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STABLE LIMITS FOR PARTIAL SUMS OF DEPENDENT RANDOM VARIABLES[1]

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Let $\{X_n\}$ be a stationary sequence of random variables whose marginal distribution F belongs to a stable domain of attraction with index α, $0 < \alpha < 2$. Under the mixing and dependence conditions commonly used in extreme value theory for stationary sequences, nonnormal stable limits are established for the normalized partial sums. The method of proof relies heavily on a recent paper by LePage, Woodroofe, and Zinn which makes the relationship between the asymptotic behavior of extreme values and partial sums exceedingly clear. Also, an example of a process which is an instantaneous function of a stationary Gaussian process with covariance function r_n behaving like $r_n \log n \to 0$ as $n \to \infty$ is shown to satisfy these conditions.

1. Introduction. There is a vast literature on central limit theorems for stationary mixing sequences (see Ibragimov and Linnik, 1971, for a convenient reference). Typically, three ingredients are needed in proving such results: the variance of the partial sums must approach ∞, a moment condition (usually the existence of a $(2 + \delta)$th moment) is required and the sequence should satisfy a suitable mixing condition with a sufficiently fast rate of mixing. Many useful results and insights into the dependence structure of stationary sequences have evolved from studying problems of this type. However, not much is known for obtaining nonnormal stable limits. Clearly, not only will the hypotheses needed for obtaining stable limits with index less than 2 be different, but many of the highly developed techniques used for normal limits will not be amenable to this case. If the common distribution function of an iid sequence belongs to a nonnormal stable domain of attraction, then the asymptotic behavior of the extreme order statistics and partial sums are closely related. This relationship is made exceedingly clear in LePage, Woodroofe, and Zinn (1981) (hereafter LWZ) which suggests that, in extending the iid results for nonnormal stable limits to dependent sequences, the dependence and mixing assumptions commonly used in extreme value theory for stationary sequences may be employed. Using these kinds of hypotheses and expoiting the ideas in LWZ, nonnormal stable limits of the normalized partial sums are established.

Let $\{X_n\}$ be a stationary sequence of random variables with marginal distribution function (df) $F(x)$. Much of the notation and formulations used in LWZ will be retained in this paper. Let $1 - G(x)$ be the distribution function of $|X_1|$. We shall assume F belongs to the stable domain of attraction with index α, $0 < \alpha < 2$. This translates into the following assumptions on F and G:

(1) $\quad G(x) = P(|X_1| > x) = x^{-\alpha}L(x)$ where $L(x)$ is a slowly varying function at ∞,

and

$$\frac{1 - F(y)}{G(y)} \to p \quad \text{and} \quad \frac{F(-y)}{G(y)} \to q \quad \text{as} \quad y \to \infty, \tag{2}$$

where $0 \le p \le 1$ and $q = 1 - p$. Then the normalizing constants, $a_n > 0$ and b_n, for the

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partial sums $S_n = X_1 + \cdots + X_n$, will be determined by

(3) $$nG(a_n x) \to x^{-\alpha} \quad \text{as } n \to \infty, \qquad x > 0,$$

and

(4) $$b_n = \int_{-a_n}^{a_n} x\, dF(x).$$

Define M_n^k and W_n^k to be, respectively, the kth largest and kth smallest among $\{X_1, \cdots, X_n\}$ and let $Y_{nk} = k$th largest among $\{|X_1|, \cdots, |X_n|\}$ for $k = 1, \cdots, n$. Also, let $\{\hat{X}_n\}$ be the associated independent sequence of $\{X_n\}$ (i.e. $\{\hat{X}_n\}$ is an iid sequence and $\hat{X}_1 =_d X_1$). Correspondingly, the analogues of M_n^k, W_n^k, and Y_{nk} for the associated iid sequence will be denoted by $\hat{M}_n^k$, $\hat{W}_n^k$, and $\hat{Y}_{nk}$. We will use throughout the convention that a "hat" over a random variable refers to a random quantity related to the associated independent sequence.

The mixing condition D which will be required is a somewhat stronger condition than originally formulated by Leadbetter (1974) (also see Davis, 1979). Let B be a finite union of disjoint intervals (possibly infinite) with nonzero endpoints of the form $(a, b]$ and set $U_{jn} = I_{\{(X_j/a_n) \in B\}}$. Observe that $EU_{1n} \to 0$ or 1 as $n \to \infty$. Then condition D is said to hold if for any choice of integers

$$1 \le i_1 < \cdots < i_p < j_1 < \cdots < j_q \le n, \qquad j_1 - i_p > \ell,$$

$$|E(U_{i_1 n} \cdots U_{i_p n} U_{j_1 n} \cdots U_{j_q n}) - E(U_{i_1 n} \cdots U_{i_p n}) E(U_{j_1 n} \cdots U_{j_q n})| \le \alpha_{n,\ell}$$

where $\alpha_{n,\ell}$ is non-increasing in ℓ and $\alpha_{n,\ell_n} \to 0$ as $n \to \infty$ for some sequence $\ell_n \to \infty$ with $\ell_n = o(n)$. Also $\alpha_{n,\ell}$ may depend on B. This condition is substantially weaker than the usual mixing conditions associated with central limit theorems (see the second example in Section 3).

In addition to the mixing hypothesis D, a local dependence assumption is also needed in extreme value theory. The condition D' is said to hold if for all $x > 0$

$$\limsup_{n\to\infty} S_{k,n}(x) = o(1) \text{ as } k \to \infty \text{ where}$$

$$S_{k,n}(x) = n \textstyle\sum_{j=2}^{[n/k]} \{P(X_1 > a_n x, X_j > a_n x) + P(X_1 > a_n x, X_j \le -a_n x)$$
$$+ P(X_1 \le -a_n x, X_j > a_n x) + P(X_1 \le -a_n x, X_j \le -a_n x)\}.$$

If $0 < p < 1$, then, under the D and D' assumptions, $P(M_n^1 \le a_n x) \to e^{-px^{-\alpha}}$ and $P(W_n^1 > -a_n x) \to e^{-qx^{-\alpha}}$ for $x > 0$ by Theorem 2.6 in Leadbetter, Lindgren and Rootzén (1979). Moreover, conditions D and D' are the same as those defined in Section 4 of Davis (1983) specialized to the present setting, and upon applying Theorem 4.2 and Remarks 1 and 2 from the same paper, we have the following result.

THEOREM 1. *If $0 < p < 1$, the joint limiting distribution of the two vectors*

$$a_n^{-1}(M_n^1, \cdots, M_n^k) \quad \textit{and} \quad a_n^{-1}(W_n^1, \cdots, W_n^k) \quad \textit{is the same as the limit of}$$

$$a_n^{-1}(\hat{M}_n^1, \cdots, \hat{M}_n^k) \quad \textit{and} \quad a_n^{-1}(\hat{W}_n^1, \cdots, \hat{W}_n^k) \quad \textit{for all positive integers } k.$$

This is the key observation in establishing the convergence of the partial sums to a stable limit. This result is proved in Section 2 provided $0 < \alpha < 1$. For the case $1 \le \alpha < 2$, a rather mild assumption is also needed in order to obtain the same result.

In Section 3, two examples are presented. The first is an instantaneous function of a stationary Gaussian sequence chosen in such a way that the marginal distribution function of this new sequence satisfies (1) and (2). If the covariance function r_n of the Gaussian process behaves like $r_n \log n \to 0$, then conditions D and D' are fulfilled by the instantaneous function process and hence the partial sums have a stable limit. The second example also satisfies D and D' yet does not satisfy any of the common mixing conditions.

2. The main result. Let $\{X_n\}$ be a stationary sequence satisfying conditions D and D' and suppose F satisfies (1) and (2).

LEMMA 1. *Theorem 1 remains valid for the additional cases $p = 0$ and $p = 1$.*

PROOF. We only handle the case $p = 0$ for the argument when $p = 1$ is similar. First, note that $n(1 - F(a_n x)) \to 0$ for all $x > 0$ and since $1 - n(1 - F(a_n x)) \le P(M_n^1 \le a_n x)$, $P(M_n^1 \le a_n x) \to 1$ for all $x > 0$. Now $M_n^1 \to x_0$ a.s. where $x_0 = \sup\{x : F(x) < 1\}$. If $x_0 = \infty$, then $P(M_n^1/a_n \le 0) \to 0$ which, together with $P(M_n^1/a_n < x) \to 1$ for $x > 0$, implies $M_n^1/a_n \to 0$ in probability. On the other hand if $x_0 < \infty$, then $X_1/a_n \le M_n^1/a_n \le x_0/a_n$ and since the outside two terms approach 0 a.s. we have the same result. Finally, for large n, $X_1/a_n \le M_n^j/a_n \le M_n^1/a_n$, so that $M_n^j/a_n \to 0$ in probability for $j = 1, 2, \cdots$. Thus the vector $a_n^{-1}(M_n^1, \cdots, M_n^k)$ has a degenerate limit while $a_n^{-1}(W_n^1, \cdots, W_n^k)$ has the same limit as $a_n^{-1}(\hat{W}_n^1, \cdots, \hat{W}_n^k)$ which completes the proof. □

As in LWZ, let $\{E_n\}$ be a sequence of independent unit exponentials and set $\Gamma_k = E_1 + \cdots + E_k$. Define $Z^n = a_n^{-1}(Y_{n1}, Y_{n2}, \cdots Y_{nn}, 0, 0, \cdots)$ and $Z = (Z_1, Z_2, \cdots)$ where $Z_i = \Gamma_i^{-1/\alpha}$.

LEMMA 2. $Z^n \to_d Z$.

PROOF. In LWZ it was shown that $\hat{Z}_n = a_n^{-1}(\hat{Y}_{n1}, \cdots \hat{Y}_{nn}, 0, 0, \cdots)$ converges in distribution to Z. The sequence $\{|X_n|\}$ satisfies conditions D and D' and using the preceding lemma with $p = 1$, $a_n^{-1}(Y_{n1}, \cdots Y_{nk})$ has the same limit as $a_n^{-1}(\hat{Y}_{n1}, \cdots, \hat{Y}_{nk})$ which is $(\Gamma_1^{-1/\alpha}, \cdots \Gamma_k^{-1/\alpha})$. The result is complete since it is enough to establish the convergence of the finite dimensional distributions. □

For some random permutation $(\sigma_{n1}, \cdots, \sigma_{nn})$ of the integers $1, 2, \cdots, n$, $Y_{nk} = |X_{\sigma_{nk}}|$. Define $\delta_{nk} = \operatorname{sign}(X_{\sigma_{nk}})$. We now state and prove the analogue of Lemma 2 in LWZ.

LEMMA 3. *$\delta^n = (\delta_{n1}, \cdots \delta_{nn}, 1, 1, \cdots) \to_d \delta = (\delta_1, \delta_2, \cdots)$ as $n \to \infty$, where $\delta_1, \delta_2, \cdots$ are* iid *random variables with $P(\delta_1 = 1) = p$ and $P(\delta_1 = -1) = q$. Furthermore, δ^n and Z^n are asymptotically independent.*

PROOF. We first do away with the case $p = 0$. The event $\{\delta_{n1} = -1, \delta_{n2} = -1, \cdots, \delta_{nk} = -1\}$ is the same as $\{M_n^1 + W_n^k < 0\}$ and $\lim_{n\to\infty} P(M_n^1 + W_n^k < 0) = \lim_{n\to\infty} P(M_n^1/a_n + W_n^k/a_n < 0) = \lim_{n\to\infty} P(W_n^k/a_n < 0) = 1$ since $M_n^1/a_n \to 0$ and $W_n^k \to -\infty$ in probability. This proves that the finite dimensional distributions of δ^n converge to those of δ and so $\delta^n \to_d \delta$ when $p = 0$. The case $p = 1$ is the same.

Now assume $0 < p < 1$. Then $M_n^j \to +\infty$ a.s. and $W_n^j \to -\infty$ a.s. for every j. Hence, $\delta_{nj} = 1$ implies for large n

$$Y_{nj} = \begin{cases} M_n^1 & \text{if } \ -W_n^j < M_n^1 \le -W_n^{j-1} \\ M_n^2 & \text{if } \ -W_n^{j-1} \ -W_n^{j-1} < M_n^2 \le -W_n^{j-2} \\ \vdots & \vdots \qquad\qquad \vdots \\ M_n^j & \text{if } \ -W_n^1 \ \ -W_n^1 \le M_n^j. \end{cases}$$

Similarly, $\delta_{nj} = -1$ implies for n large,

$$Y_{nj} = \begin{cases} -W_n^1 & \text{if } \ M_n^j < -W_n^1 \le M_n^{j-1} \\ -W_n^2 & \text{if } \ M_n^{j-1} < -W_n^2 \le M_n^{j-2} \\ \vdots & \vdots \qquad\qquad \vdots \\ -W_n^j & \text{if } \ M_n^1 < -W_n^j. \end{cases}$$

So the asymptotic properties of $(a_n^{-1} Y_{nj}, \delta_{nj})$ are completely determined by the asymptotic distribution of $a_n^{-1}(M_n^1, \cdots, M_n^j, W_n^1, \cdots, W_n^j)$. Consequently, for a compact subset B of $(0, \infty)^k$, and $(\varepsilon_1, \cdots, \varepsilon_k) \in \{-1, +1\}^k$,

$$\lim_{n\to\infty} P(a_n^{-1}(Y_{n1}, Y_{n2}, \cdots, Y_{nk}) \in B, \delta_{n1} = \varepsilon_1, \cdots, \delta_{nk} = \varepsilon_k) \tag{5}$$
$$= \lim_{n\to\infty} P(a_n^{-1}(M_n^1, \cdots, M_n^k, W_n^1, \cdots, W_n^k) \in \tilde{B})$$

where $\tilde{B}$ is a Borel subset of $(-\infty, \infty)^{2k}$ chosen so that for large n (n depends on ω), the indicator of the set $\{a_n^{-1}(Y_{n1}, \cdots, Y_{nk}) \in B, \delta_{n1} = \varepsilon_1, \cdots, \delta_{nk} = \varepsilon_k\}$ is equal to the indicator of the set $\{a_n^{-1}(M_n^1, \cdots, M_n^k, W_n^1, \cdots, W_n^k) \in \tilde{B}\}$. Using Lemma 1, the equality in (5) is equal to

$$\lim_{n\to\infty} P(a_n^{-1}(\hat{M}_n^1, \cdots, \hat{M}_n^k, \hat{W}_n^1, \cdots, \hat{W}_n^k) \in \tilde{B})$$
$$= \lim_{n\to\infty} P(a_n^{-1}(\hat{Y}_{n1}, \cdots, \hat{Y}_{nk}) \in B, \hat{\delta}_{n1} = \varepsilon_1, \cdots, \hat{\delta}_{nk} = \varepsilon_k)$$
$$= P((\Gamma_1^{-1/\alpha}, \cdots, \Gamma_k^{-1/\alpha}) \in B) P(\delta_1 = \varepsilon_1, \cdots, \delta_k = \varepsilon_k)$$

where the last equality is from Lemma 2 in LWZ. This concludes the proof of the lemma. □

Let $[x : A] = xI_A(x)$ where I_A is the indicator function of the set A.

THEOREM 2. *If* $0 < \alpha < 1$, *then* $a_n^{-1} S_n \to_d S$ *where* $S = \sum_{j=1}^{\infty} \delta_j Z_j$.

PROOF. As in LWZ, for $0 < \varepsilon < \lambda$, let $S_n\,(\varepsilon, \lambda] = \sum_{j=1}^{n} \delta_{nj} \cdot [Z_{nj} : (\varepsilon, \lambda])$ where $Z_{nj} = a_n^{-1} Y_{nj}$. Similarly define $S(\varepsilon, \lambda] = \sum_{j=1}^{\infty} \delta_j \cdot [Z_j : (\varepsilon, \lambda]]$. Since $a_n^{-1} S_n = S_n\,(o, \varepsilon] + S_n(\varepsilon, \infty)$, it is enough to show (see Theorem 4.2 in Billingsley (1968)),

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} E\,|\,S_n\,(0, \varepsilon]\,| = 0, \tag{6}$$

$$S_n\,(\varepsilon, \infty) \to_d S(\varepsilon, \infty) \text{ as } n \to \infty, \tag{7}$$

and

$$S(\varepsilon, \infty) \to_d S \text{ as } \varepsilon \to 0. \tag{8}$$

For distribution functions with regularly varying tails with index α, $0 < \alpha < 1$, we have from Karamata's Theorem (see de Haan (1970))

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{n}{a_n} \int_{-\varepsilon a_n}^{\varepsilon a_n} |\,x\,|\, dF(x) = 0.$$

Thus (6) is established since $E\,|\,S_n\,(0, \varepsilon]\,| \le (n/a_n^2) \int_{-\varepsilon a_n}^{\varepsilon a_n} |x|\, dF(x)$. Using an identical argument to the one given in LWZ, it may be shown that $S_n(\varepsilon, \infty) \to_d S(\varepsilon, \infty)$. Finally to prove (8), first observe that $Z_j / j^{-1/\alpha} \to 1$ a.s. Hence for $0 < \alpha < 1$, the series $\sum_{j=1}^{\infty} \delta_j Z_j$ converges absolutely a.s. in which case it follows that $S(\varepsilon, \infty) \to S$ a.s. as $\varepsilon \to 0$. □

In addition to the hypotheses D and D′, a further dependence assumption is needed when $1 \le \alpha < 2$. Condition D″ is said to hold if

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{n}{a_n^2} \sum_{j=2}^{n} \max(0, \mathrm{Cov}([X_1 : (-\varepsilon a_n, \varepsilon a_n)], [X_j : (-\varepsilon a_n, \varepsilon a_n)])) = 0. \tag{9}$$

Each term in the sum is bounded by $a_n^{-2} n\, \mathrm{Var}([X_1 : (-\varepsilon a_n, \varepsilon a_n)])$ which in turn is bounded by $(n/a_n^2) \int_{-\varepsilon a_n}^{\varepsilon a_n} x^2\, dF(x)$. Using properties of regularly varying distributions (deHaan, 1970),

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{n}{a_n^2}\int_{-\varepsilon a_n}^{\varepsilon a_n} x^2\,dF(x)=0 \tag{10}$$

so that each of the summands in (6) has the desired property and, thus, an m-dependent sequence satisfies (6).

Although it is believed that the following theorem is true without assuming D″, our proof, which follows closely the argument given for Theorem 1 in LWZ, requires it.

THEOREM 3. *For $1 \le \alpha < 2$, suppose the stationary sequence $\{X_n\}$ also satisfies* D″. *Then $a_n^{-1}(S_n - nb_n) \to_d S^*$ where*

$$S^* = \textstyle\sum_{j=1}^{\infty}(\delta_j Z_j - (p-q)E[Z_j\colon (0,1]]).$$

PROOF. Using the notation developed in the proof of Theorem 2, we have

$$a_n^{-1}(S_n - nb_n) = S_n(0,\infty) - ES_n(0,1] = S_n(\varepsilon,\infty) - ES_n(\varepsilon,1] + S_n(0,\varepsilon] - ES_n(0,\varepsilon].$$

It then suffices to show

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\operatorname{Var}(S_n(0,\varepsilon]) = 0 \tag{11}$$

$$S_n(\varepsilon,\infty) - ES_n(\varepsilon,1) \to_d S(\varepsilon,\infty) - ES(\varepsilon,1), \quad \text{as } n\to\infty \tag{12}$$

and

$$S(\varepsilon,\infty) - ES(\varepsilon,1) \to_d S^* \quad \text{as} \quad \varepsilon\to 0. \tag{13}$$

To prove (11), observe that

$$\operatorname{Var}(S_n(0,\varepsilon]) \le \frac{n}{a_n^2}\int_{-\varepsilon a_n}^{\varepsilon a_n} x^2\,dF(x) + \frac{2n}{a_n^2}\textstyle\sum_{j=2}^{n}\max(0, \operatorname{Cov}([X_1\colon(-\varepsilon a_n, \varepsilon a_n)], [X_j\colon(-\varepsilon a_n, \varepsilon a_n)])).$$

Now, by the D″ assumption and (10), the $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}$ of the bound is zero. In LWZ, (13) and $E\hat{S}_n(\varepsilon,1] \to ES(\varepsilon,1]$ were established. Since $ES_n(\varepsilon,1] = E\hat{S}_n(\varepsilon,1]$, the argument will be complete once we show $S_n(\varepsilon,\infty) \to_d S(\varepsilon,\infty)$. However, as in the proof of Theorem 2, this is established using the same argument as the one provided in LWZ for $\hat{S}_n(\varepsilon,\infty)$. □

The analogues of Theorem 1′ and Corollary 1 for self norming sums in LWZ also carries over to the dependent setting. Set

$$S_{n,r}^{\#} = a_n^{-1}\{\textstyle\sum_{j=1}^{n}|X_j|^r\}^{1/r}, \quad 1\le r\le\infty$$

$$S_r^{\#} = \{\textstyle\sum_{j=1}^{\infty} Z_j^r\}^{1/r}, \quad \alpha < r < \infty$$

$$S_{n,\infty}^{\#} = a_n^{-1}\max\{|X_1|, \cdots, |X_n|\} \quad\text{and}\quad S_\infty^{\#} = Z_1.$$

The proof of the following theorem and corollary is immediate from Theorems 2 and 3 and the proof of Theorem 1′ and Collary 1 in LWZ.

THEOREM 4. *Under the assumptions of Theorem 2 or Theorem 3, $(Z^n, S_n^*, S_{n,r}^{\#}) \to_d (Z, S^*, S_r^{\#})$ as $n\to\infty$ for all r, $\alpha < r \le \infty$ where $S_n^* = a_n^{-1}(S_n - nb_n)$ and S^* is defined in Theorem* 3.

COROLLARY a) *If $0<\alpha<1$, then $T_{n,r} \to_d T_r$ for all r, $\alpha < r\le\infty$ where*

$$T_{n,r} = (\textstyle\sum_{j=1}^{n} X_j)/\{\sum_{j=1}^{n}|X_j|^r\}^{1/r} \quad \textit{and} \quad T_r = \sum_{j=1}^{\infty}\delta_j Z_j/\{\sum_{j=1}^{\infty} Z_j^r\}^{1/r}.$$

b) *If, in addition to the assumptions of Theorem 3, $EX_1 = 0$, then*

$T_{n,r} \to_d T_r^*$ *for all r where* $\alpha < r \le \infty$ *and*

$$T_r^* = \{\sum_{j=1}^{\infty} (\delta_j Z_j - (p-q)E[Z_j: (0, 1]])\}/\{\sum_{j=1}^{\infty} Z_j^r\}^{1/r}.$$

3. Examples. In this section, two examples illustrating the results of the preceding section are presented. First, let $\{Y_n\}$ be a stationary Gaussian sequence with zero mean, unit variance, and covariance function $r_n = EY_1Y_{n+1}$. Set $A = \cup_{j=0}^{m}(v^{(2j)}, v^{(2j+1)})$ where $-\infty \le v^{(0)} < v^{(1)} < \cdots < v^{(2m)} < v^{(2m+1)} \le \infty$. The proof of the following lemma mimics, with minor modifications, the argument given for Lemma 3.2 in Leadbetter, Lindgren, and Rootzen (1979).

LEMMA 4. *For any choice of integers* $1 \le i_1 < i_2 < \cdots < i_s$,

$$|P(X_{i_1} \in A, \cdots, X_{i_s} \in A) - P^s(X_1 \in A)| \le Km^2 \sum_{1 \le j < k \le s} |r_{jk}| e^{-u^2/(1+|r_{jk}|)}$$

where $r_{jk} = EX_{i_j}X_{i_k}$ *and* $u = \min\{|v^{(0)}|, |v^{(1)}|, \cdots, |v^{(2m+1)}|\}$ *and K is a constant.*

For each $j = 0, 1, \cdots, 2m+1$, let $v_n^{(j)}$ be a sequence of numbers with $-\infty \le v_n^{(0)} < v_n^{(1)} < \cdots < v_n^{(2m+1)} \le \infty$ and satisfying $1 - \Phi(|v_n^{(j)}|) \sim \dfrac{\alpha^{(j)}}{n}$ as $n \to \infty$ where $0 \le \alpha^j < \infty, j = 0, 1, \cdots, 2m+1$, and Φ is the standard normal distribution function.

LEMMA 5. *For any choice of integers* $1 \le i_1 < i_2 < \cdots < i_p < j_1 < \cdots < j_q \le n$ *with* $j_1 - i_p \ge \ell$, *then*

$$|P(Y_{i_1} \in A_n, \cdots Y_{i_p} \in A_n, Y_{j_1} \in A_1, \cdots, Y_{j_q} \in A_n)$$
$$- P(Y_{i_1} \in A_n, \cdots, Y_{i_p} \in A_n)P(Y_{j_1} \in A_n, \cdots, Y_{j_q} \in A_n)| \le 3Km^2 n \sum_{j=1}^{n} |r_j| e^{-u_n^2/(1+|r_{jk}|)}$$

where A_n *is the set A defined above with* $v^{(j)}$ *replaced by* $v_n^{(j)}$ *and* $u_n = \min\{|v_n^{(0)}|, \cdots, |v_n^{(2m+1)}|\}$. *Moreover, if* $r_n \log n \to 0$ *or* $\sum |r_j|^2 < \infty$, *then this bound goes to zero for all* $\ell > 0$ *and*

$$\limsup_{n\to\infty} n \sum_{j=2}^{[n/k]} \{P(Y_1 > u_n, Y_j > u_n) + P(Y_1 \le -u_n, Y_j > u_n)$$
$$+ P(Y_1 > u_n, Y_j \le -u_n) + P(Y_1 \le -u_n, Y_j \le -u_n)\} = o(1) \quad \text{as} \quad k \to \infty.$$

The proof follows from Lemma 4 above and Lemmas 3.1 and 3.3 in [8].

Now let $X_n = H(Y_n)$ be an instantaneous function of the Gaussian process such that the df of X_1 belongs to the stable domain of attraction with index α, $0 < \alpha < 1$.

LEMMA 6. *The sequence* $\{X_n\}$ *satisfies conditions* D *and* D′ *and, consequently,* $a_n^{-1} \sum_{j=1}^{n} X_j \to_d S = \sum_{j=1}^{\infty} \delta_j Z_j$.

PROOF. Let $B = \cup_{j=0}^{m} (x_{2j}, x_{2j+1})$ where $-\infty \le x_0 < x_1 < \cdots < x_{2m+1} \le \infty$ and $x_j \ne 0$ for $j = 0, \cdots, 2m+1$. Then $\{a_n^{-1}X_j \in B\} = \{Y_j \in A_n\}$ where $A_n = \cup_{j=0}^{m} (v_n^{(2j)}, v_n^{(2j+1)})$ with $v_n^{(j)} = H^{-1}(a_n x_j)$ $j = 0, \cdots 2m+1$. Since

$$n(1 - \Phi(|v_n^{(j)}|)) \to \begin{cases} p\, x_j^{-\alpha} & \text{if } x_j > 0 \\ \\ q\, |x_j|^{-\alpha} & \text{if } x_j < 0, \end{cases}$$

by (1)–(3) and in view of Lemma 5, it is an easy task to verify both D and D′. □

It is worth remarking that if the instantaneous function of Y_n was chosen such that $\mathrm{Var}(H(Y_n)) < \infty$, then the normalized partial sums of $X_n = H(Y_n)$ may not be asymptotically normal even under the assumption $r_n \log n \to 0$. The special case where $H(Y_n) = Y_n^2 - 1$ and $r_n \approx n^{-\gamma}$, $0 < \gamma < 1/2$, was dealt with by Rosenblatt (1979). This is contrasted with Lemma 6 where H is chosen to make the df of $H(Y_1)$ belong to a stable domain of attraction.

The next example is a modification of the one given in Davis (1982). First, let $\{(Z_n^1, Z_n^2): n \geq 1\}$ be an iid sequence of 2-vectors with marginal distribution defined as follows. The density $f(x, y)$ of this distribution is supported on the unit square and is given by

$$f(x, y) = \begin{cases} 0 \text{ if } (x, y) \in (3/8, 1/2) \times (1/8, 2/8) \cup (1/2, 5/8) \times (0, 1/8) \\ 2 \text{ if } (x, y) \in (3/8, 1/2) \times (0, 1/8) \cup (1/2, 5/8) \times (1/8, 2/8) \\ 1 \text{ elsewhere on the unit square.} \end{cases}$$

In other words, $f(x, y)$ is an altered uniform density on $(0, 1) \times (0, 1)$ with twice as much mass on the squares $(3/8, 1/2) \times (0, 1/8)$ and $(1/2, 5/8) \times (1/8, 2/8)$ and no mass on the adjacent squares $(3/8, 1/2) \times (1/8, 2/8)$ and $(1/2, 5/8) \times (0, 1/8)$. It is easy to check that both marginal densities are uniform on $(0, 1)$ and $P(a < Z_1^1 \leq b, a < Z_1^2 \leq b) = (b - a)^2$ for $b > 5/8$ and $a < 3/8$. Define the sequence $(Y_1, Y_2, Y_3, \cdots)$ to be equal to $(Z_1^2, Z_3^1, Z_3^2, Z_5^1, Z_5^2, \cdots)$ and $(Z_2^1, Z_2^2, Z_4^1, Z_4^2, \cdots)$ each with probability 1/2. In [2], it was shown that $\{Y_n\}$ is stationary, ergodic, and not mixing. If T is the shift operator, a sequence of random variables is said to be mixing if for any two events A and B, $P(A \cap T^{-j}B) - P(A)P(B) \to 0$ as $j \to \infty$. However, in this example, Y_j is independent of $\{Y_{j+2}, Y_{j+3}, \cdots\}$ for all j.

Suppose A is a Borel subset of $(0, 1)$ such that

$$A \cap (3/8, 5/8) = \phi \quad \text{or} \quad (3/8, 5/8) \subset A. \tag{14}$$

Then $P(Y_{i_1} \in A, \cdots, Y_{i_s} \in A) = P^s(A)$ for all choices of integers $1 \leq i_1 < \cdots < i_s$. To complete the construction, define $X_n = F^{-1}(Y_n)$ where F is a df satisfying properties (1) and (2). If B is a finite union of disjoint intervals, then the event $\{a_n^{-1}X_j \in B\} = \{Y_j \in A_n\}$ where $A_n = \{x: a_n^{-1}F^{-1}(x) \in B\}$. For large n, A_n eventually satisfies (14) so that $P(X_{i_1} \in B, \cdots X_{i_s} \in B) = P^s(X_1 \in B)$. It follows that Conditions D and D′ are fulfilled for this sequence and since X_1 and X_j are independent for $j > 2$, Condition D″ is also satisfied by the remark preceding Theorem 3. Thus $a_n^{-1}(\sum_{j=1}^n X_j - nb_n) \to_d S^*$, even though this sequence is not mixing.

The results of Section 2 and the above examples suggest that the mixing assumptions required for proving stable limits of partial sums need not be as stringent as those for establishing central limit theorems. However, a price is paid in the form of a local dependence hypothesis (Condition D′) which occasionally is not even satisfied by 1-dependent sequences. Yet, in view of the relationship between extreme values and partial sums, the hypotheses which are commonly used in extreme value theory for dependent sequences appear appropriate for obtaining nonnormal stable limits of partial sums.

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