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CALCULATION OF THE LAPLACE TRANSFORM OF THE LENGTH OF THE BUSY PERIOD FOR THE M | G | 1 QUEUE VIA MARTINGALES

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In this paper we derive a new explicit formula for the Laplace transform of the length of the busy period for the M | G | 1 queue by a direct martingale method of independent interest. The method is probabilistic, of general character and avoids tedious calculations with complex variables.

Let d denote the length of the busy period for an M | G | 1 queue whose service time distribution is denoted by H and the arrival process $A(t)$ is a Poisson process with intensity parameter $b > 0$ i.e: $P(A(t) = j) = \exp(-bt)(bt^j/j!)$, $j = 0, 1, 2, \cdots$. We assume of course that the traffic intensity $\rho = \mu b < 1$, where $\mu = \int_0^\infty y \, dH(y)$. Let $\phi(\alpha) = E(\exp(-\alpha d))$ and $\psi(\alpha) = \int_0^\infty \exp(-\alpha y) \, dH(y)$ denote the Laplace transforms of the busy cycle and service time distributions respectively and set $\lambda(\alpha) = b(\psi(\alpha) - 1) + \alpha$. Observe that $\lambda'(0) = b\psi'(0) + 1 = -\rho + 1 \neq 0$ by hypothesis so $\lambda^{-1}(\alpha)$ exists in a neighborhood of the origin and in particular for $0 \leq \alpha \leq \delta > 0$. The purpose of this note is to derive, by a martingale method of independent interest, a new formula for $\phi(\alpha)$, namely $\phi(\alpha) = \psi(\lambda^{-1}(\alpha))$. For a different formula as well as a different proof see Takács (1962).

Let $z(t) = (\sum_{j=0}^{A(t)} S_j) - t$ where the S_j are i.i.d. random variables with common distribution H, so $z(t)$ is a compound poisson process with jump rate b and drift rate equal to -1. It is well known that if $\eta(t)$ is the virtual waiting time process for the M | G | 1 queue and $z(0) = x > 0$ then $\eta(t \wedge d) = z(t \wedge d)$ and in particular $\eta(t) = z(t)$ on the set $t \leq d$. Note that, with this definition of z, we have $z(0) = \eta(0) = S_0$ which means we are taking the time origin as the time of a customer arrival who finds the queue empty.

(1) THEOREM. *The process* $x(t) = \exp[-\lambda(\alpha)(t \wedge d) - \alpha\eta(t \wedge d)]$ *is a martingale.*

Before proving this result let us derive the formula $\phi(\alpha) = \psi(\lambda^{-1}(\alpha))$ as a consequence. If $\eta(0) = x$ then $x(0) = \exp(-\alpha x)$ and so by the martingale property

$$E(\exp[-\lambda(\alpha)(t \wedge d) - \alpha\eta(t \wedge d)] \mid \eta(0) = x) = \exp(-\alpha x). \tag{2}$$

Using the fact that $\rho < 1$ implies $P(d < \infty) = 1$ we see at once that $\lim_{t\to\infty} t \wedge d = d$ and therefore $\lim_{t\to\infty}\eta(t \wedge d) = \eta(d) = 0$. Thus letting $t \to \infty$ on the left hand side of (2) and applying the bounded convergence theorem (since $\alpha > 0$ implies $\lambda(\alpha) > 0$) yields

$$E(\exp(-\lambda(\alpha)d) \mid \eta(0) = x) = \exp(-\alpha x). \tag{3}$$

Since $\eta(0) = S_0$ and since S_0 has the distribution H it follows from (3) that

$$\phi(\lambda(\alpha)) = E(\exp(-\lambda(\alpha)d)) = \int_0^\infty \exp(-\alpha x) \, dH(x) = \psi(\alpha)$$

or, taking inverses,

$$\phi(\alpha) = \psi(\lambda^{-1}(\alpha)). \tag{4}$$

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Notice that $E(d) = -\phi'(0) = -\psi'(0)/\lambda'(0) = \mu/(1 - \rho)$ which is the well known formula for the expected length of the busy period.

Turning now to the proof of the Theorem 1 we begin with the well known result concerning compound Poisson processes with drift that

$$E(\exp[-\alpha(z(t + u) - z(t))]) = \exp(\alpha u + ub(\psi(\alpha) - 1)) = \exp(u\lambda(\alpha)).$$

Since z has independent increments, this shows that $E(\exp[-\alpha(z(t + u) - z(t))] \mid F(t)) = \exp(u\lambda(\alpha))$ where $F(t) = \sigma(z(s); s \le t)$. Hence $y(t) = \exp(-t\lambda(\alpha) - \alpha z(t))$ is a martingale with respect to the σ-fields $F(t)$. Since $d = \inf\{t \ge 0; z(t) = 0\}$ is a stopping time it follows at once from Doob's optional stopping theorem that $y(t \wedge d) = \exp(-\lambda(\alpha)(t \wedge d) - z(t \wedge d))$ is also a martingale. But $z(t \wedge d) = \eta(t \wedge d)$ and therefore $x(t) = y(t \wedge d)$. The proof is finished.

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