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STATIONARY STRONGLY MIXING SEQUENCES NOT SATISFYING THE CENTRAL LIMIT THEOREM

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For every sequence $(\varepsilon_n)_{n\in N}$ in $(0, 1)$ there exists a strictly stationary orthonormal sequence $(X_n)_{n\in N}$ of random variables with $|P(A \cap B) - P(A)P(B)| \le \varepsilon_n$ for all $A \in \sigma(X_1, \cdots, X_k)$, $B \in \sigma(X_{k+n}, X_{k+n+1}, \cdots)$, $k \in N$, $n \in N$, such that the distribution of $n^{-1/2}\sum_{i=1}^n X_i$ is not weakly convergent to the standard normal distribution.

1. Introduction and notations.

Let $(X_n)_{n\in N}$ be a sequence of random variables on some probability space $(\Omega, \mathscr{A}, P)$ with $EX_n = 0$ and $EX_n^2 < \infty$. Put $S_n = \sum_{i=1}^n X_i$ for $n \in N$. In this paper we consider strictly stationary sequences which fulfill

$$ES_n^2/n \to \sigma^2 \quad \text{as} \quad n \to \infty \quad \text{for some} \quad \sigma > 0. \tag{1.1}$$

(X_n) is said to satisfy the c.l.t., if $S_n/(\sigma n^{1/2})$ is weakly convergent to the standard normal distribution. For σ-fields $\mathscr{F}, \mathscr{G} \subset \mathscr{A}$ the coefficient of strong mixing is defined by

$$\alpha(\mathscr{F}, \mathscr{G}) = \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathscr{F}, B \in \mathscr{G}\}.$$

The mixing coefficient $\alpha(n)$ of the sequence (X_n) related to the strong mixing condition of Rosenblatt (1956) is defined by

$$\alpha(n) = \sup\{\alpha(\sigma(X_i : 1 \le i \le k), \sigma(X_i : i \ge n + k)) : k \in N\}.$$

Ibragimov (1962) proved the c.l.t. under the following assumption on $\alpha(n)$ and the moments of X_n:

$$\text{There exists} \quad \delta > 0 \quad \text{such that} \quad E|X_1|^{2+\delta} < \infty \quad \text{and} \quad \textstyle\sum_{n\in N} \alpha(n)^{\delta/(2+\delta)} < \infty. \tag{1.2}$$

Sequences (X_n) with $E|X_n|^{2+\delta} = \infty$ for every $\delta > 0$ were not covered by this theorem, but on page 420 in [7] it was mentioned that the c.l.t. holds under the following less restrictive assumption:

$$\begin{gathered}\text{There exists} \quad \delta \ge 0 \quad \text{such that} \quad E|X_1|^{2+\delta} < \infty \quad \text{and} \\ \textstyle\sum_{n\in N} \alpha(n)^{(1+\delta)/(2+\delta)} < \infty.\end{gathered} \tag{1.3}$$

This result is attributed to Gordin and has been stated again and proved by Hall and Heyde (1980), Corollary 5.3 (ii). In the present paper the following theorem will be proved.

THEOREM. *Let $\varepsilon_n > 0$, $n \in N$ be given. Then there exists a strictly stationary sequence $(X_n)_{n\in N}$ with $EX_1 = 0$, $0 < EX_1^2 < \infty$ and $EX_1X_n = 0$ for all $n \ge 2$ such that $\alpha(n) \le \varepsilon_n$ for all $n \in N$, and the sequence $(S_n)_{n\in N}$ of partial sums has the following properties*:

(A) $\inf_{n\in N} P\{S_n = 0\} > 0$,
(B) *the family of distributions of the partial sums S_n, $n \in N$, is tight,*
(C) *$S_n/b_n \to 0$ in probability as $n \to \infty$, for every sequence $(b_n)_{n\in N}$ in $(0, \infty)$ with $b_n \to \infty$.*

Since (1.1) is trivially satisfied for every uncorrelated stationary sequence, the above theorem shows that it is impossible to find any fixed numbers $\varepsilon_n > 0$ such that every

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strictly stationary sequence in L_2 which fulfills (1.1) and $\alpha(n) \le \varepsilon_n$ satisfies the c.l.t. In particular (1.1) and (1.3) for $\delta = 0$ are not sufficient for the c.l.t. The derivation of Corollary 5.3 from Theorem 5.4 in [5] is incorrect in that it was not shown that the quantity λ in Theorem 5.4 had to be positive. The proof of Theorem 5.4 in [5] contains another error: in the argument for $EY_0^2 < \infty$ the authors choose $n_0\{U(A)\}$ on page 138, but this index may also depend on u.

Counterexamples to the c.l.t. under strong mixing have been constructed in Davydov (1973), and in a forthcoming paper of Bradley [2] there is a counterexample with an arbitrarily fast mixing rate. The purpose of the present paper is the discussion of the c.l.t. under the additional assumption (1.1), which is not fulfilled in the earlier examples.

2. Auxiliary results. The first lemma shows that $\alpha(\mathscr{F}, \mathscr{F})$ is small, if $\mathscr{F}$ is a σ-field which contains a large atom.

2.1 LEMMA. *If F is an atom of the σ-field $\mathscr{F}$, then $\alpha(\mathscr{F}, \mathscr{F}) \le 1 - P(F)$.*

PROOF. Let $A, B \in \mathscr{F}$. If $A \cap F = \varnothing$ or $B \cap F = \varnothing$, then $P(A \cap B) \in [0, 1 - P(F)]$ and $P(A)P(B) \in [0, 1 - P(F)]$. If $A \supset F$ and $B \supset F$, then $P(A \cap B) - P(A)P(B) \le P(A)(1 - P(B)) \le 1 - P(F)$ and $P(A \cap B) - P(A)P(B) \ge P(F) - 1$. In both cases follows $|P(A \cap B) - P(A)P(B)| \le 1 - P(F)$.

The following lemma is Lemma 8 of Bradley (1981).

2.2 LEMMA. *If $\mathscr{F}_n$ and $\mathscr{G}_n$, $n \in N$, are σ-fields and the σ-fields $(\mathscr{F}_n \vee \mathscr{G}_n)$, $n \in N$, are independent, then $\alpha(\bigvee_{n=1}^{\infty} \mathscr{F}_n, \bigvee_{n=1}^{\infty} \mathscr{G}_n) \le \sum_{n=1}^{\infty} \alpha(\mathscr{F}_n, \mathscr{G}_n)$.*

Using 2.1 and 2.2, one can easily estimate the mixing coefficient of some "moving average sequences", which will play an important role in our example.

2.3 LEMMA. *Let $(\xi_n)_{n\in Z}$ be an* i.i.d. *sequence with $P\{\xi_n = 0\} \ge 1 - \varepsilon$ for some $\varepsilon \le 1$. Let $(X_n)_{n\in N}$ be defined by*

(2.4) $$X_n = \textstyle\sum_{j=-N}^{N} c_j \xi_{j+n} \text{ for some } N \in N \cup \{0\},\ c_j \in \mathbb{R}.$$

Then $\alpha(n) \le \varepsilon \max(2N - n + 1, 0)$.

PROOF.
$$\begin{aligned} \alpha(n) &\le \alpha(\sigma(\xi_i : i \le N), \sigma(\xi_i : i \ge n - N)) \\ &\le \textstyle\sum_{i=n-N}^{N} \alpha(\sigma(\xi_i), \sigma(\xi_i)) \\ &\le \varepsilon \max(2N - n + 1, 0). \end{aligned}$$

A nonnegative integrable function f on $[-\frac{1}{2}, \frac{1}{2}]$ is called spectral density of the stationary sequence $(X_n)_{n\in N}$ in L_2, if

$$EX_nX_{n+k} = \int_{-1/2}^{1/2} e^{2\pi ik\lambda} f(\lambda)\, d\lambda, \quad n \in N, \quad k \in N \cup \{0\}.$$

The following formula for the variance of the partial sums of a stationary sequence with spectral density f is well known.

(2.5) $$ES_n^2 = \int_{-1/2}^{1/2} \frac{\sin^2(\pi\lambda n)}{\sin^2(\pi\lambda)} f(\lambda)\, d\lambda, \quad n \in N.$$

The stationary sequences which are building blocks of our example have well known spectral densities (see [4] page 499).

2.6 LEMMA. *If $(\xi_n)_{n\in Z}$ is an orthonormal sequence in L_2 and (X_n) is defined by* (2.4), *then (X_n) has the spectral density*

$$f(\lambda) = |\textstyle\sum_{j=-N}^{N} c_j e^{2\pi ij\lambda}|^2, \quad \lambda \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

2.7 LEMMA. *If f is the spectral density of* 2.6 *and $\sum_{j=-N}^{N} c_j = 0$, then*

$$\sup_{n\in N} \int_{-1/2}^{1/2} \frac{\sin^2(\pi\lambda n)}{\sin^2(\pi\lambda)} f(\lambda)\, d\lambda < \infty.$$

PROOF. Elementary calculus shows that $f(\lambda)/\sin^2(\pi\lambda)$ is bounded.

The X_n in the example will be constructed in the following way: We choose an independent sequence of appropriate moving average processes $((X_{n,k})_{n\in N})_{k\in N}$, and put $X_n = \sum_{k\in N} X_{n,k}$ for $n \in N$. If $(X_{n,k})_{n\in N}$ has spectral density f_k and $\sum_{k\in N} f_k \equiv 1$, then we obtain an orthonormal sequence $(X_n)_{n\in N}$. The following lemma shows that the constant 1 can be written as the sum of the spectral densities of special moving average processes.

2.8 LEMMA. *There exist functions f_k on $[-\tfrac{1}{2}, \tfrac{1}{2}]$, $k \in N$, with $f_k(\lambda) = |\sum_{j=-N(k)}^{N(k)} c_{j,k} e^{2\pi ij\lambda}|^2$ for some $N(k) \in N \cup \{0\}$, $c_{j,k} \in \mathbb{R}$ with $c_{-j,k} = c_{j,k}$ and $\sum_{j=-N(k)}^{N(k)} c_{j,k} = 0$, such that $\sum_{k\in N} f_k(\lambda) = 1$ for all $\lambda \in [-\tfrac{1}{2}, \tfrac{1}{2}] - \{0\}$.*

PROOF. The f_k will be defined by induction. Put $f_1 = 0$. Assume that $f_1, \cdots, f_n$ have already been defined and fulfill

(i) $\quad \sum_{k=1}^{n} f_k(\lambda) \le 1 - 2^{-n}, \quad \lambda \in [-\tfrac{1}{2}, \tfrac{1}{2}]$

(ii) $\quad \sum_{k=1}^{n} f_k(\lambda) \ge 1 - 2^{-n+1}, \quad \lambda \in [-\tfrac{1}{2}, \tfrac{1}{2}] - [-2^{-n}, 2^{-n}]$.

We will define f_{n+1} such that (i) and (ii) remain valid when n is replaced by $n + 1$. Put $h(\lambda) = \min(1, |\lambda| 2^{n+1})$ and $g(\lambda) = (1 - 2^{-n} + 2^{-n-2} - \sum_{k=1}^{n} f_k(\lambda))h(\lambda)$. (i) implies $g(\lambda) \ge 0$. $g(\lambda)^{1/2}$ is a continuous even function on $[-\tfrac{1}{2}, \tfrac{1}{2}]$. Hence for every $\varepsilon > 0$ there exist $N \in N \cup \{0\}$ and $a_j \in \mathbb{R}$ such that $|g(\lambda)^{1/2} - \sum_{j=0}^{N} a_j \cos(2\pi j\lambda)| \le \varepsilon$ for all $\lambda \in [-\tfrac{1}{2}, \tfrac{1}{2}]$. In particular $|\sum_{j=0}^{N} a_j| \le \varepsilon$, since $g(0) = 0$. Take $\varepsilon = 2^{-n-4}$, $N(n + 1) = N$, $c_{j,n+1} = a_{|j|}/2$ for $j \in \{1, \cdots, N\} \cup \{-1, \cdots, -N\}$, $c_{0,n+1} = a_0 - \sum_{j=0}^{N} a_j$, and define $f_{n+1}(\lambda)$ with these parameters. Then follows

$$|g(\lambda)^{1/2} - \textstyle\sum_{j=-N}^{N} c_{j,n+1} e^{2\pi ij\lambda}| \le 2\,\varepsilon$$

$$|g(\lambda) - f_{n+1}(\lambda)| \le 4\,\varepsilon.$$

Using (i), (ii) and the definition of g, one easily obtains

(i′) $\quad \sum_{k=1}^{n+1} f_k(\lambda) \le 1 - 2^{-n-1}, \quad \lambda \in [-\tfrac{1}{2}, \tfrac{1}{2}]$

(ii′) $\quad \sum_{k=1}^{n+1} f_k(\lambda) \ge 1 - 2^{-n}, \quad \lambda \in [-\tfrac{1}{2}, \tfrac{1}{2}] - [-2^{-n-1}, 2^{-n-1}]$.

Now it follows by induction that $(f_k)_{k\in N}$ can be defined such that (i) and (ii) hold for every $n \in N$, and therefore $\sum_{k\in N} f_k(\lambda) = 1$ for every $\lambda \in [-\tfrac{1}{2}, \tfrac{1}{2}] - \{0\}$.

3. The example. Let $\varepsilon_n > 0$, $n \in N$, be given. W.l.g. assume $\varepsilon_{n+1} \le \varepsilon_n \le 1$ for $N \in N$. Let $f_k(\lambda) = |\sum_{j=-N(k)}^{N(k)} c_{j,k} e^{2\pi ij\lambda}|^2$ be the functions which have been defined in 2.8. Define positive real numbers a_k by

(3.1) $$a_k = 2^{k+1}(4N(k) + 2)/\varepsilon_{2N(k)}, \qquad k \in N.$$

Let $(\xi_{j,k})_{j\in Z,k\in N}$ be independent random variables with $P\{\xi_{j,k} = \pm a_k^{1/2}\} = (2a_k)^{-1}$,

$P\{\xi_{j,k} = 0\} = 1 - a_k^{-1}$. Define

$$X_{n,k} = \sum_{j=-N(k)}^{N(k)} c_{j,k}\xi_{n+j,k}, \qquad n \in N, \qquad k \in N.$$

As a consequence of 2.6 and 2.8, we obtain

$$\sum_{k\in N} EX_{n,k}^2 = \sum_{k\in N} \int_{-1/2}^{1/2} f_k(\lambda)\, d\lambda = 1.$$

Since $(X_{n,k})_{k\in N}$ is an independent sequence, it follows that $\sum_{k\in N} X_{n,k}$ converges a.s. and in L_2. Define

$$X_n = \sum_{k\in N} X_{n,k} \text{ for } n \in N.$$

Clearly $(X_n)_{n\in N}$ is strictly stationary, $EX_n = 0$ and $EX_n^2 < \infty$. Using 2.6 and 2.8, one obtains

$$EX_nX_m = \sum_{k\in N} EX_{n,k}X_{m,k} = \sum_{k\in N} \int_{-1/2}^{1/2} f_k(\lambda)e^{2\pi i|m-n|\lambda}\, d\lambda$$

$$= \int_{-1/2}^{1/2} e^{2\pi i|m-n|\lambda}\, d\lambda = \delta_{m,n}$$

i.e. the sequence $(X_n)_{n\in N}$ is orthonormal. The mixing coefficient $\alpha(n)$ of (X_n) can be estimated with the help of 2.2 and 2.3. Let $\alpha_k(n)$ denote the mixing coefficient of $(X_{n,k})_{n\in N}$. One can write:

$$\begin{aligned}\alpha(n) &\le \sum_{k\in N} \alpha_k(n) \le \sum_{k:2N(k)\ge n} \alpha_k(n)\\ &\le \sum_{k:2N(k)\ge n} a_k^{-1} 2N(k)\\ &\le \sum_{k:2N(k)\ge n} \varepsilon_{2N(k)} 2^{-k-2} \le \varepsilon_n.\end{aligned}$$

Here (3.1) and the assumption that (ε_n) is non-increasing were used. For $n, k \in N$ let $S_{n,k} = \sum_{j=1}^{n} X_{j,k}$. For each n and k

$$S_{n,k} = \sum_{j=-N(k)+1}^{N(k)+1} d_j\xi_{j,k} + \sum_{j=-N(k)+n}^{N(k)+n} d_j\xi_{j,k} \tag{3.2}$$

where the coefficients d_j depend on n and k. Here for $n \ge 2N(k) + 2$ we are using the fact that $\sum_{j=-N(k)}^{N(k)} c_{j,k} = 0$ from Lemma 2.8. It follows from (3.1), (3.2) and $\varepsilon_n \le 1$ that for each n and k,

$$\begin{aligned}P\{S_{n,k} \neq 0\} &\le (4N(k) + 2)P\{\xi_{1,k} \neq 0\}\\ &= (4N(k) + 2)\, a_k^{-1} \le 2^{-k-1}.\end{aligned} \tag{3.3}$$

Since $S_n = \sum_{k\in N} S_{n,k}$ for all n, one has

$$P\{S_n \neq 0\} \le \sum_{k\in N} P\{S_{n,k} \neq 0\} \le 1/2$$

for all n, and (A) follows. To show that (B) holds, let $\varepsilon > 0$ be given. Choose a positive integer M such that $\sum_{k=M+1}^{\infty} 2^{-k-1} \le \varepsilon/2$. Then (3.3) implies

$$\sup_{n\in N} P\{\sum_{k=M+1}^{\infty} S_{n,k} \neq 0\} \le \varepsilon/2. \tag{3.4}$$

For each fixed k, $ES_{n,k}^2$ is a bounded function of n by (2.5) and Lemma 2.7. Hence

$$\sup_{n\in N} E(\sum_{k=1}^{M} S_{n,k})^2 < \infty$$

Hence there exists $a > 0$ such that

$$\sup_{n\in N} P\{|\sum_{k=1}^{M} S_{n,k}| > a\} \le \varepsilon/2.$$

This and (3.4) together imply that $\sup_{n\in N} P\{|S_n| > a\} \leq \varepsilon$. Thus (B) holds. Clearly (B) implies (C).

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