The Annals of Probability
1983, Vol. 11, No. 2, 419–427

APPROXIMATING IMRL DISTRIBUTIONS BY EXPONENTIAL DISTRIBUTIONS, WITH APPLICATIONS TO FIRST PASSAGE TIMES[1]

BY MARK BROWN

City College, CUNY

It is shown that if F is an IMRL (increasing mean residual life) distribution on $[0, \infty)$ then:

$$\max\{\sup_t |\bar{F}(t) - \bar{G}(t)|, \sup_t |\bar{F}(t) - e^{-t/\mu}|, \sup_t |\bar{G}(t) - e^{-t/\mu}|, \sup_t |\bar{G}(t) - e^{-t/\mu_G}|\} = \frac{\rho}{\rho+1} = 1 - \frac{\mu}{\mu_G}$$

where $\bar{F}(t) = 1 - F(t)$, $\mu = E_F X$, $\mu_2 = E_F X^2$, $G(t) = \mu^{-1} \int_0^t \bar{F}(x)\, dx$, $\mu_G = E_G X = \mu_2/2\mu$, and $\rho = \mu_2/2\mu^2 - 1 = \mu_G/\mu - 1$. Thus if F is IMRL and ρ is small then F and G are approximately equal and exponentially distributed. IMRL distributions with small ρ arise naturally in a class of first passage time distributions for Markov processes, as first illuminated by Keilson. The current results thus provide error bounds for exponential approximations of these distributions.

1. Introduction. The exponential distribution satisfies $\rho = |\mu_2/2\mu^2 - 1| = 0$ where $\mu = EX$ and $\mu_2 = EX^2$. In general a small value of ρ does not in itself imply approximate exponentiality. For example the binomial distribution with $n = 1$ and $p = ½$ has $\rho = 0$. It is reasonable to conjecture, however, that within certain classes of distributions a small value of ρ does imply approximate exponentiality. In these cases, given the class and the first two moments of the distribution, it would be desirable to obtain bounds on the distance from exponentiality.

The above problem for the class of completely monotone distributions (mixtures of exponential distributions) has received some recent attention. The motivation for this interest is that completely monotone distributions with small ρ arise naturally in first passage time distributions for Markov processes, as first illuminated by Keilson ([11]–[14]). Keilson and Steutel [15] suggested ρ as a measure of departure from exponentiality within this class. Keilson [11], using results of Heyde [8], derived

$$d(F, \mu\varepsilon) = \sup |\bar{F}(x) - e^{-x/\mu}| \le k\rho^{1/4} \tag{1.1}$$

where k is bounded above by 4.41. Heyde and Leslie [9] improved the right hand side of (1.1) to 3.74ρ and Hall [7] obtained a further improvement to 2.77ρ.

This paper considers IMRL (increasing mean residual life) distributions, a class considerably larger than that of completely monotone distributions. (IMRL distributions are defined and discussed in Section 2). Defining $G(x) = \mu^{-1} \int_0^x \bar{F}(s)$ (the stationary renewal distribution), $b\varepsilon$ as an exponential distribution with mean b, $\mu_G = E_G X = \mu_2/2\mu$, $\sigma^2 = \mathrm{Var}_F X$, $d(F_1, F_2) = \sup_t |F_1(t) - F_2(t)|$ and $d^*(F_1, F_2) = \sup_\beta |F_1(B) - F_2(B)|$ where β is the collection of Borel subsets of $[0, \infty)$, we obtain:

$$d(F, \mu\varepsilon) \le \frac{\rho}{\rho+1} = 1 - \frac{\mu}{\mu_G} = 1 - \frac{2\mu^2}{\mu_2} = \frac{\sigma^2 - \mu^2}{\sigma^2 + \mu^2} \tag{1.2}$$

$$d^*(F, G) \le \frac{\rho}{\rho+1} \tag{1.3}$$

Received September 1979; revised November 1982.

[1] Research supported by Air Force Office of Scientific Research, AFSC, USAF, under Grant No. AFOSR F49620-79-C-0157 and Grant No. AFOSR 81-0038.

AMS 1970 *subject classifications.* Primary 62E10; secondary 60K10.

Key words and phrases. IMRL distributions, approximate exponentiality, first passage times, Markov processes, reliability theory, inequalities.

(1.4) $$d^*(G, \mu\varepsilon) \le \frac{\rho}{\rho + 1}$$

(1.5) $$d(G, \mu_G\varepsilon) \le \frac{\rho}{\rho + 1}.$$

Since $F = G$ if and only if F is exponential, one would expect that for small ρ, F and G are approximately equal and that G is approximately exponential. Expressions (1.2)–(1.5) give error bounds for these approximations.

The inequalities (1.2) and (1.3) are shown to be sharp even within the subclass of completely monotone distributions. Thus $\rho/(\rho + 1)$ is the best upper bound for both $d(F, \mu\varepsilon)$ and $d(F, G)$ (for F IMRL, $d(F, G) = d^*(F, G)$).

The current results thus extend the domain of applicability from completely monotone to IMRL distributions, improve the upper bound for $d(F, \mu\varepsilon)$ from 2.77ρ to its best value $\rho/(\rho + 1)$, apply to distributions of the form $G(x) = \mu^{-1}\int_0^x \bar{F}(s)\, ds$, with F IMRL, without requiring knowledge of $E_G X^2$ (which may be infinite), and bound the distance between F and G.

The inequalities (1.2)–(1.5) are derived along with related results in Section 4. The sharpness of (1.2) and (1.3) is discussed in Section 5. Further bounds for the completely monotone class are derived in Section 6. It is remarked in Section 7 that the results immediately apply to the mixtures of distributions from proportional hazard families. In Section 8 the above inequalities are applied to the class of first passage time distributions considered by Keilson ([11]–[14]) which, as mentioned above, motivated much of the interest in this problem.

2. IMRL and related distributions. A distribution F is defined to be IMRL on $[0, \infty)$ if $\mu = \int x\, dF(x) < \infty$, $F(0-) = 0$, $F(0) < 1$ and $E(X - t \mid X > t) = (\int_t^\infty \bar{F}(x)\, dx)/\bar{F}(t)$ is increasing in $t \ge 0$. A distribution F is defined to be DFR on $[0, \infty)$ if $F(0-) = 0$, $F(0) < 1$ and $\Pr(X > s + t \mid X > t) = \bar{F}(s + t)/\bar{F}(t)$ is increasing in $t \ge 0$ for each $s > 0$. In our work below all IMRL and DFR distributions will be on $[0, \infty)$ as defined above and we will leave out the phrase "on $[0, \infty)$" in referring to them.

Since stochastic ordering implies ordering of means, F DFR and $\mu < \infty$ implies F IMRL. However, it is easy to provide examples in which F is IMRL but not DFR.

Define $G(t) = \mu^{-1}\int_0^t \bar{F}(x)\, dx$, the stationary renewal distribution corresponding to F. Note that G is absolutely continuous with failure rate function $h_G(t) = \bar{F}(t)/\mu\bar{G}(t) = [E(X - t \mid X > t)]^{-1}$. Therefore F IMRL $\Leftrightarrow$ G DFR. It follows that if F is DFR with $F(0) = 0$ and $f(0+) < \infty$ (F is necessarily absolutely continuous with its pdf f possessing a version which is decreasing) then F is the stationary renewal distribution corresponding to the IMRL distribution with survival function $L(x) = f(x)/f(0+)$.

If F is IMRL then h_G is decreasing. Defining $q = \bar{F}(0)$ we thus have:

(2.1) $$h_G(t) \le h_G(0) \le q/\mu \le 1/\mu.$$

It immediately follows that:

(2.2) $$\bar{G}(t) \ge e^{-tq/\mu} \ge e^{-t/\mu}.$$

A consequence of (2.2) is that for all decreasing functions ℓ:

(2.3) $$\int \ell(t)\, dG(t) = \mu^{-1}\int \ell(t)\bar{F}(t)\, dt \le \mu^{-1}\int \ell(t)e^{-t/\mu}\, dt.$$

Since $\bar{G}(t)/\bar{F}(t) = \mu^{-1}E(X - t \mid X > t)$ it follows that for F IMRL:

(2.4) $$\bar{G}(t)/\bar{F}(t) \text{ is increasing.}$$

Furthermore since $\bar{G}(0)/\bar{F}(0) = q^{-1}$, it follows from (2.4) that:

(2.5) $$\bar{G}(t) \ge q^{-1}\bar{F}(t) \ge \bar{F}(t).$$

3. Distance between distributions. Suppose that $X \sim F_1$, and $Y \sim F_2$. Then:

$$\sup|\bar{F}_1(t) - \bar{F}_2(t)| \le \sup_\beta|F_1(B) - F_2(B)| \le \Pr(X \neq Y). \tag{3.1}$$

Thus given two distributions F_1, F_2 we try to construct random variables X, Y with $X \sim F_1$, $Y \sim F_2$ and $\Pr(X \neq Y)$ small. Then (3.1) is invoked to show that F_1 and F_2 are close. This is the approach of Hodges and LeCam [10] in their study of Poisson approximation to sums of independent Bernoulli variables. The construction for X, Y is found in Lemma 3.3. The author employed a similar construction in [6].

If F_1, F_2 are absolutely continuous with respect to μ, with Radon-Nikodyn derivatives f_1, f_2 then:

$$\int |f_1 - f_2|\, d\mu = 2\int_{f_1>f_2} (f_1 - f_2)\, du = 2\sup_\beta|F_1(B) - F_2(B)|. \tag{3.2}$$

Thus if $\Pr(X = Y)$ is small then f_1, f_2 are close in $L_1(\mu)$ norm.

(3.3) LEMMA. *Suppose that F_1, F_2 are distributions with support* $[0, \infty)$, $F_2(0) = 0$, $H_i(t) = -\mathrm{Ln}(\bar{F}_i(t)|\bar{F}_i(0))$ $i = 1, 2$ *are continuous, and $\bar{F}_2(t)/\bar{F}_1(t)$ is increasing in t. Then*:

$$\sup|F_1(B) - F_2(B)| \le 1 - \int_0^\infty \bar{F}_1(t)\, dH_2(t). \tag{3.4}$$

PROOF. Construct two independent non-homogeneous Poisson processes, $\{N_i(t), t \ge 0\}$ $i = 1, 2$, with $EN_1(t) = H_2(t)$ and $EN_2(t) = H_1(t) - H_2(t)$. Note that $\bar{F}_2/\bar{F}_1$ increasing implies $H_1 - H_2$ is increasing. Define Y_i to be the first event epoch from process i, $i = 1, 2$. Construct ρ independent of the two Poisson processes with $\Pr(\rho = 0) = p = F_1(0)$, $\Pr(\rho = 1) = \bar{F}_1(0)$. Define $X = \rho\min(Y_1, Y_2)$ and $Y = Y_1$. Now $\Pr(Y_1 > t) = \Pr(N_1(t) = 0) = e^{-H_2(t)} = \bar{F}_2(t)$ thus $Y \sim F_2$. Similarly, $\Pr(X > t) = \Pr(\rho = 1)\Pr(N_1(t) + N_2(t) = 0) = qe^{-H_1(t)} = \bar{F}_1(t)$, thus $X \sim F_1$. Finally:

$$\begin{aligned}\Pr(X = Y) &= \Pr(\rho = 1)\Pr(Y_1 < Y_2) \\ &= q\int_0^\infty e^{-(H_1(t)-H_2(t))}e^{-H_2(t)}\, dH_2(t) = \int \bar{F}_1(t)\, dH_2(t).\end{aligned} \tag{3.5}$$

The result now follows from (3.1) and (3.5)

Given two distribution functions F_1, F_2 define $d(F_1, F_2) = \sup|F_1(t) - F_2(t)| = \sup|\bar{F}_1(t) - \bar{F}_2(t)|$.

(3.6) LEMMA. *Suppose that for each t either $F_3(t) \ge \max(F_1(t), F_2(t))$ or $F_3(t) \le \min(F_1(t), F_2(t))$. Then $d(F_1, F_2) \le \max(d(F_1, F_3), d(F_2, F_3))$.*

PROOF. The above condition immediately implies that

$$|F_1(t) - F_2(t)| \le \max(|F_1(t) - F_3(t)|, |F_2(t) - F_3(t)|) \quad \text{for all} \quad t.$$

Taking the sup of both sides gives the result.

4. Inequalities. Recall that $d(F_1, F_2)$ is defined to equal $\sup|F_1(t) - F_2(t)|$. Similarly define $d^*(F_1, F_2) = \sup_\beta|F_1(B) - F_2(B)|$ where β is the collection of Borel subsets of $[0, \infty)$. Let $b\varepsilon$ denote an exponentiality distributed random variable with mean b.

(4.1) THEOREM. *Assume that F is* IMRL *with $\mu < \infty$, $G(x) = \mu^{-1}\int_0^x \bar{F}(s)\, ds$, $p = F(0)$ and $q = 1 - p$. Then*:

(i) $$d^*(F, G) \le \frac{\rho}{\rho + 1} = 1 - \frac{\mu}{\mu_G} = 1 - \frac{2\mu^2}{\mu_2} = \frac{\sigma^2 - \mu^2}{\sigma^2 + \mu^2}$$

(ii) $$\bar{F}(t) \le \bar{G}(t) \le \bar{F}(t) + \frac{\rho}{\rho + 1} \quad \textit{for all} \quad t$$

(iii) $$d^*(G, \mu\varepsilon) \le \frac{\rho}{\rho + 1}$$

(iv) $$e^{-t/\mu} \le \bar{G}(t) \le e^{-t/\mu} + \frac{\rho}{\rho + 1} \quad \textit{for all} \quad t$$

(v) $$d(F, \mu\varepsilon) \le \frac{\rho}{\rho + 1}$$

(vi) $$d(G, b\varepsilon) \le \begin{cases} \dfrac{\rho}{\rho + 1} & \textit{for} \quad \mu \le b \le \mu_G \\ 1 - \mu b^{-1} & \textit{for} \quad b \ge \mu_G. \end{cases}$$

In particular $d(G, \mu_G\varepsilon) \le \dfrac{\rho}{\rho + 1}$

(vii) $$\sup_t |\bar{F}(t) - qe^{-qt/\mu}| \le \frac{q\rho - p}{\rho + 1}$$

(viii) $$d^*(G, q^{-1}\mu\varepsilon) \le \frac{\rho - (p/q)}{\rho + 1}$$

(ix) $$e^{-qt/\mu} \le \bar{G}(t) \le e^{-qt/\mu} + \frac{\rho - (p/q)}{\rho + 1} \quad \textit{for all} \quad t \ge 0$$

(x) $$d(G, b\varepsilon) \le \begin{cases} \dfrac{\rho - (p/q)}{\rho + 1} & \textit{for} \quad q^{-1}\mu \le b \le \mu_G \\ 1 - \mu(bq)^{-1} & \textit{for} \quad b \ge \mu_G \end{cases}$$

(xi) *Assume that F is* DFR *with* $F(0) = 0$ *and* $f(0+) = a < \infty$. *Then*

$$d^*(F, a^{-1}\varepsilon) \le 1 - (a\mu)^{-1}$$

(xii) *Under the conditions of* (xi), $e^{-at} \le \bar{F}(t) \le e^{-at} + 1 - (a\mu)^{-1}$ *for all* $t \ge 0$

(xiii) *Under the conditions of* (xi):

$$d(F, b\varepsilon) \le \begin{cases} 1 - (a\mu)^{-1} & \textit{for} \quad a^{-1} \le b \le \mu \\ 1 - (ba)^{-1} & \textit{for} \quad b \ge \mu. \end{cases}$$

PROOF. (i) By (2.4), $\bar{G}(t)/\bar{F}(t)$ is increasing. Therefore Lemma (3.3) is applicable with $F_1 = F$, $F_2 = G$. Using the Cauchy-Schwartz inequality and Lemma (3.3) we obtain,

$$d^*(F, G) \le 1 - \int_0^\infty \bar{F}(t) h_G(t)\, dt = 1 - \frac{\mu_G}{\mu} \int_0^\infty \left(\frac{\bar{F}(t)}{\bar{G}(t)}\right)^2 \frac{\bar{G}(t)}{\mu_G}\, dt$$

$$\le 1 - \frac{\mu_G}{\mu} \left[\int_0^\infty \frac{\bar{F}(t)}{\bar{G}(t)} \frac{\bar{G}(t)}{\mu_G}\, dt\right]^2 = \frac{\rho}{\rho + 1}.$$

(ii) follows from (i) and (2.5).

(iii) By (2.1) and Lemma (3.3):

$$d^*(G, \mu\varepsilon) \le 1 - \int_0^\infty e^{t/\mu} h_G(t)\, dt. \tag{4.2}$$

Since h_G is decreasing it follows from (2.3) that:

$$\int_0^\infty e^{-t/\mu} h_G(t)\, dt \ge \int_0^\infty \bar{F}(t) h_G(t)\, dt. \tag{4.3}$$

The result now follows from (4.2), (4.3) and the inequality $\int \bar{F}(t) h_G(t)\, dt \ge 1/(\rho + 1)$, derived in the proof of part (i) of this theorem.

(iv) follows from (iii) and (2.2).

(v) From (2.2) and (2.5), $\bar{G}(t) \ge \max(\bar{F}(t), e^{-t/\mu})$. The result now follows from Lemma (3.6) and parts (i) and (iii) of this theorem.

(vi) For $b \ge \mu$, $e^{-t/b} \ge e^{-t/\mu}$; thus by (2.2), $e^{-t/\mu} \le \min(\bar{G}(t), e^{-t/b})$ for all $t \ge 0$. By Lemma (3.3), $d(\mu\varepsilon, b\varepsilon) \le 1 - \mu b^{-1}$; thus by Lemma (3.6) and part (iii) of this theorem:

$$d(G, b\varepsilon) \le \max\left(\frac{\rho}{\rho + 1}, 1 - \mu b^{-1}\right). \tag{4.4}$$

For $b \le \mu_G$, $\dfrac{\rho}{\rho + 1} = 1 - \mu\mu_G^{-1} \ge 1 - \mu b^{-1}$ and for $b \ge \mu_G$ the inequality reverses.

(vii) $\bar{K}(t) = q^{-1}\bar{F}(t)$ is the survival function of an IMRL distribution with

$$\mu_K = q^{-1}\mu_F, \quad \mu_{2K} = q^{-1}\mu_{2F} \quad \text{and} \quad \rho_K + 1 = q(\rho_F + 1).$$

Thus from (v):

$$\sup_t |\bar{K}(t) - e^{-qt/\mu_F}| \le \frac{\rho_K}{\rho_K + 1} = \frac{\rho_F - \dfrac{p}{q}}{\rho_F + 1}. \tag{4.6}$$

Now multiply both sides of (4.6) by q to obtain the desired result.

(viii) $\bar{K}(t) = q^{-1}\bar{F}(t)$ is the survival function of an IMRL distribution with the same stationary renewal distribution as F and the same failure rate function as F. Simply apply (iii) using $\mu_K = q^{-1}\mu_F$ and $\mu_{2K} = q^{-1}\mu_{2F}$.

(ix) follows from (viii) and (2.2).

(x) This result follows from (vi) using the distribution K as in (viii).

(xi) Take a version of f for which $f(0) = f(0+) = a$. Set $L(x) = f(x)/f(0)$, $x \ge 0$.

Then L is the survival function of an IMRL distribution on $[0, \infty)$. Moreover F is the stationary distribution corresponding to L, $\mu_L = 1/f(0) = 1/a$, and $\mu_F = \mu_{2L}/2\mu_L$. The result now follows from (iii).

(xii) Since $h_F(t) \le h_F(0) = a$, $\bar{F}(t) \ge e^{-at}$. The result thus follows from (xi).

(xiii) Follows from (vi) using a similar argument as in (xi). □

(4.7) REMARK. The inequality cited in the summary follows from (i), (iii), (v) and (vi) of Theorem (4.1).

(4.8) REMARK. If, in addition to μ_1 and μ_2, $p = F(0)$ is known then the approximation $\bar{F}(t) \approx qe^{-qt/\mu}$ is suggested. Part (vi) of Theorem (4.1) gives an error bound of $(q\rho - p)/(\rho + 1)$ for this approximation.

(4.9) REMARK. If F is DFR than its failure rate function, $h_F(t)$, has a limit as $t \to \infty$. Call this limit γ. Consider the case $\gamma > 0$. Noting that $h_F \ge \gamma$ and applying Lemma (3.3) we obtain:

$$d^*(F, \gamma^{-1}\varepsilon) \le 1 - \gamma\mu. \tag{4.10}$$

It is easily shown that h_G has the same limit as h_F, and thus applying Lemma (3.3) to G we obtain:

$$d^*(G, \gamma^{-1}\varepsilon) \le 1 - \gamma\mu_G. \tag{4.11}$$

Note that $1 - \gamma\mu_G \le 1 - \gamma\mu$. Since $\text{Ln}(e^{\gamma t}\bar{F}(t)) = \text{Ln}(q) + \int_0^t (\gamma - h_F(s))\, ds$, it follows that $e^{\gamma t}\bar{F}(t)$ is decreasing. Call its limit c; $c \ge 0$ with equality if and only if $\int_0^\infty (h_F(s) - \gamma)\, ds = \infty$.

When $c > 0$, $\gamma > 0$, the approximation $\bar{F}(t) \approx ce^{-\gamma t}$ will be preferable to $e^{-t/\mu}$ for t sufficiently large.

Note that since $\bar{F}(t) \approx c_F e^{-\gamma t}$, $\bar{G}(t) \approx c_G e^{-\gamma t}$, and $\lim_{t\to\infty} (\bar{F}(t)/\bar{G}(t)) = \mu \lim_{t\to\infty} h_G(t) = \gamma\mu$, it follows that $c_F = \gamma\mu c_G$.

(4.12) Remark. Let F be a distribution on $[0, \infty)$ with failure rate function h. Define γ to be the essential infimum of h, and assume that $\gamma > 0$. Then by Lemma (3.3):

$$d^*(F, \gamma^{-1}\varepsilon) \le 1 - \gamma\mu = 1 - \frac{E\left(\dfrac{1}{h(X)}\right)}{\sup\left(\dfrac{1}{h(x)}\right)} \tag{4.13}$$

where $\sup(1/h(x))$ is the essential supremum.

(4.14) Remark. F is defined to be NBUE (new better than used in expectation) if $E(X - t \mid X > t) \le EX < \infty$ for all $t \ge 0$. This is equivalent to F stochastically larger than G, where G is the stationary renewal distribution corresponding to F. Since $h_G(t) = [E(X - t \mid X > t)]^{-1}$ F NBUE implies that $h_G(t) \ge \mu^{-1} = h_G(0)$ for all t. Thus (4.13) applies with F replaced by G and $1 - \gamma\mu$ by ρ.

(4.15) Remark. In Brown [6], Theorem 2, it was shown that if $Z(t)$ is the forward recurrence time at t for a renewal process with IMRL interarrival time distribution F, then $F_{Z(t)}$, the distribution of $Z(t)$, is stochastically larger than F and stochastically smaller than G. Thus:

$$\max(d(F, F_{Z(t)}), d(F_{Z(t)}, G)) \le d(F, G) \le \frac{\rho}{\rho + 1}. \tag{4.16}$$

Since $\bar{G}(x) \ge \max(\bar{F}_{Z(t)}^{(x)}, e^{-x/\mu})$ it follows from Lemma (3.6), Theorem (4.1) and (4.16) that:

$$d(F_{Z(t)}, \mu\varepsilon) \le \max(d(F_{Z(t)}, G), d(\mu\varepsilon, G)) \le \max(d(F, G), d(\mu\varepsilon, G)) \le \frac{\rho}{\rho + 1}. \tag{4.17}$$

(4.18) Remark. If $a < b$ then $d(a\varepsilon, b\varepsilon) = (1 - a/b)(a/b)^{a/b - a}$. In various bounds derived above (Theorem (4.1) parts (vi), (x) and (xiii)) the upper bound, $1 - a/b$, rather than the exact distance was used. Thus the results can be somewhat strengthened at the cost of extra computation.

(4.19) Remark. Suppose that F is DFR. The following inequality will be derived:

$$\bar{F}(t) \ge e^{-((t/\mu) + \rho)} \quad \text{for} \quad t \ge 0. \tag{4.20}$$

It is known (Barlow and Marshall [2] page 1267) that $\bar{F}(t) \le e^{-t/\mu}$ for $0 \le t \le \mu$, so (4.20) leads to:

$$e^{-((t/\mu)+\rho)} \le \bar{F}(t) \le e^{-t/\mu} \quad \text{for} \quad 0 \le t \le \mu.$$

To prove (4.20), let m denote the renewal density for a renewal process with interarrival time distribution F, $A(t)$ the renewal age at time t, h the failure rate function of F, $p =$

$F(0)$ and $q = \bar{F}(0)$. Since $m(s) = q^{-1}Eh(A(s))$ with $h\downarrow$ and $A(s) \le s$, it follows that:

$$m(s) \ge q^{-1}h(s). \tag{4.21}$$

Integrate (4.21) from 0 to t to obtain:

$$M(t) - q^{-1} \ge q^{-1} \int_0^t h(s)\, ds. \tag{4.22}$$

Thus from (4.22):

$$\bar{F}(t) = qe^{-\int_0^t h(s)ds} \ge qe^{-(qM(t)-1)}. \tag{4.23}$$

From Brown [6] Lemma 2:

$$\bar{F}(t) = qe^{-\int_0^t h(s)ds} \ge qe^{-(qM(t)-1)}. \tag{4.23}$$

Thus (4.23) and (4.24) yield:

$$qe^{-(qM(t)-1)} \ge e^{-(t/\mu+\rho)}[qe^{p(t/\mu+\mu_2/2\mu^2)}]. \tag{4.25}$$

Finally for F DFR with $F(0) = p$, $\bar{F}(t) = q\bar{K}(t)$ where $\bar{K}(t) = q^{-1}\bar{F}(t)$ is DFR. But then $\rho_K \ge 0$ and $\rho_F = q^{-1}\rho_K + p/q \ge p/q$. Thus $qe^{p(\rho+1)} \ge qe^{p/q} \ge q(1 + p/q) = 1$. Therefore the quantity in brackets in (4.25) is at least 1, and (4.20) follows.

5. Sharpness of bounds. Given ρ and μ, a convenient IMRL distribution with these parameters is the one with survival function:

$$\bar{F}(t) = \frac{1}{\rho+1} e^{-(t/(\rho+1)\mu)}. \tag{5.1}$$

The stationary distribution G corresponding to F is exponential with parameter $1/(p + 1)\mu$. It follows that:

$$d^*(F, G) = |\bar{F}(0) - \bar{G}(0)| = \frac{\rho}{\rho+1} \tag{5.2}$$

$$d(F, \mu\varepsilon) = |\bar{F}(0) - 1| = \frac{\rho}{\rho+1}. \tag{5.3}$$

Since F is IMRL, (5.2) and (5.3) demonstrate that the inequalities in parts (i) and (v) of Theorem (4.1) are sharp.

The distribution F is a mixture of two exponentials, one with mean zero (failure rate ∞) and the other with mean $(\rho + 1)\mu$. The class of completely monotone distributions consists of mixtures of exponential distributions. It is not clear whether a degenerate distribution at $\{0\}$ is allowable as an exponential distribution (with failure rate ∞). Thus depending on which definition of completely monotone is employed, either F (defined in (5.1) above) is completely monotone or else is the limit in distribution of a sequence of completely monotone distributions. Either way it follows that $\rho/(\rho + 1)$ is the best upper bound for $d(F, \mu\varepsilon)$ and $d^*(F, G)$ as F ranges through the class of completely monotone distributions.

6. Further results for completely monotone distributions. It is instructive to look at the bounds of Theorem (4.1) for the class of completely monotone distributions.

Represent a completely monotone random variable Y by $Y = U\varepsilon = \Lambda^{-1}\varepsilon$ where U and ε are independent, $\Lambda = U^{-1}$, and ε is exponential with parameter 1. A failure rate of ∞ (a mean of 0) corresponds to a degenerate distribution at $\{0\}$ while a failure rate of 0 (a mean of ∞) corresponds to a degenerate distribution at $\{\infty\}$. Note that:

$$\rho = \frac{\sigma_U^2}{(EU)^2}. \tag{6.1}$$

Thus ρ is the coefficient of variation of the random mean U. Therefore (v) of Theorem (4.6) can be expressed as:

$$d(F, (EU)\varepsilon) \le \frac{\sigma_U^2}{\sigma_U^2 + (EU)^2}. \tag{6.2}$$

Suppose that $F(0) = 0$ and $f(0+) = E\Lambda = EU^{-1} < \infty$. Then (xi) and (xiii) of Theorem (4.1) give:

$$d^*(F, (E\Lambda)^{-1}\varepsilon) \le 1 - (E\Lambda EU)^{-1} \tag{6.3}$$

$$d(F, b\varepsilon) \le \begin{cases} 1 - (E\Lambda EU)^{-1} & \text{for } (E\Lambda)^{-1} \le b \le EU \\ 1 - (bE\Lambda)^{-1} & \text{for } b \ge EU. \end{cases} \tag{6.4}$$

7. Proportional hazard families. Consider a family of distributions on $[0, \infty)$ with survival functions:

$$\bar{F}_\lambda(t) = (\bar{F}(t))^\lambda \tag{7.1}$$

where $\lambda > 0$. If F is continuous with $R(t) = -\text{Ln}\bar{F}(t)$, and ε is exponential with parameter 1, then $Y_\lambda = R^{-1}(\lambda^{-1}\varepsilon)$ is easily seen to have the distribution F_λ of (7.1).

Suppose now that the parameter Λ is random with distribution H. We are interested in bounding the distance between the mixture $\int F_\lambda \, dH(\lambda)$ and F_b a distribution with fixed parameter. Note that:

$$\begin{aligned} &\sup | \Pr(R^{-1}(\Lambda^{-1}\varepsilon) > t) - \Pr(R^{-1}(b^{-1}\varepsilon) > t) | \\ &= \sup | \Pr(\Lambda^{-1}\varepsilon > R(t)) - \Pr(b^{-1}\varepsilon > R(t)) | \\ &= \sup | \bar{F}_{\Lambda^{-1}\varepsilon}(x) - e^{-bx} | \\ &= d(\Lambda^{-1}\varepsilon, b^{-1}\varepsilon). \end{aligned} \tag{7.2}$$

Consequently the desired distance is equivalent to the distance between the completely monotone distribution of $\Lambda^{-1}\varepsilon$ and the exponential distribution with parameter b. Therefore the bounds (6.2)–(6.4) apply.

8. Application to first passage times. Keilson [11] page 133 writes "If a system is modeled by a finite Markov chain which is ergodic, the passage time from some specified initial distribution over the state space to a subset B of the state space visited infrequently is often exponentially distributed to good approximation. . . . For engineering purposes, it is essential to quantify departure from exponentiality via error bounds. When one is dealing with time reversible chains, e.g., systems with independent Markov components, the complete monotonicity present permits quantification and the error bounds needed."

Keilson's interesting approach defines two special (meaning specific distributions governing the initial state) first passage times, T_V (the ergodic sojourn time) and T_E (the ergodic exit time). For finite state ergodic Markov processes in continuous time Keilson ([11], Theorem 6.7A) derives:

$$\bar{F}_E(t) = \frac{1}{\mu_V} \int_t^\infty \bar{F}_V(s) \, ds \tag{8.1}$$

where F_E, F_V are the cdf's of T_E, T_V and $\mu_V = ET_V$. Thus F_E is the stationary renewal distribution corresponding to F_V.

If in addition the process is time reversible F_E and F_V are IMRL, thus Theorem (4.1) is applicable with $F = F_V$ and $G = F_E$. We thus obtain:

$$\max(d^*(F_V, F_E), d(F_V, \mu_V\varepsilon), d^*(F_E, \mu_V\varepsilon), d(F_E, \mu_E\varepsilon)) \le \frac{\rho}{\rho + 1} = 1 - \frac{\mu_V}{\mu_E}. \tag{8.2}$$

Note that the exponential approximation to F_E is obtained without requiring knowledge or finiteness of ET_E^2. This latter quantity is needed to apply the bounds of Keilson [11], Heyde and Leslie [9], and Hall [7].

(8.3) EXAMPLE. Consider a system with three i.i.d. components. A component alternates between exponential visits to state 1, with parameter $\gamma = .01$, and exponential visits to state 0 with parameter $\mu = 1$. Of interest is the first passage time to $B = \{(0, 0, 0)\}$. In the language of reliability theory, we have a repairable three component parallel system with component failure rate .01 and component repair rate 1. The time to first system failure is the first passage time to B. Since B is a rarely visited set (the stationary probability of B is $(101)^{-3}$) we anticipate approximate exponentiality for T_E, T_V. Now (Brown [5])

$$ET_E = \frac{1}{1.01}\left(\frac{(101)^3}{(101)^3 - 1}\right) \sum_{r=1}^{3} \binom{3}{r} \frac{(100)^r}{r} = 345{,}181.85;$$

furthermore $ET_V = ((101)^3 - 1)/3 = 343{,}433.33$, thus $\rho = .005091$ and $\rho/(\rho + 1) = .005066$. Therefore by (8.3) $\bar{F}_V(t)$, $\bar{F}_E(t)$ and e^{-t/μ_V} are all within a distance of .005066 for all t.

REFERENCES

[1] BARLOW, R. E. and MARSHALL, A. W. (1964). Bounds for distributions with monotone hazard rate, I and II. *Ann. Math. Statist.* **35** 1234–1274.

[2] BARLOW, R. E., MARSHALL, A. W., and PROSCHAN, F. (1963). Properties of probability distributions with monotone hazard rate. *Ann. Math. Statist.* **34** 375–389.

[3] BARLOW, R. E. and PROSCHAN, F. (1976). Theory of maintained systems. Distribution of time to first system failure. *Math. Oper. Res.* **1** 32–42.

[4] BARLOW, R. E. and PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart, and Winston, New York.

[5] BROWN, M. (1975). The first passage time distribution for a parallel exponential system with repair. *Reliability and Fault Tree Analysis,* edited by Barlow, R. E., et. al.; SIAM, Philadelphia, 365–396.

[6] BROWN, M. (1980). Bounds, inequalities, and monotonicity properties for some specialized renewal processes. *Ann. Probability* **8** 227–240.

[7] HALL, P. (1979). On measures of the distance of a mixture from its parent distribution. *Stochastic Process. Appl.* **8** 357–365.

[8] HEYDE, C. C. (1975). Kurtosis and departures from normality. *Statistical Distributions in Scientific Work, Vol. 1,* edited by Patil, Kotz, and J. K. Ord. Reidel, Dortrecht, 193–220.

[9] HEYDE, C. C. and LESLIE, J. R. (1976). On moment measures of departure from normal and exponential laws. *Stochastic Process. Appl.* **4** 317–328.

[10] HODGES, J. L. and LECAM, L. (1960). The Poisson approximation to the Poisson binomial distribution. *Ann. Math. Statist.* **31** 737–740.

[11] KEILSON, J. (1979). *Markov Chain Models-Rarity and Exponentiality.* Springer-Verlag, New York.

[12] KEILSON, J. (1975). Systems of independent Markov components and their transient behavior. *Reliability and Fault Tree Analysis,* edited by Barlow, R. E., et al.; SIAM, Philadelphia, 351–364.

[13] KEILSON, J. (1974). Monotonicity and convexity in system survival functions and metabolic disappearance curves. *Reliability and Biometry,* edited by Proschan, F. and Serfling, R.; SIAM, Philadelphia. 81–89.

[14] KEILSON, J. (1966). A limit theorem for first passage times in ergodic regenerative processes. *Ann. Math. Statist.* **37** 866–870.

[15] KEILSON, J. and STEUTEL, F. W. (1974). Mixtures of distributions, moment inequalities and measures of exponentiality and normality. *Ann. Probability* **2** 112–130.

67 TINTERN LANE
SCARSDALE, NEW YORK 10583