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SOME COMMENTS ON THE ERDŐS-RÉNYI LAW AND A THEOREM OF SHEPP

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We show that the finiteness of the moment generating function is necessary for the finiteness of the lim sup of the moving averages considered by Shepp (1964). This also implies that the same must be true for the Erdős-Rényi law of large numbers.

1. Introduction. Recently, there was a comment attributed to me concerning the Erdős-Rényi (E-R), 1970, law of large numbers (see Csörgő and Steinebach, 1981). The gist of the comment was that the E-R law had been anticipated by Shepp in a 1964 paper which seems to have been overlooked by those working on E-R type laws. In particular, the E-R law deals with limits of maximums of certain types of partial sums of i.i.d. random variables while a specialization of Shepp's results deals with the lim sup of a subsequence of these partial sums. In both cases the limits are the same.

Here, we elaborate further on the relationship of the E-R law and Shepp's result. We show that, if the moment generating function (m.g.f.) of the underlying random variables is not finite for some $t > 0$, then the limit in Shepp's Theorem is infinity. Consequently, the same is true for the E-R law. This latter result had been proved by Steinebach (1978) using Erdős and Rényi's technique of proof of their law.

2. The result. The notation in this section is, for the most part, consistent with Csörgő and Steinebach (1981).

Let $X_1, X_2, \cdots$ be i.i.d. random variables. To avoid trivialities, we assume that they are not degenerate. Let $\phi(t)$ denote the m.g.f. of X_1 and let $\rho(\alpha) = \inf_{t\geq 0}\phi(t)e^{-\alpha t}$. Let $f(n)$ be a nondecreasing function which takes the positive integers into themselves. Let $S_n = X_1 + \cdots + X_n$ and $T_n = (S_{n+f(n)} - S_n)/f(n)$. Then, from Chernoff's Theorem (see Bahadur, 1971),

$$n^{-1}\log P(S_n \geq n\alpha) \to \log \rho(\alpha). \tag{1}$$

Let $\mathscr{A}$ = ess sup X_1 and let r denote the radius of convergence of the power series $\sum x^{f(n)}$. Then, the statement of Shepp's Theorem is that, if $\phi(t) < \infty$ for some $t > 0$,

$$\limsup T_n = \alpha_r < \infty, \tag{2}$$

where $\alpha_r = \mathscr{A}$ if $r < \rho(\mathscr{A})$ and α_r is the unique solution of $\rho(\alpha) = r$ if $r \geq \rho(\mathscr{A})$. Note that $\rho(\mathscr{A}) = 0$ if $\mathscr{A} = \infty$.

The following theorem shows the necessity of the finiteness of ϕ for (2) to hold.

THEOREM 1. *If $\phi(t) = \infty$ for all $t > 0$, then* $\limsup T_n = \infty$ *for all subsequences* $\{f(n)\}$ *for which $r < 1$.*

PROOF. If $\phi(t) = \infty$ for all $t > 0$, then $\rho(\alpha) = 1$ for $\alpha > 0$. Thus, for $r < r' < r'' < 1$, it follows from (1) that

$$P(S_{f(n)} \geq f(n)\alpha) \geq (r'')^{f(n)} \quad \text{for all sufficiently large } n. \tag{3}$$

Since $r' > r$, $\sum (r')^{f(n)} = \infty$. So, by Lemma 3.1 of Shepp (1964), there exists a subsequence $n_1 < n_2 < \cdots$ with $n_{k+1} = n_k + f(n_k)$ for which $\sum_{k=1}^{\infty} (r'')^{f(n_k)} = \infty$. Since $P(T_{n_k} \geq \alpha) =$

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$P(S_{f(n_k)} \geq f(n_k)\alpha)$, it follows from this and (3) that $\sum P(T_{n_k} \geq \alpha) = \infty$. Thus, by the Borel-Cantelli lemma, $P(T_{n_k} \geq \alpha \text{ i.o.}) = 1$ since the events $\{T_{n_k} \geq \alpha\}$, $k = 1, 2, \cdots$, are independent. So, lim sup $T_{n_k} \geq \alpha$ a.s., which implies that lim sup $T_{n_k} = \infty$ since α is arbitrary. This proves the theorem. □

REMARK. Let $f(n) = [c \log n]$ where $c > 0$ and [ ] denotes the integer part of a number. Since $T_n \leq \max_{k \leq n - f(n)}(S_{k+f(n)} - S_k)/f(n) = D_n$, then by Theorem 1, lim sup $D_n = \infty$ if $\phi(t) = \infty$ for all $t > 0$. This shows that $\phi(t) < \infty$ for some $t > 0$ is necessary for the E-R law.

As pointed out by the referee, the results in Section 4 of Csörgő and Steinebach (1981) can be viewed as refinements of Shepp's (1964) work.

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