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AN EXPLICIT FORMULA FOR THE C.D.F. OF THE L_1 NORM OF THE BROWNIAN BRIDGE

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Let $X(t)$, $0 \le t \le 1$, be the Brownian bridge and $L = \int_0^1 |X(t)|\, dt$. Using results of Shepp [5] and Rice [4], an explicit formula for $P(L \le \ell)$ is obtained.

1. Introduction. Let $X(t)$, $t \in [0, 1]$, be the Brownian bridge and

$$L = \int_0^1 |X(t)|\, dt$$

The distribution of L has been recently studied in [5] and [4]. Shepp [5] using the Kac formula obtains the moments of L and characterizes $Ee^{-\beta L}$. Rice [4] using the results of [5] gives an explicit formula for $Ee^{-\beta L}$ and obtains the density of L by a numerical inversion. In this note, we invert $\beta^{-1}Ee^{-\beta L}$ to obtain an explicit formula for the C.D.F., $P(L \le \ell)$. The C.D.F. is also tabulated.

2. Main result. We establish the following.

THEOREM.

$$P(L \le \ell) = (\pi/2)^{1/2} \textstyle\sum_{j=1}^{\infty} \dfrac{1}{\delta_j^{3/2}} \psi(\ell/\delta_j^{3/2}), \quad \textit{where} \tag{1}$$

i) $\psi(t) = \dfrac{3^{2/3} e^{-\frac{2}{27t^2}}}{t^{1/3}} Ai((3t)^{-4/3})$, *and Ai is the Airy function, and*

ii) $\delta_j = -a_j'/2^{1/3}$ *and* a_j' *is the* jth *zero of* Ai'.

To establish the above, we begin with an equivalent form of the formula (8) of [4].

$$Ee^{-\beta L} = (\pi/2)^{1/2} \textstyle\sum_{j=1}^{\infty} \dfrac{\beta^{1/3} e^{-\delta_j \beta^{2/3}}}{\delta_j} \tag{2}$$

where the δ_j are as above and are also the positive zeros of P', where

$$P(y) = \frac{(2y)^{1/2}}{3}\left(J_{-1/3}\left(\frac{(2y)^{3/2}}{3}\right) - J_{1/3}\left(\frac{(2y)^{3/2}}{3}\right)\right). \tag{3}$$

Formula (3) is required below and that $P'(\delta_j) = 0$ follows easily from the definition of the Ai.

To invert

$$\beta^{-1}Ee^{-\beta L} = (\pi/2)^{1/2} \textstyle\sum_{j=1}^{\infty} \dfrac{e^{-\delta_j \beta^{2/3}}}{\delta_j \beta^{2/3}}, \tag{4}$$

our interest centers on inverting $z^{-2/3}\exp(-z^{2/3})$. Humbert [2], in a heuristic exposition, inverts such functions as $z^{1/3}\exp(-z^{1/3})$. Pollard [3], with a judicious choice of contour, inverts $\exp(z^{-\alpha})$, $0 < \alpha < 1$. $z^{-\alpha}\exp(-z^{\alpha})$ is the convolution of $t^{\alpha-1}/\Gamma(\alpha)$ with Pollard's

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TABLE I

ℓ	$P(L \le \ell)$	ℓ	$P(L \le \ell)$	ℓ	$P(L \le \ell)$
.1531	.05	.2818	.50	.5123	.91
.1721	.10	.2975	.55	.5267	.92
.1875	.15	.3147	.60	.5427	.93
.2011	.20	.3338	.65	.5610	.94
.2142	.25	.3553	.70	.5821	.95
.2270	.30	.3804	.75	.6074	.96
.2395	.35	.4103	.80	.6391	.97
.2533	.40	.4480	.85	.6822	.98
.2671	.45	.4993	.90	.7518	.99

result. On the other hand, using Pollard's contour, it is easily seen the inverse of $z^{-\alpha}(\exp(z^{-\alpha}))$ is

$$\psi_\alpha(t) = \frac{1}{\pi} I\left\{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \exp(-i\pi\alpha(k-1)) \frac{\Gamma((k-1)\alpha+1)}{t^{(k-1)\alpha+1}}\right\}, \tag{5}$$

where $I\{\ \}$ is the imaginary part of $\{\ \}$. In the case $\alpha = 2/3$ one finds

$$\psi_{2/3}(t) = \frac{3^{1/2}}{2\pi}\left\{\frac{\Gamma(1/3)}{t^{1/3}}\Phi\left(\frac{1}{6}, \frac{1}{3}, \frac{-4}{27t^2}\right) - \frac{\Gamma(2/3)}{3t^{5/3}}\Phi\left(\frac{5}{6}, \frac{5}{3}, \frac{-4}{27t^2}\right)\right\}, \tag{6}$$

where $\Phi(a, c, \cdot)$ is the confluent hypergeometric function. Using relations between Φ and modified Bessel functions when $c = 2a$, $\psi_{2/3}$ is seen to be the ψ of the theorem.

It remains to justify the termwise inversion. ψ is clearly positive on $(0, \infty)$, though not nondecreasing. Recalling formula (3) and using the fact that the zeros of P' and P alternate and information on the zeros of $J_{-1/3}$ and $J_{1/3}$, Watson [6], page 490, lower bounds of $\delta_j \ge \frac{1}{2}(3\pi(j-2+7/12))^{2/3}, j \ge 3$, are obvious. As the series (1) converges, the proof is complete.

3. Tabulation. The series (1) converges very rapidly. With the crude bound on the δ_j, a majorizing integral for the sum from $j = 11$ on is $\le 2 \times 10^{-10}$ for $0 < t \le 2$. Table I was obtained using tables of Ai and a_j', *Handbook of Mathematical Functions* [1], and is supposed to be accurate to five places with interpolation. In the range of the argument of the table, all terms from the fifth to the tenth are zero to seven places.

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