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# ON THE SPLICING OF MEASURES

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Given probabilities $\mu$ and $\nu$ on $(X, \mathscr{A})$ and $(X, \mathscr{B})$ respectively, a probability $\eta$ on $(X, \mathscr{A} \vee \mathscr{B})$ is called a splicing of $\mu$ and $\nu$ if $\eta(A \cap B) = \mu(A)\,\nu(B)$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$. Using a result of Marczewski we give an elementary proof of Stroock's result on the existence of splicing. We also discuss the splicing problem when $\mu$ and $\nu$ are compact measures.

Let $X$ be a nonempty set and let $\mathscr{A}$ and $\mathscr{B}$ be two $\sigma$-algebras of subsets of $X$. Given probabilities $\mu$ and $\nu$ on $(X, \mathscr{A})$ and $(X, \mathscr{B})$ respectively, we say that a probability $\eta$ on $(X, \mathscr{A} \vee \mathscr{B})$ where $\mathscr{A} \vee \mathscr{B} = \sigma(\mathscr{A} \cup \mathscr{B})$ is a splicing of $\mu$ and $\nu$ if

$$\eta(A \cap B) = \mu(A)\nu(B) \quad \text{for all } A \in \mathscr{A}, B \in \mathscr{B}.$$

We denote sets from $\mathscr{A}$ by $A, A_1, A_2, \cdots$, sets from $\mathscr{B}$ by $B, B_1, B_2, \cdots$; $\sum_{i\in I} G_i$ denotes the union of sets $\{G_i, i \in I\}$ that are pairwise disjoint. With this notation, $\mathscr{C}_0 = \{\sum_{i=1}^n (A_i \cap B_i), n \geq 1\}$ is an algebra generating $\mathscr{A} \vee \mathscr{B}$ and a splicing of $\mu$ and $\nu$ exists if and only if $\eta$ on $\mathscr{C}_0$ given by

$$\eta\left(\textstyle\sum_{i=1}^n (A_i \cap B_i)\right) = \textstyle\sum_{i=1}^n \mu(A_i)\nu(B_i) \tag{1}$$

is well-defined and countably additive.

The problem of existence of splicing has been treated by several authors (see [3] and references therein). Marczewski [1] showed that the condition

$$A \cap B = \phi \Rightarrow \mu(A)\nu(B) = 0 \tag{2}$$

is necessary and sufficient for a finitely additive splicing to exist, that is, for $\eta$ given by (1) to be well-defined and finitely additive. Stroock [3] introduced the following condition

$$X = \textstyle\bigcup_{n=1}^\infty (A_n \cap B_n) \Rightarrow \sum_{n=1}^\infty \mu(A_n)\nu(B_n) \geq 1 \tag{3}$$

and showed it to be necessary and sufficient for a splicing to exist.

Stroock's proof does not utilize Marczewski's result but requires the construction of certain quotient spaces and $\sigma$-isomorphism of the given space to a subset of their product. However, (3) clearly implies (2) (by writing $X = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c)$) and so $\eta$, given by (1), is a finitely additive splicing by Marczewski's result; at this juncture, to show that $\eta$ is a splicing it only remains to check its countable subadditivity. The aim of this note is to provide an elementary proof of Stroock's result by directly verifying that the finitely additive $\eta$ is countably subadditive when (3) holds (Proposition 1). Since we use Marczewski's result we include a simple proof of it different from that of Marczewski's (Proposition 2). Finally, we add some remarks on the splicing problem when $\mu$ and $\nu$ are compact measures.

PROPOSITION 1. *A splicing exists if and only if* (3) *holds.*

PROOF. It is obvious that condition (3) is necessary. To prove sufficiency first note that (3) implies (2), by writing

$$X = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c).$$

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Thus $\eta$ defined by (1) is a finitely additive splicing by Proposition 2. Hence $\eta$ is countably superadditive and it suffices to establish its countable subadditivity.

Let $F = \sum_{i=1}^{m} (A_i \cap B_i)$, $F_N = \sum_{j=1}^{m_N} (A_j^N \cap B_j^N)$ be such that $F = \sum_{N=1}^{\infty} F_N$.

For each fixed $i$

$$A_i \cap B_i = \sum_{N=1}^{\infty} (A_i \cap B_i) \cap F_N = \sum_{N=1}^{\infty} \sum_{j=1}^{m_N} (A_i \cap A_j^N) \cap (B_i \cap B_j^N)$$

and so

$$X = (\sum_{N=1}^{\infty} \sum_{j=1}^{m_N} (A_i \cap A_j^N) \cap (B_i \cap B_j^N)) \cup (A_i^c \cap B_i) \cup (A_i \cap B_i^c) \cup (A_i^c \cap B_i^c).$$

Hence, by (3) and using the fact that $\eta$ given by (1) is finitely additive we have

$$(\sum_{N=1}^{\infty} \sum_{j=1}^{m_N} \mu(A_i \cap A_j^N)\nu(B_i \cap B_j^N)) + \mu(A_i^c)\nu(B_i) + \mu(A_i)\nu(B_i^c) + \mu(A_i^c)\nu(B_i^c) \geq 1$$

or

$$\sum_{N=1}^{\infty} \sum_{j=1}^{m_N} \mu(A_i \cap A_j^N)\nu(B_i \cap B_j^N) \geq \mu(A_i)\nu(B_i). \tag{4}$$

Now

$$\begin{aligned}\eta(F) &= \sum_{i=1}^{m} \mu(A_i)\nu(B_i) \leq \sum_{i=1}^{m} \sum_{N=1}^{\infty} \sum_{j=1}^{m_N} \mu(A_i \cap A_j^N)\nu(B_i \cap B_j^N) \quad \text{by (4)}\\ &= \sum_{N=1}^{\infty} \sum_{j=1}^{m_N} \sum_{i=1}^{m} \mu(A_i \cap A_j^N)\nu(B_i \cap B_j^N)\\ &= \sum_{N=1}^{\infty} \eta(F_N)\end{aligned}$$

proving countable subadditivity of $\eta$. □

PROPOSITION 2 (Marczewski). *A finitely additive splicing exists if and only if* (2) *holds.*

PROOF. The necessity is clear. To prove sufficiency, consider

$(X \times X, \mathscr{A} \times \mathscr{B}, \mu \times \nu)$. Let $D = \{(x, x) : x \in X\}$ and let $\mathscr{S} = \{\sum_{i=1}^{n} (A_i \times B_i), n \geq 1\}$.

Since

$$A \cap B = \phi \Leftrightarrow (A \times B) \cap D = \phi$$

and

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

(2) is equivalent to

$$(A \times B) \cap D = \phi \Rightarrow (\mu \times \nu)(A \times B) = 0. \tag{5}$$

Let (5) hold and define $\beta$ on $\mathscr{S} \cap D$ by

$$\beta(S \cap D) = (\mu \times \nu)(S), \quad (S \in \mathscr{S}).$$

If $S \in \mathscr{S}$ and $S \cap D = \phi$ then $(\mu \times \nu)(S) = 0$; for, $S = \sum_{i=1}^{n} (A_i \times B_i)$, $S \cap D = \phi$ implies $(A_i \times B_i) \cap D = \phi$ for each $i$ and so $(\mu \times \nu)(S) = \sum_{i=1}^{n} (\mu \times \nu)(A_i \times B_i) = 0$ by (5). Now if $S_1, S_2 \in \mathscr{S}$, $S_1 \cap D = S_2 \cap D$ then $(S_1 \Delta S_2) \in \mathscr{S}$, $(S_1 \Delta S_2) \cap D = \phi$ and so $(\mu \times \nu)(S_1 \Delta S_2) = 0$. Thus $(\mu \times \nu)(S_1) = (\mu \times \nu)(S_2)$ and $\beta$ is well-defined.

If $S_1, S_2, \cdots, S_n \in \mathscr{S}$, $S_i \cap S_j \cap D = \phi$ for $i \neq j$, then

$$\sum_{i=1}^{n} (S_i \cap D) = (\cup_{i=1}^{n} S_i) \cap D = (\sum_{i=1}^{n} R_i) \cap D$$

where $R_1 = S_1$, $R_i = S_i - (\cup_{j=1}^{i-1} S_j)$, $i > 1$.
Further, for each $i$,

$$(S_i - R_i) \cap D = (S_i \cap D) \cap (\cup_{j=1}^{i-1} (S_j \cap D)) = \phi$$

and so $(\mu \times \nu)(S_i) = (\mu \times \nu)(R_i)$. Hence

$$\begin{aligned}\beta(\textstyle\sum_{i=1}^n (S_i \cap D)) &= \beta((\textstyle\bigcup_{i=1}^n S_i) \cap D) = \beta((\textstyle\sum_{i=1}^n R_i) \cap D)\\ &= (\mu \times \nu)(\textstyle\sum_{i=1}^n R_i) = \textstyle\sum_{i=1}^n (\mu \times \nu)(R_i)\\ &= \textstyle\sum_{i=1}^n (\mu \times \nu)(S_i) = \textstyle\sum_{i=1}^n \beta(S_i \cap D)\end{aligned}$$

and so $\beta$ is finitely additive.

Since $\eta$ defined by (1) on $\mathscr{C}_0$ satisfies

$$\eta(\textstyle\sum_{i=1}^n (A_i \cap B_i)) = \beta(\textstyle\sum_{i=1}^n (A_i \times B_i) \cap D)$$

and since the correspondence

$$\textstyle\sum_{i=1}^n (A_i \cap B_i) \leftrightarrow \textstyle\sum_{i=1}^n (A_i \times B_i) \cap D$$

from $\mathscr{C}_0$ to $\mathscr{S} \cap D$ is 1-1, onto, preserves finite unions, finite intersections, and complements, it follows that $\eta$ is well-defined and finitely additive. □

Since the compactness of a finitely additive measure implies its countable additivity (see Proposition 1.3.1 in [2]) one might ask whether the compactness of $\mu$ and $\nu$ together with condition (2) imply that a splicing exists. The answer is in the negative and is furnished by Stroock's example which we briefly discuss.

Let $\Omega = \{0, 1\}^{Z-\{0\}}$ where $Z$ is the set of all integers. For $n \geq 1$, let

$$\mathscr{M}^{(n)} = \sigma(\{\omega_k : k \geq n\}), \quad \mathscr{N}^{(n)} = \sigma(\{\omega_k : k \leq -n\})$$

and let $\beta$ on $\Omega$ be the product of the measure $P$ on $\{0, 1\}$ given by $P(\{0\}) = P(\{1\}) = ½$. Let

$$\beta_+ = \beta|_{\mathscr{M}^{(1)}}, \qquad \beta_- = \beta|_{\mathscr{N}^{(1)}}$$

and let

$$X = \{\omega \in \Omega : \lim_{n\to\infty} |\omega_n - \omega_{-n}| = 0\}.$$

It is easy to see that $C \in \mathscr{M}^{(1)} \cup \mathscr{N}^{(1)}$, $X \subseteq C$ implies $C = \Omega$; hence $\beta_+^*(X) = \beta_-^*(X) = 1$ where $\beta_+^*, \beta_-^*$ are the corresponding outer measures. Let

$$\mathscr{A} = \mathscr{M}^{(1)} \cap X, \mu(M \cap X) = \beta_+(M), \quad M \in \mathscr{M}^{(1)}$$
$$\mathscr{B} = \mathscr{N}^{(1)} \cap X, \nu(N \cap X) = \beta_-(N), N \in \mathscr{N}^{(1)}.$$

In [3] it is shown that (2) holds in this example and that there is no splicing of $\mu$ and $\nu$.

We shall show that the measures $\mu$ and $\nu$ in this example are compact. The compactness of $\mu$, for instance, is established as follows: Let $\mathscr{K} \subset \mathscr{M}^{(1)}$ be the compact subsets in $\mathscr{M}^{(1)}$. Then, $\mathscr{K}$ is a compact class approximating $\beta_+$. Note that if $\{K_n\} \subset \mathscr{K}$, $\bigcap_{n=1}^\infty (K_n \cap X) = \phi$ then $X \subseteq \bigcup_{n=1}^\infty K_n^c \in \mathscr{M}^{(1)}$ and so $\bigcup_{n=1}^\infty K_n^c = \Omega$ which implies that $\bigcap_{n=1}^\infty K_n = \phi$. Since $\mathscr{K}$ is a compact class $\bigcap_{n=1}^m K_n = \phi$ for some $m \geq 1$; so $\bigcap_{n=1}^m (K_n \cap X) = \phi$ and hence $\mathscr{K} \cap X$ is a compact class which approximates $\mu$. Hence $\mu$ is compact and $\nu$, similarly, is compact.

The following example shows that even when $\mu$ and $\nu$ are compact and (3) holds (that is, a splicing also exists) the splicing need not be compact.

EXAMPLE. Let $I$ be the unit interval, $\mathscr{B}_I$ the Borel $\sigma$-algebra on $I$ and $\lambda$ the Lebesgue measure on $(I, \mathscr{B}_I)$. Consider $(I \times I, \mathscr{B}_I \times \mathscr{B}_I, \lambda \times \lambda)$. We need the following lemma.

LEMMA. *There exists a subset $X$ of $I \times I$ such that* (i) *$X$ intersects every closed subset of $I \times I$ of positive $\lambda \times \lambda$ measure and* (ii) *$X$ is a graph both ways, that is, for every $x \in I$ the sets $\{y: (x, y) \in X\}$ and $\{y: (y, x) \in X\}$ are exactly singletons.*

PROOF. Let $\{A_\alpha: \alpha < \omega_c\}$ be a well ordering of closed subsets of $I \times I$ of positive $\lambda \times \lambda$ measure where $\omega_c$ is the first ordinal corresponding to $c$, the cardinality of the continuum. We define a transfinite sequence $\{p_\alpha = (x_\alpha, y_\alpha): \alpha < \omega_c\}$ as follows. Take $p_1 = (x_1, y_1) \in A_1$ with $x_1 \neq y_1$. Suppose $\{p_\alpha = (x_\alpha, y_\alpha): \alpha < \beta\}$ have been defined for $\beta < \omega_c$. The set $\{x: \lambda((A_\beta)_x) > 0\}$ is an uncountable Borel set and hence has cardinality $c$. So we can find $x_\beta$ in $\{x: \lambda((A_\beta)_x) > 0\} - \{x_\alpha, y_\alpha: \alpha < \beta\}$. Again $(A_\beta)_{x^\beta}$ being an uncountable Borel set has cardinality $c$. Take $y_\beta \in (A_\beta)_{x^\beta} - \{\{x_\alpha, y_\alpha: \alpha < \beta\}, x_\beta\}$ and let $p_\beta = (x_\beta, y_\beta)$. Let $X_0 = I - \{x_\alpha, y_\alpha: \alpha < \omega_c\}$ and define

$$X = \{(x_\alpha, y_\alpha), (y_\alpha, x_\alpha): \alpha < \omega_c\} \cup \{(x, x): x \in X_0\}.$$

$X$ has the required properties.

Since both $X$ and $(I \times I) - X$ intersect every closed subset of positive $\lambda \times \lambda$ measure we have $(\lambda \times \lambda)^*(X) = 1$, $(\lambda \times \lambda)_*(X) = 0$ where $(\lambda \times \lambda)^*$ and $(\lambda \times \lambda)_*$ are the outer and inner measures induced by $\lambda \times \lambda$. Let $\mathscr{A} = \mathscr{B}^{(1)} \cap X$, $\mathscr{B} = \mathscr{B}^{(2)} \cap X$ where $\mathscr{B}^{(1)} = \{B \times I: B \in \mathscr{B}_I\}$, $\mathscr{B}^{(2)} = \{I \times B: B \in \mathscr{B}_I\}$. Let $\mu = (\lambda \times \lambda)^*|_{\mathscr{A}}$, $\nu = (\lambda \times \lambda)^*|_{\mathscr{B}}$. Let $f$ and $g$ be defined on $I \times I$ by

$$f((x_1, x_2)) = x_1, \quad g((x_1, x_2)) = x_2.$$

It can be checked that $(X, \mathscr{A}, \mu)$ and $(X, \mathscr{B}, \nu)$ are isomorphic to $(I, \mathscr{B}_I, \lambda)$ under $f$ and $g$ respectively. Since $\lambda$ is compact it follows that $\mu$ and $\nu$ are compact. Clearly $\mathscr{A} \vee \mathscr{B} = (\mathscr{B}_I \times \mathscr{B}_I) \cap X$ and $\eta = (\lambda \times \lambda)^*|_{\mathscr{A} \vee \mathscr{B}}$ is a splicing of $\mu$ and $\nu$.

If $C$ is a compact subset of $X$ then $C$ is a compact subset of $I \times I$ and so $\eta(C) = (\lambda \times \lambda)(C) = 0$ since $(\lambda \times \lambda)_*(X) = 0$. Thus $\eta$ is not tight and, hence, cannot be compact.

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