The Annals of Probability
1983, Vol. 11, No. 2, 270–276

A NEW PROOF OF THE HARTMAN-WINTNER LAW OF THE ITERATED LOGARITHM

By Alejandro de Acosta

Instituto Venezolano de Investigaciones Cientificas and University of Wisconsin, Madison

A new proof of the Hartman-Wintner law of the iterated logarithm is given. The main new ingredient is a simple exponential inequality. The same method gives a new, simpler proof of a basic result of Kuelbs on the LIL in the Banach space setting.

1. Introduction. The main object of this note is to give a new proof of one of the fundamental strong laws of classical Probability Theory, the Hartman-Wintner law of the iterated logarithm (LIL). In the original proof [6] the result was obtained from Kolmogorov's LIL via a very delicate truncation procedure; a contemporary presentation along these lines may be found in [3]. A proof given by Heyde [7] (see also [11], Chapter 10) depends on a refinement of the Berry-Esseen theorem. Strassen's proof [13] is based on an advanced tool, the Skorohod representation, and other more recent proofs by strong approximation methods (see [5]) require delicate constructions and estimates. In contrast to these proofs, the one presented here is based only on the (Lindeberg-Lévy) central limit theorem and basic probability tools, such as the Borel-Cantelli and Kronecker lemmas.

An additional feature of interest of our proof is that it extends to the case of random vectors taking values in a Banach space. We are thus able to give a new, simpler proof of a basic result of Kuelbs [9] on the LIL in the Banach space setting.

In Section 2 we prove the Hartman-Wintner law in Strassen's more complete formulation. The proof is arranged as follows. The upper bound for $\{(2n \log \log n)^{-1/2}S_n\}$ is obtained by means of a simple exponential inequality—Lemma 2.2—allied with standard elementary arguments. Instead of obtaining the lower bound by proving first an exponential inequality—which is often difficult, as in the proof of Kolmogorov's LIL—we identify directly the whole cluster set of $\{(2n \log \log n)^{-1/2}S_n\}$ by means of an elementary (asymptotic) moderate deviation result. In this part of the proof we follow ideas arising from recent work on cluster sets in more general situations ([1] and [2]).

Section 3 contains the proof of Kuelbs' theorem. The proof is based on an exponential inequality—(3.5)—similar to that given in Lemma 2.2. We also invoke the reduction given by Kuelbs' necessary and sufficient conditions for the (compact) LIL in the Banach space setting ([8], Corollary 3.1).

Notation. We put $\mathrm{LL}n = \log \log n$ for $n \geq 3$, $\mathrm{LL}n = 1$ for $n = 1, 2$. For $n \geq 1$, $a_n = (2n\, \mathrm{LL}n)^{1/2}$. The distance from a point x to a set A (in $\mathbb{R}^1$ or in a Banach space B) will be denoted $d(x, A)$. The cluster set of a sequence $\{x_n\}$ will be written $C(\{x_n\})$ (in $\mathbb{R}^1$ or in B).

2. The Hartman-Wintner LIL. For the proof of the Hartman-Wintner LIL we provide four lemmas. The first three deal with the upper bound part of the LIL and the fourth with the cluster set.

The following lemma is well-known. The proof is essentially the same as that of Theorem 5.1.1, page 256 of [12], for example; the maximal inequality needed in this case is the Ottaviani inequality,

$$P\{\max_{1\leq k\leq n}|S_k| > t + \varepsilon\}\min_{1\leq k\leq n}P\{|S_n - S_k| \leq \varepsilon\} \leq P\{|S_n| > t\},$$

Received January 1982; revised April 1982.

AMS 1970 *subject classifications*. Primary 60B05, 60F05, 60F10, 60F15.

Key words and phrases. Hartman-Wintner law of the iterated logarithm, exponential inequality, cluster set, law of the iterated logarithm in Banach spaces.

where $S_k = \sum_{j=1}^{k} X_j (k \le n)$ with $\{X_j\}$ independent (see e.g. [4], page 120).

LEMMA 2.1. *Let* $\{Y_j\}$ *be independent* r.v.'s, $T_n = \sum_{j=1}^{n} Y_j$. *Assume*

(1) $T_n/a_n \to_P 0$.
(2) *for some* $\alpha > 1, \beta > 0, c > 0, \quad n_0 \in \mathbb{N}$

$$P\{T_n/a_n > \beta\} \le c \exp\{-\alpha \mathrm{LL}n\} \quad \text{for } n \ge n_0.$$

Then $P\{\limsup_n T_n/a_n \le \beta\} = 1$.

The next lemma is the key new ingredient for the upper bound argument. The parameter τ will play a useful role in the proof of Theorem 2.5.

LEMMA 2.2. *Let* $\{Y_j\}$ *be independent* r.v.'s, $T_n = \sum_{j=1}^{n} Y_j$. *Assume*

(1) $EY_j = 0, \quad \sup_j E|Y_j|^2 < \infty$,
(2) $|Y_j| \le \tau (j/\mathrm{LL}j)^{1/2}$ a.s. *for all* j *and some* $\tau > 0$.

Then if $a \ge (\sup_j E|Y_j|^2)^{1/2}$, *for all* $t > 0, \quad n \in \mathbb{N}$,

$$P\{T_n/a_n > t\} \le \exp\{-(t/a)^2(2 - \exp(\sqrt{2}ta^{-2}\tau))\mathrm{LL}n\}.$$

PROOF. Since $e^x \le 1 + x + (x^2/2)e^{|x|}$ for all real x and for $j \le n$

$$|Y_j/a_n| \le \tau(j/\mathrm{LL}j)^{1/2}/(2n\mathrm{LL}n)^{1/2} \le \tau(n/\mathrm{LL}n)^{1/2}/(2n\mathrm{LL}n)^{1/2} \le \tau/\sqrt{2}\mathrm{LL}n,$$

we have for all $\lambda > 0$

$$\exp(\lambda Y_j/a_n) \le 1 + \frac{\lambda Y_j}{a_n} + \frac{\lambda^2 Y_j^2}{2a_n^2} \exp(\lambda\tau/\sqrt{2}\mathrm{LL}n).$$

Taking expectations, for $j \le n$

$$E \exp(\lambda Y_j/a_n) \le 1 + \frac{\lambda^2 E Y_j^2}{2a_n^2} \exp(\lambda\tau/\sqrt{2}\mathrm{LL}n) \le \exp\left\{\frac{\lambda^2 a^2}{4n\mathrm{LL}n} \exp(\lambda\tau/\sqrt{2}\mathrm{LL}n)\right\},$$

and by independence

$$E \exp(\lambda T_n/a_n) = \prod_{j=1}^{n} E \exp(\lambda Y_j/a_n) \le \exp\left\{\frac{\lambda^2 a^2}{4\mathrm{LL}n} \exp(\lambda\tau/\sqrt{2}\mathrm{LL}n)\right\}.$$

For all $\lambda > 0$, $t > 0$, by Markov's inequality

$$P\{T_n/a_n > t\} \le \exp(-\lambda t) E \exp(\lambda T_n/a_n) \le \exp\left\{-\lambda t + \frac{\lambda^2 a^2}{4\mathrm{LL}n} \exp(\lambda\tau/\sqrt{2}\mathrm{LL}n)\right\}. \tag{2.1}$$

For fixed t, we set $\lambda = a^{-2}2t\mathrm{LL}n$ in (2.1) (this choice is motivated by the fact that if the term $\exp(\lambda\tau/\sqrt{2}\mathrm{LL}n)$ were absent, then λ would minimize the right hand side). A simple computation yields now the stated inequality. □

For the sake of completeness we include a proof of the following lemma, which is elementary and classical. This lemma provides the final ingredient for the upper bound argument.

LEMMA 2.3. *Let* $\{X_j\}$ *be independent, identically distributed* r.v.'s *with* $E|X_1|^2 < \infty$. *Let* $\tau > 0$, *and define* $Z_j = X_j I\{|X_j| > \tau(j/\mathrm{LL}j)^{1/2}\}$, $U_n = \sum_{j=1}^{n} Z_j$. *Then*

$$P\{\lim_n |U_n/a_n| = 0\} = 1.$$

PROOF. By Kronecker's lemma (see e.g. [12], page 120), it is enough to prove that $\sum_j (Z_j/a_j)$ converges a.s., which in turn follows from

$$\sum_j E|Z_j|/a_j < \infty. \tag{2.2}$$

To prove (2.2) let $b_j = (j/\mathrm{LL}j)^{1/2}$. We have

$$\begin{aligned}\textstyle\sum_{j=1}^\infty E|Z_j|/a_j &= \textstyle\sum_{j=1}^\infty a_j^{-1} \sum_{i=j}^\infty E|X_j| I\{\tau b_i < |X_j| \le \tau b_{i+1}\} \\ &\le \textstyle\sum_{j=1}^\infty a_j^{-1} \sum_{i=j}^\infty (\tau b_{i+1}) P\{\tau b_i < |X_j| \le \tau b_{i+1}\} \\ &= \textstyle\sum_{i=1}^\infty (\tau b_{i+1}) P\{\tau b_i < |X_1| \le \tau b_{i+1}\} (\sum_{j=1}^i a_j^{-1}).\end{aligned}$$

By an elementary calculation, $\sum_{j=1}^i a_j^{-1} \le c b_i$ for some constant c (independent of i). Therefore for some constant c'

$$\textstyle\sum_{j=1}^\infty E|Z_j|/a_j \le c' \sum_{i=1}^\infty b_i^2 P\{\tau b_i < |X_1| \le \tau b_{i+1}\}.$$

Since $E|X_1|^2 < \infty$, (2.2) follows. □

The key to the cluster set argument is the following moderate deviation result; the proof is taken from [2]. It is through this lemma that the central limit theorem is used in the proof of the Hartman-Wintner LIL.

LEMMA 2.4. *Let $\{X_j\}$ be independent, identically distributed* r.v.'s, $EX_1 = 0$, $\sigma^2 = E|X_1|^2 < \infty$. *Let $m_k \in \mathbb{N}$, $\alpha_k > 0$, $\alpha_k/m_k \to 0$, $\alpha_k^2/m_k \to \infty$. Then for every $b \in \mathbb{R}$, $\varepsilon > 0$*

$$\liminf_k (m_k/\alpha_k^2) \log P\{|S_{m_k}/\alpha_k - b| < \varepsilon\} \ge -(1/2)(b/\sigma)^2.$$

PROOF. We first prove: if $V = (c, d)$, $t > 0$ and γ_σ is the $N(0, \sigma^2)$ distribution, then

$$\liminf_k (m_k/\alpha_k^2) \log P\{S_{m_k}/\alpha_k \in V\} \ge t^{-2} \log \gamma_\sigma(tV). \tag{2.3}$$

Let $V_\delta = (c + \delta, d - \delta)$, $U_\delta = (-\delta, \delta) (\delta > 0)$. Define $p_k = [m_k^2 t^2/\alpha_k^2]$, $q_k = [\alpha_k^2/t^2 m_k]$, $r_k = (tq_k)^{-1}\alpha_k$. Then

$$(P\{S_{p_k}/r_k \in tV_\delta\})^{q_k} \le P\{S_{p_k q_k}/r_k \in tq_k V_\delta\} = P\{S_{p_k q_k}/\alpha_k \in V_\delta\}. \tag{2.4}$$

Also

$$P\{S_{p_k q_k} | \alpha_k \in V_\delta\} \cdot P\{(S_{m_k} - S_{p_k q_k})/\alpha_k \in U_\delta\} \le P\{S_{m_k}/\alpha_k \in V\}. \tag{2.5}$$

By Chebyshev's inequality,

$$\lambda_k = P\{|S_{m_k} - S_{p_k q_k}| > \delta\alpha_k\} \le (m_k - p_k q_k)\sigma^2/\delta^2\alpha_k^2 \to 0. \tag{2.6}$$

Since $\mathscr{L}(S_{p_k}/r_k) \to_w \gamma_\sigma$ by the central limit theorem, we have from (2.4)–(2.6)

$$\begin{aligned}\liminf_k (m_k/\alpha_k^2) \log P\{S_{m_k}/\alpha_k \in V\} &\ge \liminf_k (m_k/\alpha_k^2) \log(1 - \lambda_k) \\ &\quad + \liminf_k t^{-2} \log P\{S_{p_k}/r_k \in tV_\delta\} \\ &\ge t^{-2} \log \gamma_\sigma(tV_\delta).\end{aligned}$$

Since δ is arbitrary, (2.3) follows.

Now let $c = b - \varepsilon$, $d = b + \varepsilon$. Then by a change of variable and Jensen's inequality,

$$\begin{aligned}\gamma_\sigma(tV) &= \int_{tU_\varepsilon} \exp(tbx)\gamma_\sigma(dx) \cdot \exp\left\{-\left(\frac{1}{2}\right)\frac{t^2b^2}{\sigma^2}\right\} \\ &\ge \exp\left\{-\left(\frac{1}{2}\right)\frac{t^2b^2}{\sigma^2}\right\}\gamma_\sigma(U_{t\varepsilon}).\end{aligned}$$

Thus

$$t^{-2}\log \gamma_\sigma(tV) \ge -\left(\frac{1}{2}\right)\left(\frac{b}{\sigma}\right)^2 + t^{-2}\log \gamma_\sigma(U_{t\varepsilon}),$$

$$\text{(2.7)} \qquad \liminf_{t\to\infty} t^{-2}\log \gamma_\sigma(tV) \geq -\left(\frac{1}{2}\right)\left(\frac{b}{\sigma}\right)^2.$$

Now the result follows from (2.3) and (2.7). □

We come now to the Hartman-Wintner law, which we present in Strassen's formulation.

THEOREM 2.5. *Let* $\{X_j\}$ *be independent, identically distributed* r.v.'s *with* $EX_1 = 0$, $\sigma^2 = E|X_1|^2 < \infty$, $S_n = \sum_{j=1}^n X_j$. *Then if* $K_\sigma = [-\sigma, \sigma]$,

(1) $P\{\lim_n d(S_n/a_n, K_\sigma) = 0\} = 1$,
(2) $P\{C(\{S_n/a_n\}) = K_\sigma\} = 1$.

In particular, with probability 1

$$\limsup_n S_n/a_n = \sigma, \qquad \liminf_n S_n/a_n = -\sigma, \qquad \limsup_n |S_n/a_n| = \sigma.$$

PROOF. The statements are trivially true if $\sigma = 0$, so we may assume $\sigma > 0$. We first show

$$\text{(2.8)} \qquad P\{\limsup_n S_n/a_n \leq \sigma\} = 1.$$

Let $\delta > 0$, $t = (1 + \delta)\sigma$. Choose $\tau > 0$ small enough so that

$$\alpha = (1 + \delta)^2(2 - \exp(\sqrt{2}(1 + \delta)\sigma^{-1}\tau)) > 1.$$

Then if $X_j' = X_j I\{|X_j| \leq (\tau/2)(j/\mathrm{LL}j)^{1/2}\}$ and $Y_j = X_j' - EX_j'$, we can apply Lemma 2.2 to $\{Y_j\}$ with $a = \sigma$, obtaining

$$P\{T_n/a_n > (1 + \delta)\sigma\} \leq \exp(-\alpha \mathrm{LL}n).$$

Also, by Chebyshev's inequality

$$P\{|T_n/a_n| > \varepsilon\} \leq n\sigma^2/\varepsilon^2 a_n^2 \to 0.$$

Applying now Lemma 2.1, we conclude that

$$\text{(2.9)} \qquad \limsup_n T_n/a_n \leq (1 + \delta)\sigma \quad \text{a.s.}.$$

Let $S_n' = \sum_{j=1}^n X_j'$, $Z_j = X_j - X_j'$. Then $EX_j' = -EZ_j$, and by the argument in Lemma 2.3,

$$\sum_{j=1}^\infty |EX_j'|/a_j \leq \sum_{j=1}^\infty E|Z_j|/a_j < \infty$$

so by Kronecker's lemma

$$\text{(2.10)} \qquad |E(S_n'/a_n)| \leq a_n^{-1} \sum_{j=1}^n |EX_j'| \to 0.$$

By (2.9), (2.10) and Lemma 2.3, we have

$$\begin{aligned} \limsup_n S_n/a_n &\leq \limsup_n T_n/a_n + \limsup_n E(S_n'/a_n) \\ &\quad + \limsup_n |U_n/a_n| \\ &\leq (1 + \delta)\sigma \qquad \text{a.s.} \end{aligned}$$

Since δ is arbitrary, this proves (2.8).

We prove next

$$\text{(2.11)} \qquad P\{\lim_n d(S_n/a_n, K_\sigma) = 0\} = 1.$$

In fact, (2.8) applied to $\{-X_j\}$ yields $P\{\liminf_n S_n/a_n \geq -\sigma\} = 1$, and since

$$\{\limsup_n d(S_n/a_n, K_\sigma) > 0\} \subset \{\limsup_n S_n/a_n > \sigma\} \cup \{\liminf S_n/a_n < -\sigma\},$$

(2.11) follows.

It remains to prove

$$P\{C(\{S_n/a_n\}) = K_\sigma\} = 1. \tag{2.12}$$

In order to prove (2.12), it is enough to prove that for every $b \in (-\sigma, \sigma)$

$$P\{b \in C(\{S_n/a_n\})\} = 1. \tag{2.13}$$

In fact, taking a countable dense set $D \subset (-\sigma, \sigma)$ it follows from (2.13) that $P\{C(\{S_n/a_n\}) \supset D\} = 1$. Since $C(\{S_n/a_n\})$ is closed and, as a consequence of (2.11), $P\{C(\{S_n/a_n\}) \subset K_\sigma\} = 1$, (2.12) follows.

In order to prove (2.13) we adapt an argument in [1]. Let $n_k = k^k$ and write

$$|a_{n_{k+1}}^{-1} S_{n_{k+1}} - b| \leq |a_{n_{k+1}}^{-1} S_{n_k}| + |a_{n_{k+1}}^{-1}(S_{n_{k+1}} - S_{n_k}) - b|. \tag{2.14}$$

Since $a_{n_k}/a_{n_{k+1}} \to 0$, it follows from (2.11) that

$$\limsup_k |a_{n_{k+1}}^{-1} S_{n_k}| = \limsup_k (a_{n_k}/a_{n_{k+1}}) |S_{n_k}/a_{n_k}| = 0, \quad \text{a.s.} \tag{2.15}$$

Let $m_k = n_{k+1} - n_k$; observe that $m_k \sim n_{k+1}$. Given $\varepsilon > 0$, let $A_k = \{|a_{n_{k+1}}^{-1}(S_{n_{k+1}} - S_{n_k}) - b| < \varepsilon\}$. Let $|b| < \sigma$, put $\alpha = |b/\sigma|$ and choose $\delta > 0$ so that $\alpha + \delta < 1$. By Lemma 2.4, applied with $\alpha_k = a_{n_{k+1}}$, there exists k_0 such that $k \geq k_0$ implies

$$P(A_k) = P\{|S_{m_k}/\alpha_k - b| < \varepsilon\} \geq \exp\left\{-\frac{\alpha_k^2}{m_k}\left(\frac{\alpha + \delta}{2}\right)\right\} \sim \exp\{-(\alpha + \delta)\mathrm{LL}n_{k+1}\}$$

and therefore $\sum_{k=1}^\infty P(A_k) = \infty$. Since $\{S_{n_{k+1}} - S_{n_k} : k \in N\}$ is independent, the Borel-Cantelli lemma yields:

$$P\{\liminf_k |a_{n_{k+1}}^{-1}(S_{n_{k+1}} - S_{n_k}) - b| \leq \varepsilon\} = 1. \tag{2.16}$$

Since ε is arbitrary, it follows from (2.14)–(2.16) that

$$P\{\liminf_k |a_{n_{k+1}}^{-1} S_{n_{k+1}} - b| = 0\} = 1,$$

which implies (2.13). □

3. Kuelbs' LIL for Banach space valued random vectors. Let B be a separable Banach space. Given a probability measure μ on B with $\int x \, d\mu(x) = 0$, $\int \|x\|^2 \, d\mu(x) < \infty$, we denote by K_μ the unit ball of the reproducing kernel Hilbert space of μ (see e.g. [8], Lemma 2.1). For a subspace $F \subset B$, q_F is the seminorm defined by $q_F(x) = d(x, F)(x \in B)$.

The following theorem of Kuelbs is one of the basic results on the LIL in the Banach space setting. The proof is similar to the upper bound part of the proof of Theorem 2.5.

THEOREM 3.1. *Let $\{X_j\}$ be independent, identically distributed B-valued* r.v.'s *with $EX_1 = 0$, $E\|X_1\|^2 < \infty$, $S_n = \sum_{j=1}^n X_j$. Assume that $\{\mathscr{L}(S_n/a_n)\}$ is relatively compact. Then if $\mu = \mathscr{L}(X_1)$,*

(1) $P\{\lim_n d(S_n/a_n, K_\mu) = 0\} = 1$,
(2) $P\{C(\{S_n/a_n\}) = K_\mu\} = 1$.

We shall need the following.

LEMMA 3.2. *Let $\{Y_n\}$ be a sequence of B-valued* r.v.'s. *Assume*

(1) $P\{\{Y_n\}$ *is bounded*$\} = 1$,
(2) *for every $\varepsilon > 0$, there exists a finite-dimensional subspace F such that* $P\{\limsup_n q_F(Y_n) \leq \varepsilon\} = 1$.

Then $P\{\{Y_n\}$ *is relatively compact*$\} = 1$.

The proof is straightforward.

PROOF OF THEOREM 3.1. By a result of Kuelbs ([8], Corollary 3.1), it is enough to prove:

$$P\{\{S_n/a_n\} \text{ is relatively compact}\} = 1. \tag{3.1}$$

By standard arguments, it is enough to prove (3.1) under the additional assumption of symmetry of X_1, which we adopt for the rest of the proof.

Let

$$Y_j = X_j I(\|X_j\| \le \tau(j/\mathrm{LL}j)^{1/2}), \quad Z_j = X_j - Y_j,$$
$$T_n = \textstyle\sum_{j=1}^n Y_j, \quad U_n = \sum_{j=1}^n Z_j.$$

It is easily seen that the relative compactness of $\{\mathscr{L}(S_n/a_n)\}$ implies that $T_n/a_n \to_P 0$. Then standard integrability arguments (see e.g. [10], Lemma 2.3) yield:

$$E\|T_n/a_n\| \to 0. \tag{3.2}$$

Let q be a seminorm on B such that $q \le \|\cdot\|$. In order to obtain an exponential bound for $P\{q(T_n/a_n) > t\}$ similar to that given in Lemma 2.2, we shall use a technique of Yurinskii [14], already exploited in [9]. Let $\mathscr{F}_j = \sigma(Y_1, \cdots, Y_j)$ for $j = 1, \cdots, n$ and let $\mathscr{F}_0$ be the trivial σ-algebra. Define

$$\eta_j = E\{q(T_n/a_n) \mid \mathscr{F}_j\} - E\{q(T_n/a_n) \mid \mathscr{F}_{j-1}\}, \quad (j = 1, \cdots, n).$$

Then

$$q(T_n/a_n) - Eq(T_n/a_n) = \textstyle\sum_{j=1}^n \eta_j,$$

$$\begin{aligned} E\exp\lambda(q(T_n/a_n) - Eq(T_n/a_n)) &= E(E\{\exp\lambda(\textstyle\sum_{j=1}^n \eta_j) \mid \mathscr{F}_{n-1}\}) \\ &= E\exp\lambda(\textstyle\sum_{j=1}^{n-1}\eta_j)E\{\exp(\lambda\eta_n)\mid\mathscr{F}_{n-1}\} \end{aligned} \tag{3.3}$$

for all $\lambda > 0$, and

$$|\eta_j| \le q(Y_j/a_n) + Eq(Y_j/a_n) \quad \text{a.s.}; \tag{3.4}$$

this inequality follows from elementary properties of conditional expectations.

From (3.4) and the definition of Y_j we have $|\eta_j| \le \sqrt{2}\ \tau/\mathrm{LL}n$, $(j = 1, \cdots, n)$, so proceeding as in Lemma 2.2 we get

$$\exp(\lambda\eta_n) \le 1 + \lambda\eta_n + \frac{\lambda^2\eta_n^2}{2}\exp(\sqrt{2}\lambda\tau/\mathrm{LL}n),$$

$$\begin{aligned} E\{\exp(\lambda\eta_n)\mid\mathscr{F}_{n-1}\} &\le 1 + \frac{\lambda^2 E\{\eta_n^2\mid\mathscr{F}_{n-1}\}}{2}\exp(\sqrt{2}\lambda\tau/\mathrm{LL}n) \\ &\le 1 + \frac{4\lambda^2E(q(Y_n))^2}{2a_n^2}\exp(\sqrt{2}\lambda\tau/\mathrm{LL}n) \\ &\le 1 + \frac{\lambda^2a^2}{n\mathrm{LL}n}\exp(\sqrt{2}\lambda\tau/\mathrm{LL}n) \\ &\le \exp\left\{\frac{\lambda^2a^2}{n\mathrm{LL}n}\exp(\sqrt{2}\lambda\tau/\mathrm{LL}n)\right\}, \qquad \text{where } a^2 = E(q(X_1))^2. \end{aligned}$$

By iterating this procedure we obtain from (3.3)

$$E\exp\lambda(q(T_n/a_n) - Eq(T_n/a_n)) \le \exp\left\{\frac{\lambda^2a^2}{\mathrm{LL}n}\exp(\sqrt{2}\lambda\tau/\mathrm{LL}n)\right\}.$$

By applying Markov's inequality and setting $\lambda = (2a^2)^{-1}t\mathrm{LL}n$, we obtain as in Lemma 2.2: for all $t > 0$, $n \in \mathbb{N}$

$$P\{q(T_n/a_n) - Eq(T_n/a_n) > t\} \le \exp\left\{-\left(\frac{t}{2a}\right)^2(2 - \exp(t\tau/\sqrt{2}a^2))\mathrm{LL}n\right\}. \tag{3.5}$$

Now $\lim_n q(U_n/a_n) = 0$ a.s.; in fact, the argument in Lemma 2.3 applies without change. Following now the steps in the proof of (2.8) in Theorem 2.5, and using (3.2), we get

$$P\{\limsup_n q(S_n/a_n) \le 2\{E(q(X_1))^2\}^{1/2}\} = 1. \tag{3.6}$$

In particular

(i) $P\{\limsup_n \| S_n/a_n \| \le 2(E\| X_1 \|^2)^{1/2}\} = 1$,

(ii) given $\varepsilon > 0$, choose a finite-dimensional subspace F such that $E(q_F(X_1))^2 \le (\varepsilon/2)^2$ (this is possible because $E\| X_1 \|^2 < \infty$); then $P\{\limsup_n q_F(S_n/a_n) \le \varepsilon\} = 1$.

The proof of (3.1) is now completed by applying Lemma 3.2. □

Acknowledgment. We would like to thank Jim Kuelbs for many stimulating conversations.

REFERENCES

[1] DE ACOSTA, A. (1983). Small deviations in the functional central limit theorem with applications to functional laws of the iterated logarithm. *Ann. Probability* **11** 78–101.

[2] DE ACOSTA, A. and KUELBS, J. (1981). Some new results on the cluster set $C(\{S_n/a_n\})$ and the LIL. *Ann. Probability* **11** 102–122.

[3] CHOW, Y. S. and TEICHER, H. (1978). *Probability Theory.* Springer-Verlag, New York and Berlin.

[4] CHUNG, K. L. (1974). *A Course in Probability Theory*, 2nd edition. Academic, New York.

[5] CSÖRGŐ, M. and RÉVÉSZ, P. (1980). *Strong Approximation in Probability and Statistics.* Academic, New York.

[6] HARTMAN, P. and WINTNER, A. (1941). On the law of the iterated logarithm. *American J. Math.* **63** 169–176.

[7] HEYDE, C. C. (1969). Some properties of metrics in a study on convergence to normality. *Z. Wahrsch. verw. Gebiete* **11** No. 3, 181–193.

[8] KUELBS, J. (1976). The law of the iterated logarithm and related strong convergence theorems for Banach space valued random variables. *Lecture Notes in Mathematics* **539** 224–314. Springer-Verlag, New York and Berlin.

[9] KUELBS, J. (1977). Kolmogorov's law of the iterated logarithm for Banach space valued variables. *Illinois J. of Math.* **21** No. 4, 784–800.

[10] KUELBS, J. and ZINN, J. (1979). Some stability results for vector-valued random variables. *Ann. Probability* **7** 75–84.

[11] PETROV, V. V. (1975). *Sums of Independent Random Variables.* Springer-Verlag, Berlin and New York.

[12] STOUT, W. F. (1974). *Almost Sure Convergence.* Academic, New York.

[13] STRASSEN, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrsch. verw. Gebiete* **3** 211–226.

[14] YURINSKII, V. V. (1974). Exponential bounds for large deviations. *Theor. Probability Appl.* **19** 154–155.

MATHEMATICS DEPARTMENT
VAN VLECK HALL
UNIVERSITY OF WISCONSIN-MADISON
480 LINCOLN DRIVE
MADISON WISCONSIN 53706