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EXPONENTIAL LIFE FUNCTIONS WITH NBU COMPONENTS[1]

BY MOSHE SHAKED

University of Arizona

Homogeneous nondecreasing functions of independent NBU random variables are studied. Two results of Block and Savits are improved. It is shown that if a coherent system, formed from independent NBU components, has exponential life then it is essentially a series system with exponential components. Also, it is shown that if a strictly increasing homogeneous function of independent NBU random variables has an exponential distribution then it is essentially a univariate function of one of its variables which must, then, be exponential. A new characterization of the MNBU class of distributions of Marshall and Shaked is derived, and a new proof of the closure of the class of NBU distributions under formation of nonnegative homogeneous increasing functions is given.

1. Introduction. Block and Savits (1979) showed that if a coherent system composed of components (some may be irrelevant) with increasing failure rate average (IFRA) lifetimes, has an exponential lifetime then it must be a series system and the relevant components must have exponential lives. Block and Savits (1979) also derived a similar result for sums of independent IFRA random variables. A related result was obtained by Kitchin and Proschan (1981).

In this paper we derive similar results under new better than used (NBU) assumptions. Our theorems strengthen the results of Block and Savits in two ways. First, our NBU assumption is weaker than the IFRA assumption. In addition, our second theorem applies to the class of strictly increasing nonnegative homogeneous functions on $R_+^n = \{x : x \geq 0\}^n$ (see definition below); sums are members of this class.

In Section 2 we describe some classes of homogeneous functions that will be of interest to us. The main results are given in Section 3 and some applications are briefly discussed in Section 4.

In the following we write "increasing" for "nondecreasing" and "decreasing" for "nonincreasing". For two vectors $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$ in R^n we write $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$, $i = 1, \cdots, n$ and write $\mathbf{x} < \mathbf{y}$ if $x_i < y_i$, $i = 1, \cdots, n$. Also, $\mathbf{x} \wedge \mathbf{y}$ denotes $(\min(x_1, y_1), \cdots, \min(x_n, y_n)) = (x_1 \wedge y_1, \cdots, x_n \wedge y_n)$. By $\mathbf{U} \overset{d}{=} \mathbf{V}$ we denote equality in distribution of the random vectors $\mathbf{U}$ and $\mathbf{V}$. If a random vector $\mathbf{U}$ has distribution F then the corresponding survival function $\bar{F}$ is defined by $\bar{F}(\mathbf{u}) = P(U_1 > u_1, \cdots, U_n > u_n)$.

2. Preliminaries. An n-variate function g is called homogeneous (of degree 1) if $g(\alpha\mathbf{t}) = \alpha g(\mathbf{t})$ for every $\alpha > 0$ and $\mathbf{t} \in R^n$. Let g be an n-variate nonnegative increasing homogeneous function defined on R_+^n. Clearly g is continuous on rays. From the monotonicity of g it is not hard to show that actually g is continuous on $R_{++}^n = \{\mathbf{x} : \mathbf{x} > \mathbf{0}\}$; however, as one referee noted, g need not be continuous on R_+^n. Let $\mathscr{G}^{(n)}$ be the class of all n-variate nonnegative increasing homogeneous functions which are continuous on R_+^n; such functions can be obtained by taking all non-negative increasing homogeneous functions on R_{++}^n and extending them by continuity to R_+^n. By convention, for every $m < n$, $\mathscr{G}^{(m)} \subset \mathscr{G}^{(n)}$ because every m-variate function can be thought of as an n-variate function with specific $n - m$ irrelevant arguments. Geometrically, there is a one-to-one correspondence between $\mathscr{G}^{(n)}$

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and the class of all open upper sets in R_+^n (A is an upper set if $\mathbf{x} \in A$, $\mathbf{y} \geq \mathbf{x} \Rightarrow \mathbf{y} \in A$). If $g \in \mathscr{G}^{(n)}$ then $A = A(g) = \{\mathbf{x}: g(\mathbf{x}) > 1\} \cap R_{++}^n$ is the corresponding open upper set. Conversely, if A is an open upper set in R_+^n then

$$g(\mathbf{x}) = \sup\{s > 0: \frac{1}{s}\mathbf{x} \in A\} \qquad \text{if } \{s > 0: \frac{1}{s}\mathbf{x} \in A\} \neq \varnothing$$

$$= 0 \qquad \text{otherwise}$$

is the corresponding $g \in \mathscr{G}^{(n)}$. Clearly, determination of the set $\{\mathbf{x}: g(\mathbf{x}) = 1\}$ is equivalent to determination of $g(\mathbf{x})$ for every $\mathbf{x} \in R_+^n$.

We will be interested in the following subclasses of $\mathscr{G}^{(n)}$.

The class of all $g \in \mathscr{G}^{(n)}$ which are strictly increasing in each variable when the other variables are held fixed will be denoted by $\mathscr{G}_2^{(n)}$. For example, $g(\mathbf{x}) = \sum_{i=1}^n x_i$ and, more generally, $g(\mathbf{x}) = (\sum_{i=1}^n a_i x_i^p)^{1/p}$ where $p < \infty$, $p \neq 0$ and $a_i \in (0, \infty)$, $i = 1, \cdots, n$, are members of $\mathscr{G}_2^{(n)}$.

The class of all the functions of the form $g(\mathbf{x}) = \tau(a_1x_1, \cdots, a_nx_n)$ where $a_i \in [0, \infty]$ $i = 1, \cdots, n$ and τ is a coherent life function (in the sense of Esary and Marshall (1970), allowing irrelevant components) will be denoted by $\mathscr{G}_3^{(n)}$.

The class of all the functions g which are "scaled minimum," that is, functions of the form

$$g(\mathbf{x}) = \min_{1 \leq i \leq n} a_i x_i, \tag{2.1}$$

where $a_i \in (0, \infty]$, will be denoted by $\mathscr{G}_1^{(n)}$. To avoid trivialities we do not allow all the a_i's in (2.1) to take on the value 0. Thus, $g \in \mathscr{G}_1^{(n)}$ if, and only if, the set $\{\mathbf{x}: g(\mathbf{x}) > 1\}$ is an open upper orthant (a set is an open upper orthant if it has the form $Q_\mathbf{y} = \{\mathbf{x}: \mathbf{x} > \mathbf{y}\}$ for some $\mathbf{y} \in R_+^n$).

Note that if $n > 1$ then $\mathscr{G}_2 \subset \mathscr{G}$, $\mathscr{G}_1 \subset \mathscr{G}_3 \subset \mathscr{G}$, $\mathscr{G}_2 \cap \mathscr{G}_3 = \varnothing$ and all the inclusions are strict. Here, and in the following, when a $\mathscr{G}$ is written without a superscript it is understood that the superscript is (n) unless we write otherwise.

According to the definition of Marshall and Shaked (1982), the n-variate nonnegative random vector $\mathbf{X}$ (or its distribution) is called multivariate NBU (MNBU) if $g(\mathbf{X})$ has an NBU distribution for every $g \in \mathscr{G}$. Marshall and Shaked (1982) showed that $\mathbf{X}$ is MNBU if and only if

$$Eh\left(\frac{1}{a+b}\mathbf{X}\right) \leq Eh^\gamma\left(\frac{1}{a}\mathbf{X}\right) Eh^{1-\gamma}\left(\frac{1}{b}\mathbf{X}\right)$$

whenever $a > 0$, $b > 0$, $\gamma \in (0, 1)$ and h is a nonnegative increasing function defined on R_+^n. Using a standard induction argument it can be shown that the random vector $\mathbf{X}$ is MNBU if and only if

$$Eh\left(\frac{1}{a_1 + \cdots + a_m}\mathbf{X}\right) \leq \prod_{i=1}^m Eh^{\lambda_i}\left(\frac{1}{a_i}\mathbf{X}\right) \tag{2.2}$$

whenever m is a positive integer, $a_i > 0$, $\gamma_i \in (0, 1)$, $i = 1, \cdots, m$, $\sum_{i=1}^m \gamma_i = 1$ and h is a nonnegative increasing function defined on R_+^n. Setting $a_i = \gamma_i = m^{-1}$, $i = 1, \cdots, m$ in (2.2) it follows that if $\mathbf{X}$ is MNBU (and, in particular, if its components $X_1, \cdots, X_n$ are independent NBU random variables) then

$$Eh(\mathbf{X}) \leq [Eh^\alpha(\alpha^{-1}\mathbf{X})]^{1/\alpha}, \quad \alpha = m^{-1}, \quad m = 1, 2, \cdots. \tag{2.3}$$

This result should be compared with Definition 2.1 of Block and Savits (1980).

It follows that if F is a univariate NBU distribution then for every nonnegative increasing function h defined on $[0, \infty)$,

$$\left(\int_0^\infty h(x)\, dF(x)\right)^\alpha \leq \int_0^\infty h^\alpha\left(\frac{x}{\alpha}\right) dF(x) \tag{2.4}$$

where $\alpha = m^{-1}$, $m = 1, 2, \cdots$. Block and Savits (1976) showed that F is IFRA if and only if (2.4) holds for every $\alpha \in (0, 1]$.

3. The main results. The first theorem is an improvement of Theorem 2.1 of Block and Savits (1979). Block and Savits (1979), page 915 observed that their Theorem 2.1 can be improved and so did Mark Brown. However, no proof of the stronger result has been written so far. From personal communication with Henry Block we gather that the proof which is given here is different than the one Block, Savits and Brown had in mind.

THEOREM 3.1. *Let $X_1, \cdots, X_n$ be independent* NBU *random variables, having life distributions $F_1, \cdots, F_n$, and let $g \in \mathscr{G}_3$. If $g(\mathbf{X})$ is exponential then there exists $\tilde{g} \in \mathscr{G}_1$ (that is, for some $a_i \in (0, \infty]$, $i = 1, \cdots, n$, $\tilde{g}(\mathbf{x}) = \min_{1 \le i \le n} a_i x_i$, $\mathbf{x} \ge 0$) such that $g(\mathbf{X}) \overset{d}{=} \tilde{g}(\mathbf{X})$ and X_i is exponential if $a_i < \infty$, $i = 1, \cdots, n$. In other words, there exists a subcollection $1 \le i_1 \le \cdots \le i_k \le n$ such that the distribution F of $g(\mathbf{x})$ satisfies*

$$\bar{F}(t) = \textstyle\prod_{j=1}^{k} \bar{F}_{i_j}(t), \quad t \ge 0,$$

and each F_{i_j} is exponential.

REMARK. In Theorem 3.1 we cannot conclude that $g \in \mathscr{G}_1$. For example, if X_1 is degenerate at 0 and X_2 and X_3 are exponential then $\max(X_1, \min(X_2, X_3))$ is exponential.

THEOREM 3.2. *Let $g \in \mathscr{G}_2$ and let $X_1, \cdots, X_n$ be independent NBU, random variables, $n > 1$. If $g(\mathbf{X})$ is exponential then for some $j \in \{1, \cdots, n\}$ and $a > 0$,*

$$g(0, \cdots, 0, x_j, 0, \cdots, 0) = ax_j,$$

and, with probability one, X_j is exponential and $X_i = 0$ for $i \ne j$.

Note that Theorem 2.8 of Block and Savits (1979) follows from Theorem 3.2 by recalling that every IFRA distribution is NBU and that $g(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i$ is a member of $\mathscr{G}_2$.

4. Applications. Marshall and Shaked (1981) have introduced some classes of multivariate NBU distributions which are analogous to the classes of multivariate IFRA distributions of Esary and Marshall (1979). In particular, $(T_1, \cdots, T_n)$ is said to belong to the class $\tilde{G}_3$ if there exist independent NBU random variables $X_1, \cdots, X_k$ and functions $g_1, \cdots, g_n \in \mathscr{G}_3^{(k)}$ such that

$$(T_1, \cdots, T_n) \overset{d}{=} (g_1(\mathbf{X}), \cdots, g_n(\mathbf{X})).$$

Also, $(T_1, \cdots, T_n)$ is said to belong to the class $\tilde{G}_2$ if there exist independent NBU random variables $X_1, \cdots, X_k$ and functions $g_1, \cdots, g_n$ of the form

$$g_i(x_1, \cdots, x_k) = \textstyle\sum_{j=1}^{k} a_j^{(i)} x_j, \quad i = 1, \cdots, n, \tag{4.1}$$

where $a_j^{(i)} \in [0, \infty)$, such that

$$(T_1, \cdots, T_n) \overset{d}{=} (g_1(\mathbf{X}), \cdots, g_n(\mathbf{X})).$$

Another class which will be discussed in this section is the class of MNBU distributions which were defined in Section 2.

Example 3.2 of Block and Savits (1979) can be used to show that MNBU $\not\subset \tilde{G}_3$. That example is MNBU because it is MIFRA (see Block and Savits, 1980). But, by Theorem 3.1 if this distribution belonged to the class $\tilde{G}_3$, its survival would be of the form

$$\bar{F}(t_1, t_2) = \exp\{-\textstyle\sum_{i=1}^{k} \max(t_1/a_i, t_2/b_i)\}, \quad t_1, t_2 \ge 0$$

for some $a_i \in (0, \infty]$, $b_i \in (0, \infty]$, $i = 1, \cdots, k$ and $k \in \{1, 2, \cdots\}$. However, it is not.

In a similar manner, using the fact that g's of the form (4.1) with positive $a_j^{(i)}$'s belong to $\mathscr{G}_2^{(k)}$ and using Theorem 3.2, it can be shown that Example 3.3 of Block and Savits (1979) yields a counterexample which proves MNBU $\not\subset \tilde{G}_2$.

It should be mentioned that Marshall and Shaked (1981) showed that $\tilde{G}_2 \subset$ MNBU and $\tilde{G}_3 \subset$ MNBU.

Further applications of Theorems 3.1 and 3.2 are discussed in Marshall and Shaked (1981).

5. Proofs. The standard proof of the first lemma is omitted.

LEMMA 5.1. *Let F be a nondegenerate univariate life distribution and let $\bar{F} = 1 - F$. If*

$$\bar{F}^{\alpha}(x) = \bar{F}(\alpha x) \text{ for all } x \geq 0, \quad \text{and} \quad \alpha = m^{-1}, \quad m = 1, 2, \cdots$$

then F is an exponential distribution.

PROOF OF THEOREM 3.1. Let $g \in \mathscr{G}_3$; then for some $b_i \in [0, \infty]$, $i = 1, \cdots, n$, and some coherent life function τ,

$$g(x_1, \cdots, x_n) = \tau(b_1 x_1, \cdots, b_n x_n), \quad \mathbf{x} \geq 0.$$

If $g(x)$ is exponential, then there is at least one b_i in $(0, \infty)$. Hence, without loss of generality, it can be assumed that for all $i = 1, \cdots, n$, $b_i \in (0, \infty)$ because otherwise the problem reduces to a similar one in a lower dimension. But then, without loss of generality, it can be assumed that $b_i = \cdots = b_n = 1$ because the property of NBU is preserved under changes of scale.

Under these assumptions, the assertion of Theorem 3.1 is equivalent to the assertion of Theorem 2.1 of Block and Savits (1979) except that "IFRA" of Block and Savits (1979) is replaced here by "NBU". To prove the NBU for version of Theorem 2.1 of Block and Savits (1979) one can follow word for word the proof of Block and Savits (1979) using some of the results here. Their Lemma 2.2 is used now in exactly the same manner by taking $\alpha = m^{-1}$ for some integer m. The inequalities which are needed follow from (2.4). The value α_0 at the top of page 913 in Block and Savits (1979) is replaced now by m_0^{-1} for some integer m_0. The exponentiality of F_1 follows in Block and Savits (1979) from the equation $\bar{F}_i(\alpha T) = \bar{F}_i^{\alpha}(t)$ for all $0 < \alpha < 1$ and $t > 0$. It follows here, by Lemma 5.1, from the equation $\bar{F}_i((1/m)t) = [\bar{F}_i(t)]^{1/m}$ for all integers m and $t > 0$. □

The next results will be needed in the proof of Theorem 3.2.

LEMMA 5.2. *If $g \in \mathscr{G}^{(n)} - \mathscr{G}_1^{(n)}$, $n \geq 2$, and $g \neq 0$ then*

$$\mu\{(\mathbf{x}, \mathbf{y}) : g(\mathbf{x} \wedge \mathbf{y}) \leq 1 < g(\mathbf{x}) \wedge g(\mathbf{y})\} > 0 \tag{5.1}$$

where μ is the Lebesgue measure on R_+^{2n}.

PROOF. We will actually show that there exist $\mathbf{x}, \mathbf{y} \in R_+^n$ such that $g(\mathbf{x}) = 1$, $g(\mathbf{y}) = 1$ and $g(\mathbf{x} \wedge \mathbf{y}) < 1$; (5.1) then follows from the continuity and homogeneity of g.

Pick a point $\mathbf{z}$ such that $g(\mathbf{z}) = 1$. Since $\{\mathbf{u} : g(\mathbf{u}) \geq 1\}$ is a closed upper set and $\mathbf{z} \in \{\mathbf{u} : g(\mathbf{u}) \geq 1\}$, it follows that there exists at least one closed upper orthant containing $\mathbf{z}$ which is contained in $\{\mathbf{u} : g(\mathbf{u}) \geq 1\}$. Let $\mathbf{x}$ be the vertex of such an orthant which is not contained in any other orthant which contains $\mathbf{z}$ (see Figure 1). By continuity and monotonicity of g,

$$g(\mathbf{x}) = 1.$$

Since $g \notin \mathscr{G}_1^{(n)}$ it follows that $\{\mathbf{u} : g(\mathbf{u}) \geq 1\}$ is not a closed upper orthant (see discussion in Section 2), thus $\{\mathbf{u} : g(\mathbf{u}) \geq 1\} - \bar{Q}_{\mathbf{x}} \neq \varnothing$ and hence $\{\mathbf{u} : g(\mathbf{u}) = 1\} - \bar{Q}_{\mathbf{x}} \neq \varnothing$. Let $\mathbf{y} \in$

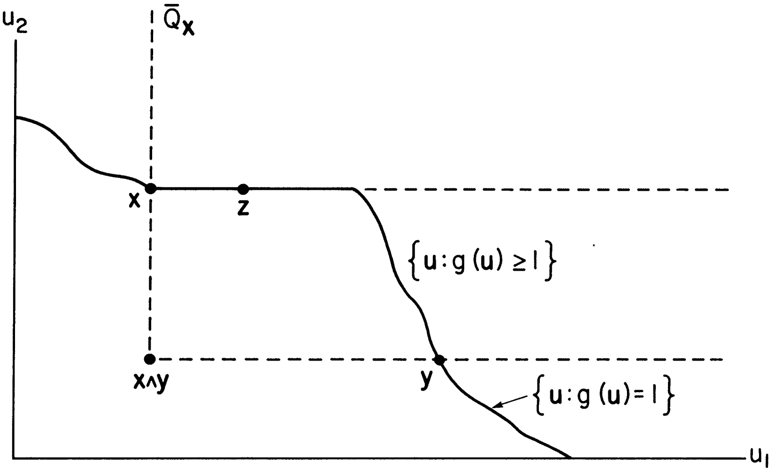

FIG. 1

$\{\mathbf{u}: g(\mathbf{u}) = 1\} - \bar{Q}_{\mathbf{x}}$, then

$$g(\mathbf{y}) = 1.$$

Since $\mathbf{y} \notin Q_{\mathbf{x}}$ it follows that at least one coordinate of $\mathbf{y}$ is less than the corresponding coordinate of $\mathbf{x}$. Hence $\mathbf{x} \wedge \mathbf{y} \leq \mathbf{x}$ and at least one coordinate of $\mathbf{x} \wedge \mathbf{y}$ is strictly less than the corresponding coordinate of $\mathbf{x}$. Thus, by definition of $\mathbf{x}$,

$$g(\mathbf{x} \wedge \mathbf{y}) < 1. \quad \square$$

LEMMA 5.3. *Let $X_1, \cdots, X_n$ be independent* NBU *random variables with support R_+ (that is, each X_i has an absolutely continuous portion which has a density which is positive on R_+). Let $g \in \mathscr{G}$. Then $g(\mathbf{X})$ is exponential if and only if $g \in \mathscr{G}_1$ (that is, for some $a_i \in (0, \infty]$, $i = 1, \cdots, n$, $g(\mathbf{x}) = \min_{1 \leq i \leq n} a_i x_i$) and X_i is exponential if $a_i < \infty$, $i = 1, \cdots, n$.*

PROOF. Assume $g(\mathbf{X})$ is exponential. Let $Y_1, \cdots, Y_n$ be independent of the X_i's such that $\mathbf{X} \overset{d}{=} \mathbf{Y}$. Let $s > 0$, $t > 0$ then

$$p\{g(\mathbf{X}) > s + t\} = P\left\{g\left(\frac{1}{s+t}\mathbf{X}\right) > 1\right\} \leq p\left\{g\left(\left(\frac{1}{s}\mathbf{X}\right) \wedge \left(\frac{1}{t}\mathbf{Y}\right)\right) > 1\right\} \tag{5.2}$$

where the equality follows from the homogeneity of g and the inequality follows from the monotonicity of g and the fact that (since the X_i's and the Y_i's are NBU) $X_i/(s + t)$ is stochastically smaller than $\min(X_i/s, Y_i/t)$, $i = 1, \cdots, n$.

Denote $\tilde{X}_i = X_i/s$, $\tilde{Y}_i = Y_i/t$ and let ϕ be the indicator function of the set $\{\mathbf{x}: g(\mathbf{x}) \geq 1\}$. Then, from (5.2) obtain

$$P\{g(\mathbf{X}) > s + t\} \leq E\phi(\tilde{\mathbf{X}} \wedge \tilde{\mathbf{Y}}). \tag{5.3}$$

For every $\mathbf{x}, \mathbf{y} \in R_+^n$

$$\phi(\mathbf{x} \wedge \mathbf{y}) \leq \phi(\mathbf{x})\, \phi(\mathbf{y}). \tag{5.4}$$

Let

$$A = \{(\mathbf{x}, \mathbf{y}): \phi(\mathbf{x} \wedge \mathbf{y}) < \phi(\mathbf{x})\phi(\mathbf{y})\} = \{(\mathbf{x}, \mathbf{y}): g(\mathbf{x} \wedge \mathbf{y}) \leq 1 < g(\mathbf{x}) \wedge g(\mathbf{y})\}.$$

By Lemma 5.2, if $g \notin \mathscr{G}_1$ then $\mu(A) > 0$ where μ is the Lebesgue measure on R_+^{2n}. Thus

$$E\phi(\tilde{\mathbf{X}}) \wedge \phi(\tilde{\mathbf{Y}}) - E\phi(\tilde{\mathbf{X}} \wedge \tilde{\mathbf{Y}}) = P\{(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) \in A\} > 0, \tag{5.5}$$

because the support of $(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}})$ is R_+^{2n}. Combining (5.3) and (5.5) obtain, using the independence of $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$,

$$\begin{aligned} P\{g(\mathbf{X}) > s+t\} &< E\phi(\tilde{\mathbf{X}})\phi(\tilde{\mathbf{Y}}) = E\phi(\tilde{\mathbf{X}})E\phi(\tilde{\mathbf{Y}}) \\ &= P\left\{g\left(\frac{1}{s}\mathbf{X}\right) > 1\right\}P\left\{g\left(\frac{1}{t}\mathbf{Y}\right) > 1\right\} \\ &= P\{g(\mathbf{X}) > s\}P\{g(\mathbf{Y}) > t\}. \end{aligned}$$

But this contradicts the exponentiality of $g(\mathbf{X})$, hence $g \in \mathscr{G}_1$. It is easy to see now that if $g \in \mathscr{G}_1$ and $g(\mathbf{X})$ is exponential then the relevant X_i's must be exponential. □

REMARK. The proof of Lemma 5.3 suggests an alternative direct proof of a weak version of Corollary 3.6(b) of Marshall and Shaked (1982). The following proof is due to A. W. Marshall.

PROPOSITION 5.4. *If* $X_1, \cdots, X_n$ *are independent* NBU *random variables and* $g \in \mathscr{G}$ *then* $g(\mathbf{X})$ *is* NBU.

PROOF. Use the notations of the proof of Lemma 5.3. It is shown there that

$$P\{g(\mathbf{X}) > s+t\} \le E\phi(\tilde{\mathbf{X}} \wedge \tilde{\mathbf{Y}}).$$

But, from (5.4), using the independence of $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$,

$$E\phi(\tilde{\mathbf{X}} \wedge \tilde{\mathbf{Y}}) \le E\phi(\tilde{\mathbf{X}})\, E\phi(\tilde{\mathbf{Y}}) = P\{g(\mathbf{X}) > s\}P\{g(\mathbf{Y}) > t\}. \quad \square$$

In the proof of Theorem 3.2 the following notation will be used. Let $g \in \mathscr{G}_2$ then for fixed nonnegative $x_2, \cdots, x_n$, $g(x_1, x_2, \cdots, x_n)$ is a strictly increasing continuous function of $x_1 \in [0, \infty)$, hence it has an inverse which will be denoted by $g^{-1}_{x_2,\cdots,x_n}$, that is, for $x_1 \ge 0$, $t \ge 0$,

$$g(x_1, \cdots, x_n) = t \Leftrightarrow g^{-1}_{x_2,\cdots,x_n}(t) = x_1.$$

It is easy to verify, using the homogeneity of g that for every $\alpha > 0$, $t \ge 0$,

$$g^{-1}_{x_2,\cdots,x_n}(\alpha t) = \alpha g^{-1}_{x_2/\alpha,\cdots,x_n/\alpha}(t). \tag{5.6}$$

PROOF OF THEOREM 3.2. Assume first that none of the X_i's is degenerate at zero. If all the X_i's are exponential then, by Lemma 5.3, $g \in \mathscr{G}_1$, thus $g \in \mathscr{G}_2$—a contradiction. Hence at least one of the X_i's, X_1 say, is not exponential. Denote by F_i the distribution of X_i. By Lemma 5.1 there exist x and $\alpha = m^{-1}$, where m is some positive integer, such that

$$\bar{F}_1(\alpha x) > \bar{F}_1^{\alpha}(x)$$

and by right-continuity there exists a nonempty open interval $(b, c) \subset [0, \infty)$ such that

$$\bar{F}_1(\alpha x) > \bar{F}_1^{\alpha}(x) \text{ for every } x \in (b, c). \tag{5.7}$$

Since, by assumption, the X_i's are nondegenerate at zero there exist $y_2, \cdots, y_n$ such that, for some $\varepsilon' > 0$,

$$F_i(y_i + \tilde{\varepsilon}) - F_i(y_i - \tilde{\varepsilon}) > 0 \quad \text{whenever} \quad \tilde{\varepsilon} \in (0, \varepsilon'] \tag{5.8}$$

and $y_i - \varepsilon' > 0$, $i = 2, \cdots, n$. By assumption, g is strictly increasing, thus

$$g\left(b, \frac{y_2}{\alpha}, \cdots, \frac{y_n}{\alpha}\right) < g\left(c, \frac{y_2}{\alpha}, \cdots, \frac{y_n}{\alpha}\right),$$

and by continuity of g, for some $\varepsilon'' > 0$,

$$g\left(b, \frac{y_2+\eta}{\alpha}, \cdots, \frac{y_n+\eta}{\alpha}\right) < g\left(c, \frac{y_2-\eta}{\alpha}, \cdots, \frac{y_n-\eta}{\alpha}\right)$$

whenever $\eta \in (0, \varepsilon'']$ and $y_i - \varepsilon'' > 0$, $i = 2, \cdots, n$. Let $\varepsilon = \min(\varepsilon', \varepsilon'')$.

Choose t such that

$$g\left(b, \frac{y_2+\varepsilon}{\alpha}, \cdots, \frac{y_n+\varepsilon}{\alpha}\right) < t < g\left(c, \frac{y_2-\varepsilon}{\alpha}, \cdots, \frac{y_n-\varepsilon}{\alpha}\right),$$

then $g^{-1}_{z_2/\alpha,\cdots,z_n/\alpha}(t) \in (b, c)$ whenever $z_i \in (y_i - \varepsilon, y_i + \varepsilon)$, $i = 2, \cdots, n$. Thus, by (5.7) and (5.8)

$$(5.9) \qquad \int_{y_2-\varepsilon}^{y_2+\varepsilon} \cdots \int_{y_n-\varepsilon}^{y_n+\varepsilon} \bar{F}_1(\alpha g^{-1}_{z_2/\alpha,\cdots,z_n/\alpha}(t))\, dF_n(z_n) \cdots dF_2(z_2)$$

$$> \int_{y_2-\varepsilon}^{y_2+\varepsilon} \cdots \int_{y_n-\varepsilon}^{y_n+\varepsilon} \bar{F}_1^{\alpha}(g^{-1}_{z_2/\alpha,\cdots,z_n/\alpha}(t))\, dF_n(z_n) \cdots dF_2(z_2).$$

Hence, for α and t as above, and for some $\lambda > 0$, by the exponentiality of $g(\mathbf{X})$,

$$e^{-\lambda\alpha t} = \int_{z_2=0}^{\infty} \cdots \int_{z_n=0}^{\infty} \bar{F}_1(g^{-1}_{z_2,\cdots,z_n}(\alpha t))\, dF_n(z_n) \cdots dF_2(z_2)$$

$$= \int_{z_2=0}^{\infty} \cdots \int_{z_n=0}^{\infty} \bar{F}_1(\alpha g^{-1}_{z_2/\alpha,\cdots,z_n/\alpha}(t))\, dF_n(z_n) \cdots dF_2(z_2), \quad \text{by (5.6)}$$

$$> \int_{z_2=0}^{\infty} \cdots \int_{z_n=0}^{\infty} \bar{F}_1^{\alpha}(g^{-1}_{z_2/\alpha,\cdots,z_n/\alpha}(t))\, dF_n(z_n) \cdots dF_2(z_2), \quad \text{by (5.9)}$$

$$\geq \left(\int_{z_2=0}^{\infty} \cdots \int_{z_n=0}^{\infty} \bar{F}_1(g^{-1}_{z_2,\cdots,z_n}(t))\, dF_n(z_n) \cdots dF_2(z_2)\right)^{\alpha}, \quad \text{by (2.3)}$$

$$= e^{-\lambda\alpha t}.$$

But this is a contradiction, hence X_1 is an exponential random variable. Similarly it can be shown that $X_2, \cdots, X_n$ are exponential random variables. But this is impossible as argued above. Thus, at least one of the X_i's is degenerate at zero.

Assuming now that $n - 1$ of the X_i's are nondegenerate at zero, $X_1, \cdots, X_{n-1}$ say, apply the same proof to $g(x_1, \cdots, x_{n-1}, 0)$. This is also a strictly increasing function and the proof goes through to show that at least two X_i's are degenerate at zero.

Continue this way to obtain that $n - 1$ of the X_i's are degenerate at zero. Hence g must satisfy $g(0, \cdots, 0, x_j, 0, \cdots, 0) = ax_j$ for some $a > 0$ and $j \in \{1, \cdots, n\}$. □

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MATH DEPARTMENT
UNIVERSITY OF ARIZONA
TUCSON, ARIZONA 85721