The Annals of Probability
1983, Vol. 11, No. 3, 814–816

THE NATURAL BOUNDARY PROBLEM FOR RANDOM POWER SERIES WITH DEGENERATE TAIL FIELDS

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If the sequence of coefficients of a random power series has a degenerate tail field, then either its circle of convergence is a natural boundary, or this situation can be achieved by subtracting a fixed series. This generalises the known result for independent coefficient sequences.

1. Let $\{A_n, n = 0, 1, \cdots\}$ be a complex random sequence and $F(z) = \sum A_n z^n$ the corresponding random power series in the complex variable z. Following a conjecture by Blackwell, Ryll-Nardzewski [11] proved the following result:

If the A_n are independent, there is a fixed (i.e. non-random or degenerately random) power series $f(z) = \sum a_n z^n$ such that

(i) with probability one, $F(z) - f(z)$ is singular at every point of its circle of convergence (i.e. it has a natural boundary);

(ii) the radius of convergence $r(F - f)$ of $F(z) - f(z)$ (which is almost surely (a.s.) constant) is maximal with respect to the choice of fixed series;

(iii) if $g(z)$ is another fixed series such that $r(F - g) = r(F - f)$ a.s. then $F(z) - g(z)$ has a.s. a natural boundary.

By convention, an entire function is considered to have a "natural boundary" at infinity.

The background and many detailed results associated with the problem are described by Arnold [2].

Ryll-Nardzewski's proof is based on symmetrisation and median-centering (see Loève [9, vol. 1, section 18] and in this context particularly Walk [12]), and a theorem of Marcinkiewicz and Zygmund [10, Theorem 7] which in turn depends on the Riesz-Fischer Theorem. Kahane [6, Chapter 4] has given a neat alternative proof, also based on symmetrisation. Ahmad [1] has provided a criterion for $F(z)$ itself to have a.s. a natural boundary, and his result has been strengthened by Hinderer and Walk [6, Section 5] (see also Walk [12, 13]). Extensive studies of random power series, including results about boundary behaviour that implies singularity, can be found in the work of Walk and Hinderer [6, 12–15].

There have been a few references to the situation where the A_n are not independent. Walk has obtained results for properties implying a natural boundary subject to the assumptions of two types of weak dependence [11, 14] and a further kind of multiplicative dependence [15].

In this note, a complement to [7] in which I discussed the distribution of the radius of convergence of a random power series with non-independent coefficients, I show that Ryll-Nardzewski's theorem continues to hold in the dependent case, provided only that the sequence of coefficients has a degenerate tail field. The essential role of the tail field was foreshadowed by Borel [4].

The most important non-independent sequences whose tail fields are degenerate are certain, not necessarily homogeneous, Markov chains. If $\{A_n\}$ is such a sequence, let $M_{s,t}(w) = \sup\{\Pr(S) - \Pr(S \mid F_t)\}$ where F_t is the sub field in the probability space generated by A_t, and S runs through the sets of the subfield generated by A_s, for $t > s$. Cohn [5] has shown that $\lim_{k\to\infty}\lim_{n\to\infty} M_{s,t}(w)$ exists a.s., and that the tail field of the chain is degenerate if and only if the limit is a.s. zero. Particular cases of degeneracy were established earlier, e.g. by Blackwell and Freedman [3].

Received May 1982.
AMS 1970 *subject classification*. Primary 60H99; secondary 30A12.
Key words and phrases. Random power series, natural boundary, Blackwell's conjecture.

2. The tail field of a random sequence is said to be degenerate if every set in it has probability 0 or 1.

PROPOSITION. *If the tail field of $\{A_n\}$ is degenerate, the propositions of Ryll-Nardzewski's Theorem hold for the random power series $F(z) = \sum A_n z^n$.*

PROOF. For any integer k, let $\{A_n^{(s)}\}$, $s = 0, \cdots, k-1$ be random sequences, independent of each other, with the same distribution as $\{A_n\}$. Let $F_s(z) = \sum A_n^{(s)} z^n$. Write $\beta = 2\pi/k$ and set

$$H_k(z) = \sum_{s=0}^{k-1} e^{i\beta s} \sum_{n=0}^{\infty} A_n^{(s)} z^n \tag{1}$$

$$= \sum_{n=0}^{\infty} z^n \sum_{s=0}^{k-1} e^{i\beta s} A_n^{(s)}. \tag{2}$$

Since the tail field of $\{A_n\}$ is degenerate, each point z is either a.s. regular or a.s. singular for the random series $F(z)$. Let $\gamma(F, r)$ denote the angular measure of the largest arc of a.s. regularity of F on the circle $|z| = r$. Let z be an a.s. regular point for $F(z)$. Since $H_k(z)$ is the sum of k independent series, each of which is a.s. regular at z, it is itself a.s. regular at z. Hence

$$\gamma(H_k, r) \geq \gamma(F, r). \tag{3}$$

Now

$$H_k(ze^{i\beta}) = \sum_{n=0}^{\infty} z^n e^{i\beta n} \sum_{s=0}^{k-1} e^{i\beta s} A_n^{(s)} \tag{4}$$

$$= \sum_{n=0}^{\infty} z^n \sum_{s=0}^{k-1} e^{i\beta s} A_n^{(s-n) \bmod k}.$$

Let $\{\alpha_n\}$, $\{\mu_n\}$ be any sequences of complex numbers, and consider the random sequences $\{t_n\}$, $\{\tau_n\}$ defined by

$$t_n = \left(\sum_{s=0}^{k-1} e^{i\beta s} A_n^{(s)} - \alpha_n\right)/\mu_n.$$

$$\tau_n = \left(\sum_{s=0}^{k-1} e^{i\beta s} A^{(s-n) \bmod k} - \alpha_n\right)/\mu_n.$$

Now by our hypothesis any complex number λ is a.s. a limit point of $\{t_n\}$, or a.s. not. If it is, it must be an a.s. limit point of the subsequence $\{t_{nk+\ell}\}$ for at least one $\ell = 0, \cdots, k-1$. Since the joint distribution of the $\{A_n^{(s)}\}$, $s = 0, \cdots, k-1$ is invariant under permutation of the s, $\{\tau_{nk+\ell}\}$ is equidistributed with $\{t_{nk+\ell}\}$ and hence also has λ as a.s. limit point. Hence λ is an a.s. limit point of $\{\tau_n\}$. The k subsequences for $\ell = 0, \cdots, k-1$ have different dependence structures for the two sequences, but this does not affect the occurrence of an event whose probability is one. The argument is reversible, and since the choice of $\{\alpha_n\}$, $\{\beta_n\}$ is arbitrary, the standard tests for regularity of points with respect to power series show that $H_k(z)$ and $H_k(ze^{i\beta})$ have the same singularities. In other words, if $H_k(z)$ is a.s. singular at z_0, it is also a.s. singular at $z_0 e^{i\beta}$, and hence at $z_0 e^{i\beta s}$, $s = 2, \cdots, k-1$. Thus for any $r > 0$ we have three possibilities:

(5)
(a) $H_k(z)$ is a.s. regular everywhere on $|z| = r$.
(b) $0 < \gamma(H_k, r) \leq 2\pi/k = \beta$
(c) $\gamma(H_k, r) = 0$, i.e. $H_k(z)$ is a.s. singular everywhere on $|z| = r$.

We choose $k > 2\pi/\gamma(F, r)$. Then there is a contradiction between (3) and (5), so case (b) cannot occur. In case (a), as in [8, page 34], we take a sample realisation of $\{A_n^{(s)}\}$, $s = 1, \cdots, k-1$ and form

$$f(z) = -\sum_{s=1}^{k-1} e^{i\beta s} \sum_{n=0}^{\infty} A_n^{(s)} z^n. \tag{6}$$

Then $F(z) - f(z)$ is a.s. regular on $|z| = r$, with probability one in respect of the sampling procedure. In particular, one such $f(z)$ can be found, and we say that it clears the disk $|z| \leq r$ of singularities.

In case (c), it follows from (3) that $F(z)$ is also a.s. singular at every point on $|z| = r$. Suppose it were possible to find a fixed function $g(z)$ such that the a.s. singularity set of $F(z) - g(z)$ on $|z| = r$ is smaller than the whole circle. Then by (3), so is that of $\sum e^{i\beta s}\{F^{(s)}(z) - g(z)\} = H_k(z)$, which is a contradiction.

Thus for each r it is either possible to find a fixed function $f(z)$, possibly zero, such that $F(z) - f(z)$ is a.s. regular everywhere on $|z| = r$; or it is not possible and $F(z)$ is a.s. singular on $|z| = r$. If the second alternative holds for some r, it clearly holds for all greater values. Let ρ be the infimum of these case (c) values. It then follows by a contradiction argument that there is some $f(z)$ such that $F(z) - f(z)$ has a.s. no singularities in $|z| < \rho$, and $|z| = \rho$ is a.s. a natural boundary. The other assertions of the theorem are easily checked.

The truth of the conclusions of Ryll-Nardzewski's theorem does not imply the degeneracy of the tail field of $\{A_n\}$. However the proposition does enable us to summarise fairly satisfactorily the natural boundary problem for general random power series, in terms of the following obvious corollary.

COROLLARY. *Corresponding to every random power series* $\sum A_n z^n$ *there corresponds a random power series* $\Phi(z)$, *measurable with respect to the tail field of* $\{A_n\}$, *such that the conclusions of Ryll-Nardzewski's Theorem hold, with "maximal" in* (ii) *and the equality in* (iii) *being required to hold almost everywhere.*

The degree of complication of $\Phi(z)$ can therefore be expressed in term of the results given in [5].

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