuity, which is possible by the attractiveness assumption. Note that $\mathcal{L}g \in C(\Omega)$ and $\lim_{n\to\infty,m\to\infty}\mathcal{L}_{m,n}\, g(\eta) = \mathcal{L}g(\eta)$ uniformly on Ω for $g \in \cup_{k\le\ell}X_{k,\ell}$.

THEOREM 2.2 (a) *$S(t)g \in C(\Omega)$ for $g \in \cup_{k\le\ell}\mathcal{M}_{k,\ell}$ and $t > 0$.* (b) *$S(t)$ can be extended uniquely to all of $C(\Omega)$, and then it maps $C(\Omega)$ to $C(\Omega)$.* (c) *$S_{m,n}(t)g(\eta) \to S(t)g(\eta)$ uniformly in $\eta \in \Omega$ as $m \to -\infty$ and $n \to +\infty$ for any $g \in C(\Omega)$.* (d) *$S(t)$ satisfies the semigroup property on $C(\Omega)$.* (e) *For $g \in \cup_{k\le\ell}X_{k,\ell}$,*

$$\frac{S(t)g - g}{t} \to \mathcal{L}g$$

as $t \to 0$ uniformly on Ω.

PROOF. (a) By the monotonicity of the convergence in (2.1), if $\eta_n \to \eta$,

$$S(t)g(\eta) \ge \limsup_{n\to\infty} S(t)g(\eta_n)$$

for $g \in \cup_{k\le\ell}\mathcal{M}_{k,\ell}$. In order to show that

$$S(t)g(\eta) \le \liminf_{n\to\infty} S(t)g(\eta_n) \tag{2.3}$$

when $\eta_n \to \eta$, it suffices to consider the case in which $\eta_n \le \eta$ for all n, since otherwise η_n can be replaced by $\eta_n \wedge \eta$ since $S(t)g$ is a monotone function. By the nearest particle nature of the interaction and the fact that the death rates are $= 1$, if $g \in X_{k,\ell}$, $|g| \le 1$, $m \le k \le \ell \le j$ and $\eta = \zeta$ on $Z_{k,\ell}$, then

$$|S_{m,j}(t)g(\eta) - S_{m,j}(t)g(\zeta)| \le (1 - e^{-t})^{R_j} + (1 - e^{-t})^{L_m}$$

where

$$R_j = \begin{cases} \sum_{x=\ell+1}^{j^*} \eta(x) & \text{if } j^* < j \\ \infty & \text{if } j^* = j \end{cases}$$

$$L_m = \begin{cases} \sum_{x=m^*}^{k-1} \eta(x) & \text{if } m^* > m \\ \infty & \text{if } m^* = m \end{cases}$$

$$j^* = \max\{x \le j : \eta = \zeta \quad \text{on} \quad Z_{\ell+1,x}\}$$

$$m^* = \min\{x \ge m : \eta = \zeta \quad \text{on} \quad Z_{x,k-1}\}.$$

Letting $m \to -\infty$, $j \to \infty$, we see that if $g \in \mathcal{M}_{k,\ell}$, $|g| \le 1$ and $\eta = \zeta$ on $Z_{k,\ell}$, then

$$|S(t)g(\eta) - S(t)g(\zeta)| \le (1 - e^{-t})^R + (1 - e^{-t})^L \tag{2.4}$$

where $R = \lim_{j\to\infty} R_j$ and $L = \lim_{m\to-\infty} L_m$. Inequality (2.3) now follows quickly when $\eta_n \to \eta$ and $\eta_n \le \eta$ by considering separately the four cases: $\sum_x \eta(x) < \infty$; $\sum_{x<0}\eta(x) < \infty$ and $\sum_{x>0}\eta(x) = \infty$; $\sum_{x<0}\eta(x) = \infty$ and $\sum_{x>0}\eta(x) < \infty$; and $\sum_{x<0}\eta(x) = \sum_{x>0}\eta(x) = \infty$. In the first case, for example, η_n is eventually the same as η, while in the last case, one can use (2.4) with $\zeta = \eta_n$ and note that the right side tends to zero as $n \to \infty$.

(b) This is a consequence of the fact that linear combinations of elements in $\cup_{k\le \ell} \mathcal{M}_{k,\ell}$ are dense in $C(\Omega)$ and that the norm of $S(t)$ is at most one.

(c) For $g \in \cup_{k\le \ell} \mathcal{M}_{k,\ell}$, this is a consequence of (a) and Dini's theorem (page 162 of [12]). Then use the argument in (b) above.

(d) This follows from (c) and the semigroup property for $S_{m,n}(t)$.

(e) For $m \le k \le \ell \le n$ and $g \in X_{k,\ell}$,

$$S_{m,n}(t)g - g = \int_0^t S_{m,n}(u)\mathcal{L}_{m,n} g\, du.$$

Since $\mathcal{L}_{m,n} g \to \mathcal{L} g$ uniformly on Ω for $g \in X_{k,\ell}$, property (c) allows one to take limits to obtain

$$S(t)g - g = \int_0^t S(u)\mathcal{L} g\, du$$

for $g \in \cup_{k\le\ell} X_{k,\ell}$. Therefore $\lim_{t\to 0} S(t)g = g$ uniformly on Ω for $g \in \cup_{k\le\ell} X_{k,\ell}$ and hence for $g \in C(\Omega)$ since $S(t)$ is a contraction. Hence $\lim_{t\to 0}(S(t)g - g)/t = \lim_{t\to 0}(1/t)\int_0^t S(u)\mathcal{L} g\, du = \mathcal{L} g$ uniformly on Ω.

REMARKS. (a) Since the limit in (2.1) exists for each $g \in \cup_{k\le\ell} \mathcal{M}_{k,\ell}$, the process with semigroup $S(t)$ is invariant under translation on Z. (b) By (e) of Theorem 2.2, $\cup_{k\le\ell} X_{k,\ell}$ is a subset of the domain of the generator of $S(t)$. However it is not in general a core for the generator, since if it were, δ_0 would be invariant for the process whenever $\lim_{\ell,r\to 0}\beta(\ell, r) = 0$. But by Theorem 1.8, this is often not the case.

PROOF OF THEOREM 1.4. The Markov chain on $\Omega_{m,n}$ with semigroup $S_{m,n}(t)$ is a finite state irreducible chain which is reversible with respect to the measure $\mu_{m,n}$ defined by

$$\mu_{m,n}(\eta) = c_{m,n} \prod_{i=1}^{k+1} f(x_i - x_{i-1})$$

where $k = \sum_{x\in Z_{m,n}} \eta(x)$ and the x_i's are determined by $\{x_1, \cdots, x_k\} = \{x \in Z_{m,n} : \eta(x) = 1\}$ and $m - 1 = x_0 < x_1 < \cdots < x_k < x_{k+1} = n + 1$. By the general theory of attractive spin systems (see Section I.2.2 of [11], for example), $\mu = \lim_{m\to-\infty, n\to+\infty} \mu_{m,n}$ exists and $\mu = \lim_{t\to\infty} \delta_1 S(t)$, and it is easy to see that either $\mu = \delta_0$ or $\mu(\Omega') = 1$. It therefore suffices to show that the second alternative is ruled out by the assumptions of the theorem. So, suppose that $\mu(\Omega') = 1$. Since $\mu_{m,n}$ is reversible for $S_{m,n}(t)$,

$$\int hS_{m,n}(t)g\, d\mu_{m,n} = \int gS_{m,n}(t)h\, d\mu_{m,n}$$

for $g, h \in X_{k,\ell}$ with $m \le k \le \ell \le n$. By (c) of Theorem 2.2,

$$\int hS(t)g\, d\mu = \int gS(t)h\, d\mu$$

for $g, h \in \cup_{k\le\ell} X_{k,\ell}$, so that μ is reversible for $S(t)$. But then by Theorem 1.1, there is a Θ so that $f_\Theta(\ell)$ satisfies (1.3). That contradicts the assumption of Theorem 1.4.

3. Free energy computations. This section is devoted to the proof of Theorem 1.6. The first and last stages of the argument follow Holley and Stroock [10]. The new complications which arise in the present context involve the estimates which are needed for the terms on the right of the identity in Lemma 3.2 below. If $\beta(\ell, r)$ were bounded below (it is automatically bounded above by attractiveness), the right side of that identity would be trivially $o(n)$ as $n \to \infty$, and that is what is needed to carry out the final part of the proof. Since $\beta(\ell, r)$ is not bounded below in most cases of interest, more delicate estimates are needed.

Perhaps a few remarks on the ideas behind the use of free energy in Markov processes would be helpful in interpreting the somewhat formal expressions and computations in this section. If P_t is the transition operator for a finite state Markov chain with stationary distribution π, one can define the "free energy" $H(\nu)$ for any probability measure ν on the state space by

$$H(\nu) = -\sum_x \nu(x)\log\left[\frac{\nu(x)}{\pi(x)}\right].$$

A simple application of the convexity of the function $u \log u$ shows that $H(\nu P_t)$ is nondecreasing in t. In fact, the derivative of $H(\nu P_t)$ with respect to t can be written as the sum of nonnegative terms. Of course, if ν happens to be stationary for the process, then each of these terms must be zero, and under appropriate irreducibility conditions, this implies that $\nu = \pi$. In applying these ideas to processes with uncountable state spaces such as Ω, one begins by looking at the process just on $\Omega_{0,n}$ which is finite. As a result of the interactions between sites inside and outside $Z_{0,n}$, the process on $\Omega_{0,n}$ is not Markovian, and the derivative of the corresponding free energy contains terms which are not necessarily nonnegative. In the statement of Lemma 3.2 below, the terms which are automatically nonnegative are written on the left, while the terms which arise from the interaction with sites outside $Z_{0,n}$ are on the right. The equality follows from the assumption that μ is invariant, so the time derivative is zero.

Throughout this section, we will assume that $\beta(\ell, r)$ is given by (1.2) (with $\delta(\ell, r) \equiv 1$) where f satisfies (1.3),

$$\frac{f(n)}{f(n+1)} \downarrow \gamma \geq 1 \quad \text{as} \quad n \to \infty \tag{3.1}$$

and f satisfies (1.7) if $\gamma = 1$. Let ν be the renewal measure corresponding to the density f, and let μ be any invariant measure which is translation invariant and concentrates on Ω'. For $n \geq 0$, $x \in Z_{o,n}$, and $\eta \in \Omega_{0,n}$, let

$$a_n(\eta, x) = \int_{\{\zeta:\zeta=\eta \text{ on } Z_{0,n}\}} c(x, \zeta)\mu(d\zeta), \qquad M_n(\eta) = \mu\{\zeta : \zeta = \eta \text{ on } Z_{0,n}\}$$

$$N_n(\eta) = \nu\{\zeta : \zeta = \eta \text{ on } Z_{0,n}\}.$$

LEMMA 3.2.

$$\sum_{\eta\in\Omega_{0,n},x\in Z_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)]\log\frac{a_n(\eta_x, x)}{a_n(\eta, x)}$$
$$= 2\sum_{(x,\eta)\in B_n} [a_n(\eta_x, x) - a_n(\eta, x)]\log\frac{a_n(\eta_x, x)N_n(\eta_x)}{M_n(\eta_x)N_n(\eta)}$$

where

$$B_n = \{(x, \eta) \in Z_{0,n} \times \Omega_{0,n} : \eta(x) = 1 \text{ and } \eta = 0 \text{ on } Z_{0,x-1} \text{ or on } Z_{x+1,n}\}.$$

PROOF. For a fixed $\eta \in \Omega_{0,n}$, let

$$g(\zeta) = \begin{cases} 1 & \text{if} \quad \zeta = \eta \quad \text{on} \quad Z_{0,n} \\ 0 & \text{otherwise.} \end{cases}$$

Apply part (e) of Theorem 2.2 to this function to obtain

$$0 = \int \mathscr{L}g\, d\mu = \sum_{x\in Z_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)].$$

Multiplying this by $\log[M_n(\eta)/N_n(\eta)]$ and summing gives

$$0 = \sum_{x\in Z_{0,n},\eta\in\Omega_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)]\log\frac{M_n(\eta)}{N_n(\eta)}.$$

By making the change of variable $\eta \to \eta_x$, we see that for $x \in Z_{0,n}$,

$$\sum_{\eta\in\Omega_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)]\log\frac{N_n(\eta_x)}{M_n(\eta)} = -\sum_{\eta\in\Omega_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)]\log\frac{N_n(\eta)}{M_n(\eta)}.$$

Therefore, by the previous two identities,

$$0 = \sum_{x\in Z_{0,n},\eta\in\Omega_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)]\log\frac{M_n(\eta)N_n(\eta_x)}{M_n(\eta_x)N_n(\eta)},$$

and hence

$$\sum_{x\in Z_{0,n},\eta\in\Omega_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)]\log\frac{a_n(\eta_x, x)}{a_n(\eta, x)}$$

$$= \sum_{x\in Z_{0,n},\eta\in\Omega_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)]\log\frac{M_n(\eta)N_n(\eta_x)a_n(\eta_x, x)}{M_n(\eta_x)N_n(\eta)a_n(\eta, x)}.$$

Since replacing η by η_x in the summand on the right has no effect, we can replace the right side by twice the same sum over just those η and x for which $\eta(x) = 1$. The required statement then follows from the observations that if $\eta(x) = 1$, then $a_n(\eta, x) = M_n(\eta)$, while if $\eta(u) = \eta(x) = \eta(v) = 1$ for some $0 \le u < x < v \le n$, then

$$\frac{a_n(\eta_x, x)}{M_n(\eta_x)} = \frac{f(x-u)f(v-x)}{f(v-u)} = \frac{N_n(\eta)}{N_n(\eta_x)}.$$

The main work now is to show that the expression on the right of the identity in Lemma 3.2 is $o(n)$ as $n \to \infty$, at least along a subsequence of n's. We do this via the following lemmas.

LEMMA 3.3. $\sum_{(x,\eta)\in B_n} a_n(\eta, x) \le 2$.

PROOF. Since $c(x, \zeta) = 1$ when $\zeta(x) = 1$,

$$\sum_{(x,\eta)\in B_n} a_n(\eta, x) = \sum_{(x,\eta)\in B_n} M_n(\eta) \le 2 \sum_{\eta\in\Omega_{0,n}} M_n(\eta) = 2.$$

Here we have used the fact that for each $\eta \in \Omega_{0,n}$, there are at most 2 x's for which $(x, \eta) \in B_n$.

LEMMA 3.4. *For* $(x, \eta) \in B_n$,

$$\frac{N_n(\eta_x)}{N_n(\eta)} \ge [\beta(1, 1)]^{-1} \quad \textit{and} \quad \frac{a_n(\eta_x, x)}{M_n(\eta_x)} \le \beta(1, 1).$$

PROOF. For $(x, \eta) \in B_n$,

$$N_n(\eta_x) = \sum_{u<x<v} \nu\{\zeta \in \Omega : \zeta = \eta_x \text{ on } Z_{0,n}, \zeta(u) = 1, \zeta(v) = 1, \zeta = 0 \text{ on } Z_{u+1,v-1}\}$$

and

$$N_n(\eta) = \sum_{u<x<v} \nu\{\zeta \in \Omega : \zeta = \eta \text{ on } Z_{0,n}, \zeta(u) = 1, \zeta(v) = 1, \zeta = 0 \text{ on } Z_{u+1,v-1}\}.$$

Therefore for an appropriate choice of coefficients $c(u, v)$,

$$N_n(\eta_x) = \sum_{u<x<v} f(v-u)c(u, v)$$

and

$$N_n(\eta) = \sum_{u<x<v} f(v-x)f(x-u)c(u, v)$$
$$= \sum_{u<x<v} \beta(x-u, v-x)f(v-u)c(u, v) \le \beta(1, 1) \sum_{u<x<v} f(v-u)c(u, v)$$

by the monotonicity of $\beta(\ell, r)$. This monotonicity also gives $a_n(\eta_x, x) \leq \beta(1, 1)M_n(\eta_x)$ for $(x, \eta) \in B_n$.

LEMMA 3.5. *If $\ell, m \geq 1$ and $k \geq \ell + m$, then*

$$\sum_{u+v=k, u\geq \ell, v\geq m} \beta(u, v) \geq f(\ell + m)\gamma^{\ell+m}.$$

PROOF. $\sum_{u+v=k, u\geq \ell, v\geq m} \beta(u, v) \geq \beta(\ell, k-\ell) = \dfrac{f(\ell)f(k-\ell)}{f(k)}$.

Therefore it suffices to show that

$$\frac{f(\ell)f(k-\ell)}{f(\ell+m)f(k)} \geq \gamma^{\ell+m},$$

which is a consequence of the fact that

$$\frac{f(n)}{f(n+1)} \geq \gamma \quad \text{for all} \quad n.$$

Now put $F(k) = \sum_{\ell=k}^{\infty} f(\ell)$,

$$g(k) = \mu\{\zeta \in \Omega : \zeta(0) = 1, \zeta(k) = 1, \text{ and } \zeta = 0 \text{ on } Z_{1,k-1}\},$$

and $G(k) = \sum_{\ell=k}^{\infty} g(\ell)$. Note that $\sum_k F(k) < \infty$ by (1.3) and $\sum_k G(k) < \infty$ since μ is translation invariant and concentrates on Ω'.

LEMMA 3.6. *For $(x, \eta) \in B_n$,*

$$\frac{a_n(\eta_x, x)}{M_n(\eta_x)} \geq \frac{f(n+2)\gamma^{n+2}G(n+2)}{\sum_{k\geq n+2} G(k)}, \qquad \text{and} \qquad \frac{N_n(\eta_x)}{N_n(\eta)} \leq \frac{\sum_{k\geq n+2} F(k)}{f(n+2)\gamma^{n+2}F(n+2)}.$$

PROOF. Suppose $(x, \eta) \in B_n$. If $\eta = 0$ on $Z_{0,n}\backslash\{x\}$, then since μ is translation invariant and concentrates on Ω',

$$\begin{aligned} a_n(\eta_x, x) &= \sum_{u<0, v>n} \beta(x-u, v-x)g(v-u) \\ &= \sum_{k\geq n+2} g(k) \sum_{\ell+r=k, \ell\geq x+1, r\geq n-x+1} \beta(\ell, r) \geq f(n+2)\gamma^{n+2}G(n+2) \end{aligned}$$

by Lemma 3.5. On the other hand,

$$M_n(\eta_x) = \sum_{u<0, v>n} g(v-u) = \sum_{k\geq n+2} g(k)(k-n-1) = \sum_{k\geq n+2} G(k),$$

so the first claim holds for that η. Now suppose that $(x, \eta) \in B_n$, but $\sum_{u\in Z_{0,n}} \eta(u) \geq 2$. Then we may assume without loss of generality that there is some y with $x < y \leq n$ so that $\eta(y) = 1$ and $\eta(z) = 0$ on $Z_{0,y-1}\backslash\{x\}$. Then

$$a_n(\eta_x, x) = \sum_{u<0} \beta(x-u, y-x)\, \mu\{\zeta \in \Omega: \zeta(u) = 1, \zeta = \eta_x \text{ on } Z_{0,n} \text{ and } \zeta = 0 \text{ on } Z_{u+1,0}\}$$

and

$$M_n(\eta_x) = \sum_{u<0} \mu\{\zeta \in \Omega : \zeta(u) = 1, \zeta = \eta_x \text{ on } Z_{0,n} \text{ and } \zeta = 0 \text{ on } Z_{u+1,0}\}.$$

Therefore the first claim follows from

$$\beta(x-u, y-x) \geq f(y-x)\gamma^{y-x} \geq f(n+2)\gamma^{n+2}.$$

The proof of the second claim is similar.

LEMMA 3.7. $\lim_{n\to\infty} (1/n) \log f(n) = -\log \gamma$.

PROOF. $f(n) = f(0) \prod_{k=1}^{n} f(k)/f(k-1)$, so

$$\frac{1}{n}\log f(n) = \frac{1}{n}\log f(0) - \frac{1}{n}\sum_{k=1}^{n}\log\frac{f(k-1)}{f(k)},$$

which converges to $-\log\gamma$ by (3.1).

LEMMA 3.8. *Suppose $u(k) > 0$ and $\sum_{k=1}^{\infty} u(k) < \infty$. Then*

$$\liminf_{n\to\infty}\frac{1}{n}\log\left[\frac{\sum_{k=n}^{\infty} u(k)}{u(n)}\right] = 0.$$

PROOF. Suppose not. Then there is an $\varepsilon > 0$ so that

$$u(n) \le e^{-\varepsilon n}\sum_{k=n}^{\infty} u(k)$$

for all n. Summing for $n \ge m$ gives

$$\sum_{n=m}^{\infty} u(n) \le \sum_{k=m}^{\infty} u(k)\sum_{n=m}^{k} e^{-\varepsilon n} \le \left[\sum_{k=m}^{\infty} u(k)\right]\frac{e^{-\varepsilon m}}{1-e^{-\varepsilon}}$$

and hence

$$1 - e^{-\varepsilon} \le e^{-\varepsilon m}$$

for all m. Letting $m \to \infty$ yields a contradiction.

LEMMA 3.9.

$$\liminf_{n\to\infty}\left\{\frac{1}{n}\sum_{(x,\eta)\in B_n}[-a_n(\eta, x)]\log\frac{a_n(\eta_x, x)N_n(\eta_x)}{M_n(\eta_x)N_n(\eta)}\right\} \le 0.$$

PROOF. By Lemmas 3.3, 3.4 and 3.6, the expression in braces is bounded above by

$$\frac{2}{n}\left|\log\beta(1, 1) - \log f(n+2) - (n+2)\log\gamma + \log\left[\frac{\sum_{k\ge n+2} G(k)}{G(n+2)}\right]\right|.$$

The result now follows from Lemmas 3.7 and 3.8.

LEMMA 3.10. (a) $\lim_{n\to\infty}(1/n)\sum_{(x,\eta)\in B_n} a_n(\eta_x) = 0$.

(b) *If $\gamma = 1$, then* $\sup_n \sum_{(x,\eta)\in B_n} a_n(\eta_x, x) < \infty$.

PROOF. By Lemma 3.4, it is enough to prove (a) with $a_n(\eta_x, x)$ replaced by $M_n(\eta_x)$. But

$$\begin{aligned}\sum_{(x,\eta)\in B_n} M_n(\eta_x) &= \sum_{x=0}^{n}\sum_{\eta:(x,\eta)\in B_n} M_n(\eta_x) \le 2\sum_{x=0}^{n}\mu\{\zeta:\zeta = 0 \text{ on } Z_{0,x}\}\\ &= 2\sum_{x=0}^{n}\sum_{k=x+2}^{\infty} G(k) = 2\sum_{k=2}^{\infty} G(k)(n+1)\wedge(k-1),\end{aligned}$$

so (a) follows from $\sum_k G(k) < \infty$ and the dominated convergence theorem. For part (b), assume $\gamma = 1$. By assumption (1.7) and the attractiveness assumption,

$$\sum_{u+v=n,u,v\ge1}\beta(u, v) \le 2\sum_{1\le k\le(n/2)}\beta(k, k) \le 2\sum_{k=1}^{\infty}\frac{f^2(k)}{f(2k)} < \infty.$$

Therefore

$$\begin{aligned}\sum_{(x,\eta)\in B_n} a_n(\eta_x, x) &= \sum_{\eta\in\Omega_{0,n}}\sum_{x:(x,\eta_x)\in B_n} a_n(\eta, x)\\ &\le 2\left[2\sum_{k=1}^{\infty}\frac{f^2(k)}{f(2k)}\right]\sum_{\eta\in\Omega_{0,n}} M_n(\eta) \le 4\sum_{k=1}^{\infty}\frac{f^2(k)}{f(2k)} < \infty.\end{aligned}$$

LEMMA 3.11. *If $\gamma = 1$, then*

$$\limsup_{n\to\infty}\left\{\frac{1}{n}\sum_{(x,\eta)\in B_n} a_n(\eta_x, x)\log\frac{a_n(\eta_x, x)N_n(\eta_x)}{M_n(\eta_x)N_n(\eta)}\right\} \le 0.$$

PROOF. By Lemmas 3.4, 3.6 and 3.10 (b), the expression in braces is bounded above by a constant multiple of

$$\frac{1}{n}\left|\log\beta(1, 1) + \log\frac{\sum_{k\geq n+2} F(k)}{f(n+2)F(n+2)}\right|.$$

The result now follows from Lemma 3.7 since $\gamma = 1$ and $f(n) \leq F(n) \leq \sum_{k\geq n} F(k)$.

LEMMA 3.12. *Suppose $\gamma > 1$. Then*

(a) $$\lim_{n\to\infty}\frac{F(n)}{f(n)} = \frac{\gamma}{\gamma - 1} < \infty \quad \textit{and}$$

(b) $$\lim_{n\to\infty}\frac{\sum_{k=n}^{\infty} F(k)}{f(n)} = \left(\frac{\gamma}{\gamma-1}\right)^2 < \infty.$$

PROOF. $\frac{F(n)}{f(n)} = \sum_{k=n}^{\infty}\frac{f(k)}{f(n)}$, AND $\frac{f(k)}{f(n)} \leq \frac{1}{\gamma^{k-n}}$ for $k \geq n$, so that part (a) follows from (3.1) and the dominated convergence theorem. Part (b) is similar.

LEMMA 3.13 *Suppose $\gamma > 1$. Then for some constant M independent of n, x and η, for all $(x, \eta) \in B_n$,*

(a) $$\frac{N_n(\eta_x)}{N_n(\eta)} \leq M[\beta(x+1, n-x+1)]^{-1} \qquad \textit{if } \eta = 0 \textit{ on } Z_{0,n}\backslash\{x\}.$$

(b) $$\frac{N_n(\eta_x)}{N_n(\eta)} \leq M[\beta(x+1, v-x)]^{-1} \qquad \textit{if } x < v \leq n \textit{ and}$$

$$\eta = 0 \textit{ on } Z_{0,v-1}\backslash\{x\}.$$

PROOF. For part (a), suppose $\eta = 0$ on $Z_{0,n}\backslash\{x\}$. Then

$$N_n(\eta_x) = \sum_{u<0,v>n}\nu\{\zeta:\zeta(0) = 1\}f(v-u) = \nu\{\zeta:\zeta(0) = 1\}\sum_{k=n+2}^{\infty} F(k), \quad \text{and}$$

$$N_n(\eta) = \sum_{u<0,v>n}\nu\{\zeta:\zeta(0) = 1\}f(x-u)f(v-x) = \nu\{\zeta:\zeta(0) = 1\}F(n+1-x)F(x+1).$$

Therefore

$$\frac{N_n(\eta_x)}{N_n(\eta)} = \frac{\sum_{k=n+2}^{\infty} F(k)}{f(n+2)}\cdot\frac{f(n+1-x)f(x+1)}{F(n+1-x)F(x+1)}[\beta(x+1, n-x+1)]^{-1},$$

so part (a) follows from Lemma 3.12. Part (b) is similar.

LEMMA 3.14. *If $\gamma > 1$, then*

$$\limsup_{n\to\infty}\left\{\frac{1}{n}\sum_{(x,\eta)\in B_n} a_n(\eta_x, x)\log\frac{a_n(\eta_x, x)N_n(\eta_x)}{M_n(\eta_x)N_n(\eta)}\right\} \leq 0.$$

PROOF. By Lemmas 3.4, 3.10, 3.13 and by the monotonicity of $\beta(\ell, r)$, it suffices to prove that the limits of

$$\frac{1}{n}\sum_{x=0}^{n}\beta(x+1, n-x+1)\mu\{\zeta:\zeta = 0 \text{ on } Z_{0,n}\}\left|\log\beta(x+1, n-x+1)\right|$$

and

$$\frac{1}{n}\sum_{x=0}^{n-1}\sum_{v=x+1}^{n}\beta(x+1, v-x)\,\mu\{\zeta:\zeta = 0 \text{ on } Z_{0,v}, \zeta(v) = 1\}\left|\log\beta(x+1, v-x)\right|$$

as $n \to \infty$ are both zero. Since the function $t\log t$ is bounded on compact subsets of

$[0, \infty)$ and $\beta(\ell, r)$ is uniformly bounded, these statements reduce to

$$\lim_{n\to\infty} \mu\{\zeta : \zeta = 0 \text{ on } Z_{0,n}\} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v=1}^{n} v\mu\{\zeta : \zeta = 0 \text{ on } Z_{0,v} \quad \text{and} \quad \zeta(v) = 1\} = 0.$$

But these are both true because $\sum_k G(k) < \infty$.

For $x \in Z_{0,n}$, let

$$D_n(x) = \sum_{\eta\in\Omega_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)]\log \frac{a_n(\eta_x, x)}{a_n(\eta, x)}.$$

Note that $D_n(x) \geq 0$ for all x and n.

COROLLARY 3.15.

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{x=0}^{n} D_n(x) = 0.$$

PROOF. This follows immediately from Lemmas 3.2, 3.9, 3.11 and 3.14.

PROOF OF THEOREM 1.6. As observed in [10],

$$D_m(x) \leq D_n(x)$$

for $m \leq n$ and $x \in Z_{0,m}$ since the function

$$(s, t) \to (s - t)\log \frac{s}{t}$$

is homogeneous and convex. Therefore, using the translation invariance of μ, we see that for $x \in Z_{0,m}$ and $n \geq m$,

$$D_m(x) \leq \frac{1}{n - m + 1} \sum_{k=x}^{n-m+x} D_n(k).$$

So, by Corollary 3.15, $D_m(x) = 0$ for all $m \geq 0$ and all $x \in Z_{0,m}$. From this definition of $D_m(x)$, we see that

$$a_n(\eta_x, x) = a_n(\eta, x)$$

for all $n \geq 0$, $x \in Z_{0,n}$ and $\eta \in \Omega_{0,n}$. This easily implies that $\mu = \nu = \nu_f$.

We conclude this section with two remarks which are intended to show that Assumption (1.7) is almost always satisfied when $\gamma = 1$. (Of course there is no additional assumption in Theorem 1.6 when $\gamma > 1$.) First note that (1.7) is immediate if $\sup_n f(n)/f(2n) < \infty$, so in particular it holds whenever $f(n)$ is regularly varying. More generally, (1.7) can only fail if $n^2 f^2(n)/f(2n)$ oscillates between 0 and ∞ as $n \to \infty$. To see this, note that if $\sup_n[n^2 f^2(n)/f(2n)] < \infty$, then (1.7) is certainly satisfied, while if $\inf_n[n^2 f^2(n)/f(2n)] > 0$, then

$$f^2(n) \geq \varepsilon \frac{f(2n)}{n^2}$$

for all n and some $\varepsilon > 0$. But then

$$\log f(n) \geq \tfrac{1}{2} \log \varepsilon + \tfrac{1}{2} \log f(2n) - \log n,$$

which after iterating and passing to the limit gives

$$\log f(n) \geq \log \varepsilon - 2 \log n - 2 \log 2.$$

Here we have used the assumption that $\gamma = 1$ and Lemma 3.7 to show that the error term tends to zero in the limit. Thus

$$f(n) \geq \frac{\varepsilon}{4n^2},$$

which contradicts the assumption in (1.3) that $\sum_n nf(n) < \infty$.

4. Spontaneous entry into Ω'. This section is devoted to the proof of Theorem 1.8, which appears here as Theorems 4.9 and 4.14. The proof proceeds via a series of lemmas. Let $E_{m,n}$ denote the expectation operator for the process with semigroup $S_{m,n}(t)$ which was described at the beginning of Section 2, and let $\tilde{E}_{m,n}$ be the expectation for this process which is modified by suspending the deaths (i.e., by taking $\delta(\ell, r) \equiv 0$). Put

$$\tau = \min\{t > 0 : \eta_t(0) = 1\}.$$

The symbol $\mathbf{o}$ will be used to denote the configuration $\eta \equiv 0$.

LEMMA 4.1. *δ_0 is invariant for the nearest particle process if and only if*

$$\lim_{m \to -\infty, n \to +\infty} E^{\mathbf{o}}_{m,n}(e^{-\tau}) = 0. \tag{4.2}$$

PROOF. Since the deaths occur independently of the rest of the process at rate one,

$$E^{\mathbf{o}}_{m,n}[\eta_t(0) \mid \tau \leq t] \geq e^{-t}.$$

By the definition of τ, when the initial configuration is $\mathbf{o}$,

$$\{\eta_t(0) = 1\} \subset \{\tau \leq t\}.$$

Hence

$$e^{-t} P^{\mathbf{o}}_{m,n}(\tau \leq t) \leq P^{\mathbf{o}}_{m,n}(\eta_t(0) = 1) \leq P^{\mathbf{o}}_{m,n}(\tau \leq t).$$

Therefore $P^{\mathbf{o}}[\eta_t(0) = 1] = \lim_{m \to -\infty, n \to +\infty} P^{\mathbf{o}}_{m,n}[\eta_t(0) = 1] = 0$ if and only if

$$\lim_{m \to -\infty, n \to +\infty} P^{\mathbf{o}}_{m,n}[\tau \leq t] = 0,$$

which gives the required result.

LEMMA 4.3. $\lim_{m \to -\infty, n \to +\infty} E^{\mathbf{o}}_{m,n}(e^{-\tau}) = 0$ *if and only if*

$$\lim_{m \to -\infty, n \to +\infty} \tilde{E}^{\mathbf{o}}_{m,n}(e^{-\tau}) = 0.$$

PROOF. First note that both limits exist by attractiveness. Couple together in the natural way copies of η_t, $\tilde{\eta}_t$ and W where η_t and $\tilde{\eta}_t$ are processes with semigroup $S_{m,n}(t)$ and $\tilde{S}_{m,n}(t)$ respectively, $\eta_0 = \tilde{\eta}_0 = \mathbf{o}$, and W is an exponentially distributed random variable with mean ½, in such a way that $\eta_t \leq \tilde{\eta}_t$ for all t, $\eta_t(0) = \tilde{\eta}_t(0)$ for all $t < W$, and W is independent of the process $\{\tilde{\eta}_t, t > 0\}$. The role of W is to give the time of the first death for the η_t process at one of the two sites containing the nearest one to the right or left of site 0. Note that in general the two sites in question vary with time, but this does not create difficulties because the distribution is exponential. Put $\tilde{\tau} = \min\{t > 0 : \tilde{\eta}_t(0) = 1\}$. Then $\tilde{\tau} \leq \tau$ and $\{\tilde{\tau} < W\} = \{\tau < W\}$, so

$$E(e^{-\tau}) \leq E(e^{-\tilde{\tau}})$$

and for any $M > 0$,

$$\begin{aligned}[E(e^{-\tilde{\tau}})]^2 \leq E(e^{-2\tilde{\tau}}) &= P(\tilde{\tau} < W) = P(\tau < W) \\ &\leq P(\tau \leq M) + P(W > M) \leq e^M E(e^{-\tau}) + e^{-2M}.\end{aligned}$$

These inequalities imply the statement of the lemma.

LEMMA 4.4. *For $m, n \geq 1$, put*

$$h(m, n) = \tilde{E}^{o}_{-(m-1),n-1}(e^{-\tau}).$$

Then $h(m, n)$ is a nonincreasing function of m and n and satisfies the recursion

$$\begin{aligned} h(m, n)[1 + \textstyle\sum_{\ell+r=m+n} \beta(\ell, r)] &= \beta(m,n) + \textstyle\sum_{k=1}^{m-1} \beta(m-k, n+k)h(k,n) \\ &\quad + \textstyle\sum_{k=1}^{n-1} \beta(m+k, n-k)h(m, k). \end{aligned} \tag{4.5}$$

PROOF. The monotonicity of $h(m, n)$ is a consequence of the attractiveness assumption, as mentioned earlier in the proof of Lemma 4.3. The recursion follows from the fact that the first birth in $[-m + 1, n - 1]$ occurs at an exponential time with mean $[\sum_{\ell+r=n+m} \beta(\ell, r)]^{-1}$ and at a site j with probability

$$\beta(m + j, n - j)[\textstyle\sum_{\ell+r=n+m} \beta(\ell, r)]^{-1}.$$

Recall that there are no deaths in the tilde process.

LEMMA 4.6. *For $N \geq 1$, put*

$$h(N) = \frac{1}{N} \textstyle\sum_{k=1}^{N} h(k, N + 1 - k).$$

(a) *$h(N)$ is nonincreasing in N.*
(b) *$h(N)$ satisfies the recursion*

$$\begin{aligned} h(N)[1 + \textstyle\sum_{\ell+r=N+1} \beta(\ell, r)] &= \frac{1}{N} \textstyle\sum_{\ell+r=N+1} \beta(\ell, r) \\ &\quad + \frac{2}{N} \textstyle\sum_{k=1}^{N-1} kh(k)\beta(k + 1, N - k). \end{aligned} \tag{4.7}$$

(c) $\lim_{N\to\infty} h(N) = \lim_{m,n\to\infty} h(m, n)$.

PROOF. Part (a) is a consequence of the monotonicity of $h(m, n)$, since

$$\begin{aligned} h(N) - h(N + 1) &= \textstyle\sum_{k=1}^{N} \frac{k}{N(N + 1)} [h(k, N + 1 - k) - h(k + 1, N + 1 - k)] \\ &\quad + \textstyle\sum_{k=1}^{N} \frac{N - k + 1}{N(N + 1)} [h(k, N + 1 - k) - h(k, N + 2 - k)]. \end{aligned}$$

The recursion in (b) is obtained by summing (4.5) on those $m, n \geq 1$ for which $m + n = N + 1$. Part (c) is immediate.

Now let $\alpha(n) = \sum_{\ell+r=n+1} (\ell \wedge r)\beta(\ell, r)$ for $n \geq 1$ as in the introduction.

LEMMA 4.8. $\dfrac{\alpha(n + 1)}{\alpha(n)} \leq \dfrac{n + 2}{n}$.

PROOF. $(n + 2) \sum_{\ell+r=n+1} (\ell \wedge r)\beta(\ell, r)$

$$\geq \textstyle\sum_{\ell=1}^{n} \{\ell[(\ell+1) \wedge (n + 1 - \ell)] + (n + 1 - \ell) [(n + 2 - \ell) \wedge \ell]\}\beta(\ell, n + 1 - \ell)$$

and

$$\begin{aligned} n \textstyle\sum_{\ell+r=n+2} (\ell \wedge r)\beta(\ell, r) &\leq \textstyle\sum_{\ell=1}^{n+1} \{(\ell - 1)[\ell \wedge (n - \ell + 2)] \\ &\quad + (n + 1 - \ell)[(n + 2 - \ell) \wedge \ell]\}\beta(\ell, n + 2 - \ell) \\ &= \textstyle\sum_{\ell=1}^{n} \ell[(\ell + 1) \wedge (n - \ell + 1)]\beta(\ell + 1, n + 1 - \ell) \\ &\quad + \textstyle\sum_{\ell=1}^{n} (n + 1 - \ell)[(n + 2 - \ell) \wedge \ell]\beta(\ell, n + 2 - \ell). \end{aligned}$$

The result then follows from the attractiveness assumption.

THEOREM 4.9. *If* $\sum_n 1/\alpha(n) < \infty$, *then*

$$\lim_{N\to\infty} h(N) > 0 \tag{4.10}$$

and hence δ_0 *is not invariant.*

PROOF. Use (4.7) for $N = 2^{n+1}$ together with $h(k) \ge h(2^n)$ if $k \le 2^n$ and $h(k) \ge h(2^{n+1})$ if $k \le 2^{n+1}$ to obtain

$$h(2^{n+1})[1 + \textstyle\sum_{\ell+r=2^{n+1}+1} \beta(\ell, r)]$$
$$\ge \frac{h(2^n)}{2^n} \sum_{k=1}^{2^n} k\beta(k+1, 2^{n+1}-k) + \frac{h(2^{n+1})}{2^n} \sum_{k=2^n+1}^{2^{n+1}-1} k\beta(k+1, 2^{n+1}-k)$$
$$+ \frac{h(2^{n+1})}{2^{n+1}} \sum_{\ell+r=2^{n+1}+1} \beta(\ell, r).$$

Since

$$\sum_{\ell+r=n+1} \beta(\ell, r) = \frac{2}{N-1} \sum_{k=1}^{N-1} k\beta(k+1, N-k), \tag{4.11}$$

this gives

$$h(2^{n+1})[1 + \frac{1}{2^n} \sum_{k=1}^{2^n} k\beta(k+1, 2^{n+1}-k)] \ge \frac{h(2^n)}{2^n} \sum_{k=1}^{2^n} k\beta(k+1, 2^{n+1}-k).$$

Iterating this and using the summation test for the convergence of infinite products, one sees that $\lim_{n\to\infty} h(2^n) > 0$ provided that

$$\sum_{n=1}^{\infty} \left[1 + \frac{1}{2^n} \sum_{k=1}^{2^n} k\beta(k+1, 2^{n+1}-k)\right]^{-1} < \infty. \tag{4.12}$$

By the monotonicity of $\beta(k, \ell)$,

$$\frac{1}{2}\alpha(2^{n+1}) \le \sum_{k=1}^{2^n} k\beta(k+1, 2^{n+1}-k) + \sum_{k=1}^{2^n} \beta(k, 2^{n+1}-k+1)$$
$$\le 2^n\beta(1, 1) + \sum_{k=1}^{2^n} k\beta(k+1, 2^{n+1}-k).$$

Therefore $\sum_{n=0}^{\infty} 2^{n+1}[\alpha(2^{n+1})]^{-1} < \infty$ implies (4.12). But by Lemma 4.8,

$$\frac{1}{4} \sum_{n=0}^{\infty} \frac{2^n}{\alpha(2^n)} \le \sum_{n=0}^{\infty} \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{\alpha(k)} = \sum_{n=2}^{\infty} \frac{1}{\alpha(n)}.$$

Thus $\sum_n 1/\alpha(n) < \infty$ implies $\lim_{n\to\infty} h(2^n) > 0$ and hence (4.10) by (a) of Lemma 4.6. The final statement now follows from Lemmas 4.1 and 4.3 and part (c) of Lemma 4.6.

LEMMA 4.13. *Suppose* $\sum_n 1/\alpha(n) = \infty$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell+r=n+1} \beta(\ell, r) = 0.$$

PROOF. By the attractiveness assumption, $(1/n) \sum_{\ell+r=n+1} \beta(\ell, r)$ is nonincreasing in n. The proof of this is the same as that of part (a) of Lemma 4.6. Let

$$c = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell+r=n+1} \beta(\ell, r).$$

Then

$$\sum_{\ell+r=n+1} \beta(\ell, r) \ge nc$$

for all n. By attractiveness again, $\beta(\ell, r)$ is bounded above by $\beta(1, 1)$, so for any $\varepsilon > 0$,

$$\sum_{\ell+r=n+1,\ell,r\le\varepsilon n} \beta(\ell, r) \le 2\varepsilon n\beta(1, 1),$$

and hence

$$\sum_{\ell+r=n+1,\ell,r>\varepsilon n} \beta(\ell, r) \ge n[c - 2\varepsilon\beta(1, 1)].$$

Therefore

$$\alpha(n) \ge \varepsilon n \sum_{\ell+r=n+1,\ell,\ r>\varepsilon n} \beta(\ell, r) \ge \varepsilon n^2[c - 2\varepsilon\beta(1, 1)].$$

Therefore if $c > 0$, $\inf_n[\alpha(n)n^{-2}] > 0$, which contradicts the hypothesis.

THEOREM 4.14. *Put $\tilde{\alpha}(n) = \max_{k\le n}\alpha(k)$. Suppose*

$$\sum_n \frac{1}{\tilde{\alpha}(n)} = \infty.$$

Then

$$\lim_{N\to\infty} h(N) = 0 \tag{4.15}$$

and hence δ_0 is invariant.

PROOF. Let $\Delta(k) = h(k-1) - h(k)$, which is nonnegative by part (a) of Lemma 4.6. Then for $k < N$,

$$h(k) = h(N) + \sum_{\ell=k+1}^{N} \Delta(\ell). \tag{4.16}$$

Substitute for $h(k)$ on the right side of (4.7) the expression in (4.16) and use (4.11) to obtain

$$h(N)\left[1 + \frac{1}{N}\sum_{\ell+r=N+1} \beta(\ell, r)\right] = \frac{1}{N}\sum_{\ell+r=N+1} \beta(\ell, r) + \frac{2}{N}\sum_{\ell=2}^{N} \Delta(\ell) \sum_{k=1}^{\ell-1} k\beta(k+1, N-k).$$

Now assume that $\lim_{N\to\infty} h(N) > 0$, from which we will deduce a contradiction. By this assumption and Lemma 4.13, we can deduce from the above identity that there is a $c > 0$ and an N_0 so that for $N \ge N_0$,

$$\frac{1}{N}\sum_{\ell=2}^{N} \Delta(\ell) \sum_{k=1}^{\ell-1} k\beta(k+1, N-k) \ge c. \tag{4.17}$$

Let $u(N)$ be a nonnegative function. Multiplying (4.17) by $Nu(N)$ and summing the resulting inequality for $N \ge N_0$ gives

$$\sum_{N=N_0}^{\infty} u(N) \sum_{\ell=2}^{N} \Delta(\ell) \sum_{k=1}^{\ell-1} k\beta(k+1, N-k) \ge c\sum_{N=N_0}^{\infty} Nu(N).$$

Since $\sum_n \Delta(n) < \infty$, we will reach a contradiction if we can choose $u(N) \ge 0$ in such a way that

$$\sum_N Nu(N) = \infty \quad \text{and} \tag{4.18}$$

$$\sup_{\ell\ge 2} \sum_{1\le k<\ell\le N} u(N)k\beta(k+1, N-k) < \infty. \tag{4.19}$$

If $\alpha(k)$ is bounded, then it is easily seen that $u(N) = N^{-2}$ satisfied both (4.18) and (4.19). Therefore, we can assume that $\tilde{\alpha}(k) \uparrow \infty$. Now put

$$u(N) = \frac{1}{\tilde{\alpha}(N)} - \frac{1}{\tilde{\alpha}(N+1)}.$$

Then

$$\sum_{N=1}^{\infty} Nu(N) = \sum_{N=1}^{\infty} u(N) \sum_{k=1}^{N} 1 = \sum_{k=1}^{\infty} \sum_{N=k}^{\infty} u(N) = \sum_{k=1}^{\infty} \frac{1}{\tilde{\alpha}(k)},$$

so (4.18) follows from the assumption of the theorem. To verify (4.19), consider separately the sums over the following three ranges of summation: (a) $N \geq 2\ell$, (b) $\ell \leq N < 2\ell$ and $K \leq N/2$, and (c) $\ell \leq N < 2\ell$ and $N/2 < k < \ell$. By attractiveness, the sum over range (a) is dominated by

$$\textstyle\sum_{1 \leq k < \ell} k\beta(k+1, 2\ell - k) \sum_{N \geq 2\ell} u(N) \leq \frac{\alpha(2\ell)}{\tilde{\alpha}(2\ell)} \leq 1.$$

The sum over range (b) is dominated by

$$\textstyle\sum_{N=\ell}^{2\ell-1} \alpha(N) u(N)$$

which by Lemma 4.8 is bounded by

$$4\alpha(\ell) \textstyle\sum_{N=\ell}^{2\ell-1} u(N),$$

which is at most 4. Finally, the sum over range (c) is dominated by

$$\ell \sup_{\ell < N < 2\ell} u(N) \textstyle\sum_{\ell \leq N < 2\ell, N/2 < k < \ell} \beta(k+1, N-k). \tag{4.20}$$

By attractiveness, if $[x]$ denotes the integer part of x,

$$\begin{aligned} &\textstyle\sum_{\ell \leq N < 2\ell, N/2 < k < \ell} \beta(k+1, N-k) \\ &\quad \leq \textstyle\sum_{\ell \leq N < 2\ell, N/2 < k < \ell} \beta\left(k + 1 - \left[\frac{N-\ell}{2}\right], \ell - k + \left[\frac{N-\ell}{2}\right]\right) \\ &\quad \leq \alpha(\ell) \leq \tilde{\alpha}(\ell). \end{aligned} \tag{4.21}$$

For a given N, either $\tilde{\alpha}(N) = \tilde{\alpha}(N+1)$ or

$$\alpha(N) \leq \tilde{\alpha}(N) < \tilde{\alpha}(N+1) = \alpha(N+1),$$

so by Lemma 4.8,

$$\frac{\tilde{\alpha}(N+1)}{\tilde{\alpha}(N)} \leq \frac{N+2}{N}.$$

Therefore

$$u(N) = \frac{1}{\tilde{\alpha}(N)}\left[1 - \frac{\tilde{\alpha}(N)}{\tilde{\alpha}(N+1)}\right] \leq \frac{2}{(N+2)\tilde{\alpha}(N)},$$

so

$$\sup_{\ell \leq N < 2\ell} u(N) \leq \frac{2}{(\ell + 2)\tilde{\alpha}(\ell)}.$$

The boundedness of (4.20) then follows from this and (4.21), thus completing the proof.

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