*The Annals of Probability*
1983, Vol. 11, No. 3, 765–771

# SHARP BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF TWO I.I.D. RANDOM VARIABLES

BY DAVID C. COX AND J. H. B. KEMPERMAN[1]

*Battelle Memorial Institute and University of Rochester*

We determine the exact optimal bounds $A_p$ and $B_p$ such that

$$A_p[E|X|^p + E|Y|^p] \le E|X+Y|^p \le B_p[E|X|^p + E|Y|^p],$$

$(p \ge 1)$, whenever $X$, $Y$ are i.i.d. random variables with mean zero. We give examples of random variables attaining equality or nearly so. Exactly the same bounds $A_p$ and $B_p$ are obtained in the more general case where it is only assumed that $E(X \mid Y) = E(Y \mid X) = 0$.

**1. Introduction.** We determine the exact optimal bounds $A_p$ and $B_p$ such that

$$A_p \le E|X+Y|^p/\{E|X|^p + E|Y|^p\} \le B_p,$$

$(p \ge 1)$, whenever $X$, $Y$ are i.i.d. random variables with mean 0. One has $0 \le A_p \le 1 \le B_p$. It is shown that $A_p = 1$ for $p \ge 3$; $B_p = 2^{p-2}$ for $p \ge 2$, while $A_p = 2^{p-2}$ for $1 \le p \le 2$. A more complicated formula (2.8) holds for $A_p$ with $2 \le p \le 3$. A similar formula (2.11) yields $B_p$ with $1 \le p \le 2$. The latter constants $A_p$, $B_p$ are very close to 1. Exactly the same bounds $A_p$ and $B_p$ are obtained in the more general case where it is only assumed that $E(X \mid Y) = E(Y \mid X) = 0$. We note that Esseen (1975) obtained related bounds for the third moment $E|X+Y|^3$, in the i.i.d. case.

**2. The results.** In the sequel, $p$ will be a fixed constant, $1 \le p < \infty$. We are interested in the best possible constants $A_p$ and $B_p$ such that

$$A_p[E|X|^p + E|Y|^p] \le E|X+Y|^p \le B_p[E|X|^p + E|Y|^p] \tag{2.1}$$

for each pair of random variables $X$ and $Y$ satisfying $E|X|^p < \infty$, $E|Y|^p < \infty$ and

$$E(Y \mid X) = 0; \quad E(X \mid Y) = 0. \tag{2.2}$$

The inequalities (2.1) remain *sharp* when (2.2) is strengthened by requiring in addition that $X$ and $Y$ are i.i.d. For, it will be shown that the equality signs in (2.1) are either assumed or nearly assumed, depending on the value of $p$, by suitable non-zero pairs $X$, $Y$ of i.i.d. random variables with mean zero. On the other hand, these same inequalities (2.1) remain *valid* when (2.2) is weakened to

$$E\{X(\text{sgn } Y)|Y|^{p-1} + Y(\text{sgn } X)|X|^{p-1}\} = 0; \tag{2.3}$$

(sgn $x = -1$, 0 or $+1$, depending on whether $x < 0$, $x = 0$ or $x > 0$, respectively). Namely, it will be shown that there exist real constants $c_p$ and $d_p$ such that

$$|x+y|^p \ge A_p\{|x|^p + |y|^p\} + c_p\{x(\text{sgn } y)|y|^{p-1} + y(\text{sgn } x)|x|^{p-1}\} \tag{2.4}$$

and

$$|x+y|^p \le B_p\{|x|^p + |y|^p\} + d_p\{x(\text{sgn } y)|y|^{p-1} + y(\text{sgn } x)|x|^{p-1}\} \tag{2.5}$$

for all $x, y \in R$. For example,

$$(x+y)^6 \le 16(x^6 + y^6) + 16(xy^5 + x^5y); \ge (x^6 + y^6) + 6(xy^5 + x^5y).$$

Received June 1982.

[1] The research of the second author was supported in part by the National Science Foundation.

*AMS* 1980 *subject classifications*. Primary, 60E15; secondary, 60G50, 60G42.

*Key words and phrases*. Sharp bounds on moments, sum of two i.i.d. random variables, martingales.

(2.6) THEOREM. *Inequality* (2.4) *holds with the following constants* $A_p$ *and* $c_p$. *Moreover,* $A_p$ *is the best possible constant in* (2.1), *subject to condition* (2.2). *First*

(2.7) $$A_p = 2^{p-2}; \quad c_p = 2^{p-2} \quad \text{if } 1 \le p \le 2.$$

*Next, if* $2 \le p \le 3$ *then*

(2.8) $$A_p = \inf_{0 \le z \le 1}[2^{p-1}(z + z^{p-1}) + (1 - z)^p]/[(1 + z)(1 + z^{p-1})]$$

*and* $c_p = 2^{p-1} - A_p$. *Finally,*

(2.9) $$A_p = 1; \quad c_p = p \quad \text{if } 3 \le p < \infty.$$

(2.10) THEOREM. *Inequality* (2.5) *holds with the following choice of the constants* $B_p$ *and* $d_p$. *Moreover, this constant* $B_p$ *is the best possible constant in* (2.1) *subject to condition* (2.2).

*If* $1 \le p \le 2$ *then*

(2.11) $$B_p = \sup_{0 \le z \le 1}[2^{p-1}(z + z^{p-1}) + (1 - z)^p]/[(1 + z)(1 + z^{p-1})],$$

*while* $d_p = 2^{p-1} - B_p$, *Moreover,*

(2.12) $$B_p = 2^{p-2} \text{ and } d_p = 2^{p-2} \quad \text{if } 2 \le p < \infty.$$

(2.13) REMARK. The constants $A_p$ for $2 \le p \le 3$ and the $B_p$ for $1 \le p \le 2$ are extremely close to 1 as Table I shows.

TABLE I.
*The Constants* $A_p$, $2 \le p \le 3$, *and* $B_p$, $1 \le p \le 2$.

| $p$ | $A_p$ | $p$ | $B_p$ |
|---|---|---|---|
| 2.1 | 0.99586 | 1.1 | 1.02300 |
| 2.2 | 0.99285 | 1.2 | 1.03292 |
| 2.3 | 0.99102 | 1.3 | 1.03551 |
| 2.4 | 0.99038 | 1.4 | 1.03374 |
| 2.5 | 0.99085 | 1.5 | 1.02938 |
| 2.6 | 0.99228 | 1.6 | 1.02363 |
| 2.7 | 0.99441 | 1.7 | 1.01729 |
| 2.8 | 0.99685 | 1.8 | 1.01098 |
| 2.9 | 0.99900 | 1.9 | 1.00511 |

(2.14) REMARK. Note that, for $p \ge 2$, inequality (2.5) with $B_p = 2^{p-2}$ implies that

$$|x + y|^p + |x - y|^p \le 2B_p(|x|^p + |y|^p) = 2^{p-1}(|x|^p + |y|^p).$$

This is exactly the well-known Clarkson inequality.

In the sequel, for convenience, we assume that $p \ne 1, p \ne 2, p \ne 3$. The cases $p = 1$, 2 or 3 are rather trivial anyway. When $p = 1$, the inequalities (2.4), (2.5) become

$$\tfrac{1}{2}\{|x| + |y| + x(\operatorname{sgn} y) + y(\operatorname{sgn} x)\} \le |x + y| \le |x| + |y|.$$

The left equality sign holds if and only if either $y = -x$ or $xy \ge 0$; the right equality sign holds if and only if $xy \ge 0$. When $p = 2$, (2.4), (2.5) reduce to the simple identity

$$(x + y)^2 = (x^2 + y^2) + (xy + yx),$$

and (2.1) (with $A_2 = B_2 = 1$) always holds with the equality sign, (even if only $E(Y|X) = 0$). When $p = 3$, (2.4) reduces to

$$|x + y|^3 \ge \{|x|^3 + |y|^3\} + 3\{x(\operatorname{sgn} y)y^2 + y(\operatorname{sgn} x)x^2\},$$

which holds with the equality sign if and only if $xy \geq 0$. Further, (2.5) with $p = 3$ becomes

$$|x + y|^3 \leq 2\{|x|^3 + |y|^3\} + 2\{x(\operatorname{sgn} y)y^2 + y(\operatorname{sgn} x)x^2\},$$

where the equality sign holds if and only if $y = \pm x$.

(2.15) From now on either $1 < p < 2$ or $2 < p < 3$ or $p > 3$. Let further $A_p$, $B_p$, $c_p$ and $d_p$ be *defined* as in the formulation of the Theorems 2.6 and 2.10. In the Sections 3 and 4 we will establish the inequalities (2.4) and (2.5). Obviously, they imply that (2.1) holds for any pair of random variables $X$, $Y$ satisfying (2.3).

We first show that the constants $A_p$, $B_p$ in (2.1) cannot be improved by requiring that $X$ and $Y$ be independent (possibly with different distributions). It suffices to prove that the inequalities (2.1) can hold with the equality sign in a non-trivial way; (the trivial way would be to take $X = Y = 0$). Let $S_p'$ and $S_p''$, respectively, denote the set of points $(x, y)$ at which (2.4) (or (2.5), respectively) holds with the equality sign.

Assuming that (2.4) has been established, it is obvious that a pair of random variables $X$, $Y$ satisfying (2.3) will satisfy the lower bound (2.1) with the equality sign if and only if the distribution of $(X, Y)$ is carried by $S_p'$. Hence, in order to show that the lower bound (2.1) is optimal for the independent case, it suffices to show that there exists a pair of independent random variables $X$ and $Y$, not both zero, such that $P((X, Y) \in S_p') = 1$ and $EX = 0$, $EY = 0$.

Similarly, the upper bound (2.1) is optimal as soon as $P((X, Y) \in S_p'') = 1$ holds for some non-trivial pair of independent random variables $X$, $Y$, each with mean zero.

(2.16) EQUALITY. As will be shown, the set $S_p'$ is as follows.

(i) If $1 < p < 2$ then $S_p'$ consists of all the points $(x, x)$ and $(x, -x)$, (where $x$ is an arbitrary real number). Hence, in this case, $S_p'$ supports the pair $(X, Y)$ with $X$, $Y$ i.i.d. and $P(X = 1) = P(X = -1) = ½$.

(ii) Let $2 < p < 3$. Then $S_p'$ consists of all points $(x, x)$, $(x, -z_0 x)$ and $(x, -x/z_0)$; here, $z_0$ is the unique value at which the infimum in (2.8) is attained, $0 < z_0 < 1$. Hence, $S_p'$ supports the pair $(X, Y)$ with $X$, $Y$ i.i.d. and $P(X = -z_0) = 1/(1 + z_0)$; $P(X = +1) = z_0/(1 + z_0)$.

(iii) If $p > 3$ then $S_p'$ consists of all points $(x, 0)$ and $(0, x)$. In this case one may take $X = 0$ and $Y$ as an arbitrary random variable with mean zero.

The set $S_p''$ will turn out to be as follows.

(iv) Suppose $1 < p < 2$. Then $S_p''$ consists of all points $(x, x)$, $(x, -z_0 x)$ and $(x, -x/z_0)$, with $z_0$ as the unique value at which the supremum in (2.11) is attained, $0 < z_0 < 1$. Thus, $S_p''$ supports a pair $(X,Y)$ with $X$, $Y$ i.i.d. such that $P(X = -z_0) = 1/(1 + z_0)$ and $P(X = 1) = z_0/(1 + z_0)$.

(v) Suppose that $2 < p < \infty$, $p \neq 3$. Then $S_p''$ consists of all points $(x, x)$ and $(x, -x)$. Thus, $S_p''$ carries the distribution of the pair of i.i.d. random variables $X$ and $Y$ with $P(X = 1) = P(X = -1) = ½$.

Finally, we note that, except for $A_1$, $B_1$ and $A_p$, $p \geq 3$, the examples given above establish the sharpness of (2.1) for the i.i.d. case also. The exceptions are taken care of by considering a distribution taking the values $1, -a$, with mean zero, and letting $a \to 0$.

**3. Proofs.** Note that the infimum in (2.8) and the supremum in (2.11) involve a strictly positive valued continuous function on $[0, 1]$. It follows from the definitions of $A_p$ and $B_p$ (letting $z = 0$ in (2.8), (2.11)) that

$$0 < A_p \leq 1 \leq B_p < \infty. \tag{3.1}$$

In particular, (2.4), (2.5) hold when $x = 0$ or $y = 0$.

In proving (2.4) and (2.5), because of the obvious symmetry in $x$ and $y$, one only needs to consider the case that $0 \leq |x| \leq |y|$, $y \neq 0$. Dividing by $|y|^p$ and introducing $z =$

$(\operatorname{sgn} xy)|x/y| = x/y$, it suffices to show that

$$|1+z|^p \geq A_p(1+|z|^p) + c_p(\operatorname{sgn} z)(|z| + |z|^{p-1}) \tag{3.2}$$

and

$$|1+z|^p \leq B_p(1+|z|^p) + d_p(\operatorname{sgn} z)(|z| + |z|^{p-1}), \tag{3.3}$$

whenever $-1 \leq z \leq +1$.

Let $Z_p'$ (and $Z_p''$, respectively) denote the set of values $z \in [-1, +1]$ such that (3.2) (or (3.3), respectively) holds with the equality sign. Observe that $S_p'$ (or $S_p''$, respectively) consists precisely of all points $(x, zx)$ and $(zx, x)$, where $x \in R$ is arbitrary, while $z \in Z_p'$ (or $z \in Z_p''$, respectively).

(3.4) We conclude that in proving the Theorems 2.6 and 2.10, as well as the assertions (i)–(v) in 2.16, it only remains to prove the following results. Here, the variable $z$ can take any value with $-1 \leq z \leq +1$.

(I) Suppose $1 < p < 2$. Then

$$(1+z)^p \geq 2^{p-2}[1 + |z|^p + (\operatorname{sgn} z)(|z| + |z|^{p-1})]. \tag{3.5}$$

Moreover, the equality sign in (3.5) holds if and only if either $z = -1$ or $z = +1$.

(II) Suppose $2 < p < 3$. Then

$$(1+z)^p \geq A_p(1+|z|^p) + (2^{p-1} - A_p)(\operatorname{sgn} z)(|z| + |z|^{p-1}). \tag{3.6}$$

Here, $A_p$ is defined by (2.8). Moreover, the equality sign holds in (3.6) if and only if either $z = 1$ or $z = -z_0$, with $z_0$ as the unique value at which the infimum (2.8) is attained. One has $0 < z_0 < 1$. This as well as the uniqueness of $z_0$ remains to be shown.

(III) Suppose $p > 3$. Then

$$(1+z)^p \geq 1 + |z|^p + p(\operatorname{sgn} z)(|z| + |z|^{p-1}). \tag{3.7}$$

Here, the equality sign holds only at $z = 0$.

(IV) Suppose $1 < p < 2$. Then

$$(1+z)^p \leq B_p(1+|z|^p) + (2^{p-1} - B_p)(\operatorname{sgn} z)(|z| + |z|^{p-1}), \tag{3.8}$$

with $B_p$ as defined in (2.11). Moreover, the equality sign in (3.8) holds if and only if either $z = 1$ or $z = -z_0$, with $z_0$ as the unique value at which the supremum (2.11) is attained. One has $0 < z_0 < 1$. This and the uniqueness of $z_0$ remains to be shown.

(V) Suppose $2 < p < \infty$, $p \neq 3$. Then

$$(1+z)^p \leq 2^{p-2}[1 + |z|^p + (\operatorname{sgn} z)(|z| + |z|^{p-1})], \tag{3.9}$$

where the equality sign holds if and only if either $z = -1$ or $z = +1$.

(3.10) PROOF OF (I) AND (V). Let $-1 \leq z \leq +1$ and put $w = (\operatorname{sgn} z)|z|^{p-1}$. As to (3.5) and (3.9), observe that

$$1 + |z|^p + (\operatorname{sgn} z)(|z| + |z|^{p-1}) = (1+z)(1+w).$$

Dividing by $1 + z$, one must show that

$$\left(\frac{1+z}{2}\right)^{p-1} \geq \frac{1+w}{2} \quad \text{if } 1 < p < 2, \tag{3.11}$$

and also the opposite inequality when $p > 2$.

However, these two inequalities are equivalent as may be seen by raising (3.11) to the power $q - 1$, where $q > 2$ is defined by $p^{-1} + q^{-1} = 1$. Note that $(p-1)(q-1) = 1$; $z = (\operatorname{sgn} w)|w|^{q-1}$.

As is well-known, if $Z \geq 0$ is a non-constant random variable with moments $\beta_t = EZ^t$ then $\beta_t^{1/t}$ is strictly increasing in $t > 0$. Hence, if $0 < z < 1$ then $(\frac{1}{2} + \frac{1}{2}z^t)^{1/t}$ is a strictly increasing function of $t > 0$. This proves the case $z \geq 0$ of (3.11).

As to the case $z = -x < 0$ of (3.11), it suffices to observe that

$$(1-x)^{p-1} \geq 1 - x^{p-1} > 2^{p-2}(1-x^{p-1}),$$

when $0 \leq x < 1$ and $1 < p < 2$; (after all, the excess in the first inequality is a concave function equal to 0 at $x = 0$ and $x = 1$). It also follows from the above remarks that (3.5) and (3.9) hold with the equality sign if and only if $z = 1$ or $z = -1$.

(3.12) Proof of (III). Let $p > 3$. It suffices to show that $g(z) > 0$ and $h(z) > 0$, for $0 < z \leq 1$, where

$$g(z) = (1-z)^p - 1 - z^p + pz + pz^{p-1};$$

$$h(z) = (1+z)^p - 1 - z^p - pz - pz^{p-1}.$$

The first follows from $(1-z)^p > 1 - pz$ and $pz^{p-1} > z^p$. The second follows from $h(0) = 0$; $h'(0) = 0$ and the strict convexity of the function $h$. The latter follows from

$$h''(z) = p(p-1)z^{p-2}\{(1+t)^{p-2} - (p-2)t - 1\} > 0,$$

where $t = 1/z$.

(3.13) Remark. This completes the proof of the lower bound (2.1) with $A_p = 1$ in the case $p > 3$. More generally, (2.2) implies that

$$EU(X+Y) \geq EU(X) + EU(Y), \tag{3.14}$$

when $U$ is a (not necessarily even) function on $R$ having a convex second derivative $U''$ and such that $U(0) = 0$; $U'(0) = 0$; $U''(0) = 0$; (example: $U(x) = x^4$ if $x \geq 0$; $U(x) = |x|^5$ if $x \leq 0$). We assume that all needed expectations exist.

It suffices to prove that

$$U(x+y) \geq U(x) + U(y) + xU'(y) + yU'(x), \tag{3.15}$$

for all $x, y \in R$. Let $x$ be fixed and let $f(y)$ denote the difference between the two sides of the inequality. One must prove that $f \geq 0$. Since $f(0) = 0$ and $f'(0) = 0$, it suffices to show that $f$ is convex. In fact,

$$f''(y) = U''(x+y) - U''(y) - xU'''(y) \geq 0,$$

by the convexity of $U''$. Here f″ and U‴ denote the right hand derivatives of $f'$ and $U''$, respectively.

**4. Proof, second part.** The proofs of assertions (II) and (IV), (see(3.6) and (3.8)), are somewhat harder. We need the following two lemmas.

(4.1) Lemma. *Consider the function*

$$\phi(z) = (1+z)^p - (1+z^p) - (2^{p-1}-1)(z+z^{p-1}). \tag{4.2}$$

*Here, $0 \leq z \leq 1$ and $p \geq 1$. Note that $\phi(0) = 0$; $\phi(1) = 0$, while $\phi(z) \equiv 0$ if $p = 1, 2$ or 3. We assert that*

$$\phi(z) > 0 \quad \textit{if} \quad 0 < z < 1 \quad \textit{and} \quad 2 < p < 3 \tag{4.3}$$

*and*

$$\phi(z) < 0 \quad \textit{if} \quad 0 < z < 1 \quad \textit{and} \quad 1 < p < 2 \quad \textit{or} \quad p > 3. \tag{4.4}$$

(4.5) Proof. Observe that $\phi(0) = \phi(1) = \phi'(1) = 0$. Moreover,

$$\begin{aligned} \phi''(z) &= p(p-1)[(1+z)^{p-2} - z^{p-2} - cz^{p-3}] \\ &= p(p-1)z^{p-2}[(1+t)^{p-2} - 1 - ct], \end{aligned} \tag{4.6}$$

where $c = (2^{p-1} - 1)(p-2)/p$ and $t = 1/z$. In particular,

$$\phi''(1) = (p-1)\theta(p) \quad \text{with} \quad \theta(p) = (4-p)2^{p-2} - 2.$$

Since $\theta(2) = \theta(3) = 0$ and $\theta'(p)$ changes sign only at $p_0 = 4 - (\log 2)^{-1} = 2.557$ (from positive to negative), one has that

$$\phi''(1) > 0 \quad \text{if} \quad 2 < p < 3;$$

$$< 0 \quad \text{if} \quad 1 < p < 2 \quad \text{or} \quad p > 3.$$

Consider first the case $2 < p < 3$. Then the function $(1+t)^{p-2}$ in (4.6) is strictly concave. Hence, $\phi''(z)$ might possibly have a $(-, +, -)$ type of sign change somewhere (as $z$ moves to the right) but not a $(+, -, +)$ type of sign change. Since $\phi''(1) > 0$, the sign of $\phi''(z)$ can actually change only once, for $z \in (0, 1)$, and then only from negative to positive. Since $\phi''(0) = -\infty$, it follows that $\phi'(z)$ first decreases and then increases toward its final value $\phi'(1) = 0$. Consequently, if $\phi'(z_0) = 0$ with $0 < z_0 < 1$ then $\phi'(z)$ is decreasing near $z_0$. In particular, $\phi$ cannot have any interior local minimum. Since $\phi(0) = \phi(1) = 0$, one must have $\phi(z) > 0$ for $0 < z < 1$.

The proof for the case $1 < p < 2$ or $p > 3$ is entirely analogous. In this case $\phi''(1) < 0$ while the function $(1+t)^{p-2}$ in (4.6) is strictly convex. Also using $\phi'(1) = 0$, it follows that $\phi$ cannot have any interior local maximum in (0, 1). This proves (4.4).

(4.7) LEMMA. *Consider the function*

$$\psi(z) = (1-z)^p - (a + bz + cz^{p-1} + dz^p), \tag{4.8}$$

*where* $0 < z < 1$. *Further, a, b, c, d denote arbitrary real constants.*

*We assert for the cases* $0 < p < 1$ *and* $2 < p < 3$ *that the function* $\psi$ *can have at most one (interior) local minimum in* (0, 1). *Similarly, if* $p < 0$ *or* $1 < p < 2$ *or* $p > 3$ *then* $\psi$ *can have at most one local maximum in* (0, 1).

(4.9) PROOF. Suppose first that $0 < p < 1$ or $2 < p < 3$. One has for $0 < z < 1$ that

$$\psi''(z) = p(p-1)z^{p-2}\{(t-1)^{p-2} - c't - d'\},$$

where $c'$, $d'$ are constants and $t = 1/z$. Presently, the function $p(p-1)(t-1)^{p-2}$ is strictly concave. It follows that $\psi''(z)$ cannot have a $(+, -, +)$ type of sign change anywhere, which implies that $\psi(z)$ cannot have two (or more) local minima in (0, 1).

If instead $p < 0$ or $1 < p < 2$ or $p > 3$ then $\psi''(z)$ cannot have a $(-, +, -)$ type of sign change, hence, $\psi$ cannot have more than one local maximum.

(4.10) PROOF OF (II). Let $2 < p < 3$. In order that (3.6) holds in the interval $-1 \le z \le 0$, it is necessary and sufficient that $\psi \ge 0$ on [0, 1]. Here, $\psi$ is defined by

$$\psi(z) = (1-z)^p + 2^{p-1}(z + z^{p-1}) - A_p(1+z)(1+z^{p-1}). \tag{4.11}$$

And $\psi \ge 0$ is indeed true because of the definition (2.8) of $A_p$.

There is at least one value $z_0 \in [0, 1]$ at which the infimum (2.8) is assumed. Equivalently, there is at least one value $z_0 \in [0, 1]$ with $\psi(z_0) = 0$.

Since $\psi(1) = 4(2^{p-2} - A_p) > 0$, we have $z_0 < 1$. If $z$ is small then

$$\psi(z) = (1 - A_p)(1 + z) - v(p)z + O(z^{p-1}),$$

where $v(p) = p + 1 - 2^{p-1} > 0$; (namely, the function $v(p)$ is strictly concave while $v(3) = 0$, $v(0) > 0$, hence, $v(p) > 0$ for $0 \le p < 3$). Since $\psi(z) \ge 0$ on [0, 1], it follows that $A_p < 1$, thus, $\psi(0) = 1 - A_p > 0$, hence, $z_0 > 0$.

We have shown that any zero $z_0$ of $\psi$ must be an interior zero and thus an interior minimum. It follows from Lemma 4.7 that $z_0$ is *unique*. Moreover, $-z_0$ is the unique value $z \in [-1, 0]$ at which (3.6) holds with the equality sign.

Next, we must show that (3.6) holds when $0 \le z \le 1$. This is equivalent to

$$\phi(z) + (1 - A_p)(1 - z)(1 - z^{p-1}) \ge 0, \tag{4.12}$$

for $0 \le z \le 1$, where $\phi$ is defined by (4.2). In fact, (4.12) is obvious from $A_p < 1$ and the conclusion (4.3) of Lemma 4.1. In fact, we see that $z = 1$ is the only value in [0, 1] at which (4.12) holds with the equality sign. This completes the proof of (II).

(4.13) PROOF OF (IV). This is completely analogous to the proof of (II).

5. CONCLUDING REMARKS. Let $p > 1, p \ne 2$, and consider $n \ge 2$ random variables $X_1, \cdots, X_n$ with $E|X_i|^p < \infty$. Put $S_j = X_1 + \cdots + X_j$. For $n \ge 3$, it is an open problem to describe the best possible constants $A_p(n)$ and $B_p(n)$ such that

$$A_p(n) \sum_{i=1}^n E|X_i|^p \le E|S_n|^p \le B_p(n) \sum_{i=1}^n E|X_i|^p, \tag{5.1}$$

under the condition that

$$E(X_i | X_j) = 0 \text{ for } i \ne j, \quad (i, j = 1, \cdots, n). \tag{5.2}$$

Similarly for the stronger condition that the $X_i$ be independent, or even i.i.d., with mean zero. Unlike the case $n = 2$, it is doubtful that the latter conditions are equivalent to (5.2) for any $n \ge 3$ in determining the optimal constants $A_p(n)$, $B_p(n)$.

A related problem is to determine the best possible constants $A_p'(n)$ and $B_p'(n)$ such that

$$A_p'(n) \sum_{i=1}^n E|X_i|^p \le E|S_n|^p \le B_p'(n) \sum_{i=1}^n E|X_i|^p, \tag{5.3}$$

under the condition that

$$E(X_i | S_{i-1}) = 0 \quad \text{for } i = 2, 3, \cdots, n. \tag{5.4}$$

These constants, and their asymptotic behavior for large $n$, were determined in Kemperman and Smit (1974). It turns out that the inequalities (5.3) remain sharp when (5.4) is strengthened to the full martingale condition

$$E(X_i | X_1, \cdots, X_{i-1}) = 0 \quad \text{for } i = 2, \cdots, n. \tag{5.5}$$

Note that neither of the conditions (5.2), (5.5) implies the other (and similarly for (5.2) and (5.4)).

For $n = 2$, the constant $A_p' = A_p'(2)$ is precisely the largest constant for which one can find a constant $c_p'$ such that

$$|1 + z|^p \ge A_p'(1 + |z|^p) + c_p' z \quad \text{for all } z. \tag{5.6}$$

Similarly, $B_p' = B_p'(2)$ is the smallest constant for which one can find a constant $d_p'$ such that

$$|1 + z|^p \le B_p'(1 + |z|^p) + d_p' z \quad \text{for all } z. \tag{5.7}$$

These are the analogues of (3.2) and (3.3), respectively. For example, $A_4' = (5 - \sqrt{17})/2 = .43845$ while $A_4 = 1$. Further, $B_4' = (5 + \sqrt{17})/2 = 4.56155$ while $B_4 = 4$. Similarly, $B_3' = 2.169773708$ while $B_3 = 2$.

## REFERENCES

ESSEEN, C. G. (1975). Bounds for the absolute third moment. *Scand. J. Statist.* **2** 149–152.

KEMPERMAN, J. H. B., and SMIT, J.C. (1974). Sharp upper and lower bounds for the moments of a martingale. *Adv. Appl. Probab.* **6** 244–245.

BATTELLE MEMORIAL INSTITUTE
2030 M ST. N.W.
WASHINGTON, D.C. 20036

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ROCHESTER
ROCHESTER, NEW YORK 14627