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SITE RECURRENCE FOR ANNIHILATING RANDOM WALKS ON Z_d.[1]

BY RICHARD ARRATIA

University of Southern California

Consider a system of identical particles moving on the integer lattice with mutual annihilation of any pair of particles which collide. Apart from this interference, all particles move independently according to the same random walk p. A system will be called *site recurrent* if a.s. each site is occupied at arbitrarily large times. The following generalization of a conjecture by Erdos and Ney was open: the system of annihilating simple random walks on Z_2, starting with all sites except the origin occupied, is site recurrent. We prove, for general p and a reasonably broad class of initial distributions, that the annihilating system is site recurrent. Loosely speaking, this condition is that the initial configuration does not have any *fixed* sequence of holes with diameters tending to infinity.

1. Introduction.

The study of annihilating particle systems was initiated in a paper by Erdos and Ney (1974). They considered a system of discrete-time simple random walks on the integers Z in which particles annihilate in pairs if they collide *or cross paths*. They conjectured that, starting with all sites except the origin occupied, a.s. the origin is occupied at some time $t \in \{1, 2, \cdots\}$. Lootgieter (1977) verified this conjecture, along with its generalization to the case $p(x, x+1) = 1 - p(x, x-1) \in [0, 1]$. Schwartz (1978) handled the corresponding question in continuous time. Schwartz exploits two distinct connections between the system of annihilating random walks and the voter model: a duality relation, given in Holley and Stroock (1979), which holds for a general random walk p, and a border relation, which only holds for one-dimensional, nearest-neighbor p. This border connection is exploited further in Bramson and Griffeath (1980) and Arratia (1982).

Consider the system of annihilating random walks in continuous time corresponding to a transition kernel $p(x, y) = p(0, y-x)$ on Z_d. A particle at site x waits an exponentially distributed time with mean one, then chooses a site y with probability $p(x, y)$ and jumps there. If site y is already occupied by another particle, the two particles mutually annihilate, leaving both sites x and y vacant. All the waiting times and choices according to p are independent of each other and of the initial configuration η_0. Denote by η_t the set of sites occupied at time t. Griffeath (1978) proved, for irreducible p, that if L is a sublattice of Z_d (i.e. Z_d can be partitioned into finitely many translates of L) and the initial configuration η_0 satisfies:

$$(1)\qquad \begin{aligned} &\text{a.s. } |\eta_0 \Delta L| \text{ is finite and even} && \text{with } p \text{ recurrent;} \\ &\text{or a.s. } |\eta_0 \Delta L| \text{ is finite} && \text{with } p \text{ transient,} \end{aligned}$$

then the annihilating system is *site recurrent* — a.s. every site will be occupied at arbitrarily large times. Consider a two-dimensional version of the Erdos-Ney conjecture: start a system of annihilating simple random walks on Z_2 with all sites *except the origin* occupied at time zero. No conclusion about site recurrence can be drawn from (1), because the initial configuration differs from a lattice by an *odd* number of sites.

Let p be any random walk on Z_d whose symmetrization is irreducible, and let η_t be the corresponding system of annihilating random walks in continuous time. Theorem 1 below gives the following sufficient condition, on the distribution of the initial configuration η_0,

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to imply site recurrence:

$$\exists r \text{ such that } \forall x_1, x_2, \cdots \in Z_d \text{ with } |x_n| \to \infty, \quad P(\eta_0 \cap B_r(x_n) \neq \varnothing \quad \text{i.o.}) = 1. \tag{2}$$

Here $B_r(x)$ is the ball of radius r centered at x. For a *deterministic* initial configuration $\eta_0 = A \subset Z_d$ this condition is that A be *dense*, in the terminology of Harris (1978). If η_0 has a distribution which is translation invariant, and *mixing* {in the sense that for all finite A, $B \subset Z_d, \lim_{|x|\to\infty} | P(A \subset \eta_0, (x + B) \subset \eta_0) - P(A \subset \eta_0) P((x + B) \subset \eta_0) | = 0$} and if $P(\eta_0 \neq \varnothing) > 0$, then η_0 satisfies (2). Note that product measure with $P(x \in \eta_0) = \rho \in (0, 1)\ \forall x \in Z_d$ satisfies (2) but is concentrated on a class of sets $A \subset Z_d$ which *do not* satisfy (2).

Our proof that (2) implies site recurrence involves two steps. The first step, following Griffeath (1978), is to estimate the density of particles, in a system of coalescing random walks starting from all sites occupied (Lemma 1). From the estimate it follows that the expected occupation time per site is infinite; site recurrence for the coalescing system is a consequence of this and monotonicity (Lemma 2). The second step is to analyze the coupling between annihilating and coalescing random walks, relation (3) below. Fix $e \in Z_d$ with $p(0, e) > 0$. Site recurrence for the coalescing system implies that there exist large times t, and sites x for which $\xi_t^x = 0$ and $\xi_t^{x+e} \neq 0$ (Lemma 3). Condition (2) guarantees that for infinitely many of these x, the initial configuration η_0 will have an occupied site y nearby. The particle at y may feed up to site x or to site $x + e$ when the system begins to run; this choice, localized in space and time, controls the parity of the number of coalescing paths to reach the origin at time t. We have an infinite number of these localized choices, which are sufficiently independent to conclude that a.s., there exist arbitrarily large t at which the annihilating system has a particle at the origin (Theorem 1). This same general strategy was used in Arratia (1981) to establish a one-half thinning relation between the limiting point processes for rescalings of systems of annihilating and coalescing random walks, both started with all sites occupied.

The method used in Theorem 1 also works for systems in *discrete time*; we give this as Theorem 2. The same method could also be used to handle the original Erdos-Ney system, in which particles annihilate if they collide *or cross paths*.

Systems with annihilations between two different types of particles are discussed in Erdos-Ney (1974) and Holley (1982). In these systems, particles of the same type move independently of each other, while two particles of opposite type annihilate upon collision. These systems are substantially different from the systems studied in this paper; e.g. a coupling such as (3) below is not available.

2. Further discussion. In this section we present the basic coupling between annihilating systems η_t^A with various initial configurations $A \subset Z_d$. This will explain the source of Griffeath's condition (1) that $|A \Delta L|$ be even in the case that p is recurrent. We discuss an example from Griffeath (1978) which shows how site recurrence, for annihilating simple random walks on Z, depends delicately on the initial configuration A. This same example will show that our sufficient condition (2) is *not necessary* for site recurrence.

Here is the *basic coupling* of the annihilating systems corresponding to a given p. Start with a system $(\xi_t^x: x \in Z_d, t \geq 0)$ of coalescing random walks. In this latter system particles live forever; ξ_t^x is the position at time t of a particle initially at site x. Particles move independently before collision, and move together afterwards. Thus for each x, ξ_t^x is a random walk based on p, and for all $\omega \in \Omega$; $x, y \in Z_d$; $s, t \geq 0$; $\xi_t^x = \xi_t^y$ implies that $\xi_{t+s}^x = \xi_{t+s}^y$. The annihilating system η_t^A starting with particles on $A \subset Z_d$ is realized by defining

$$\eta_t^A \equiv \{y \in Z_d: \ |\{x \in A: \xi_t^x = y\}| \text{ is odd}\}. \tag{3}$$

This coupling has the *cancellative* property: $\forall A, B \subset Z_d$; $t \geq 0$;

$$\eta_t^{A \Delta B} = \eta_t^A \,\Delta\, \eta_t^B. \tag{4}$$

Now suppose that the symmetrization of p is recurrent and irreducible, so that $\forall x, y \in Z_d$, $P(\xi_t^x = \xi_t^y$ eventually$) = 1$. Then for B finite and even, $P(\eta_t^B = \varnothing$ eventually$) = 1$ and hence $P(\eta_t^{A \Delta B} = \eta_t^A$ eventually$) = 1$. This explains why an *even* perturbation is allowed in the recurrent case of condition (1).

For an example, consider annihilating simple random walks on Z starting from $A \equiv \bigcup_{n \geq 1} \{n^4, n^4 + 1\}$. The system starting from $A \cup \{0\}$ is site recurrent, and $A \cup \{0\}$ does not satisfy our condition (2). Starting from A instead of $A \cup \{0\}$, a.s. each site is eventually empty. The proof is an easy exercise, working with the event E_n that the coalescing paths ξ for the particles initially at n^4 and $n^4 + 1$ collide with each other before either one is displaced n^3 from its initial position. By the Borel-Cantelli lemma, $P(E_n$ eventually$) = 1$. All our claims follow from the cancellative property (4) together with recurrence of the underlying simple random walk.

3. Theorems and proofs. The continuous-time cases of Lemmas 1 and 2 below appear in Griffeath (1978). In that paper, the bound $P(0 \in \xi_t) \geq 1/(1 + t)$ is obtained by comparing the voter model, which is dual to the system ξ_t of coalescing random walks, with the critical binary branching process in continuous time. Such a comparison does not seem to be available for the discrete-time voter model. The argument we give here to prove Lemma 1 is taken from an excellent paper by Kelly (1977) which gives better bounds for $P(0 \in \xi_t)$ when p is simple symmetric random walk on Z_d, $d \geq 2$. Bramson and Griffeath (1980b) build on Kelly's work to obtain asymptotics for $P(0 \in \xi_t)$; and these asymptotics are exploited in Arratia (1981). Finally, the stopping time argument given to prove Lemma 2 below is taken from Griffeath (1978), where it is used in a more complicated setting.

LEMMA 1. *Let p be an arbitrary random walk on Z_d, and let $(\xi_t^x, x \in Z_d)$ be the corresponding system of coalescing random walks, in either discrete or continuous time. Let $\xi_t \equiv \{\xi_t^x : x \in Z_d\}$ be the set of sites occupied at time t, starting from all sites occupied. Then*

$$P(0 \in \xi_t) \geq 1/(1 + 2t).$$

PROOF. For $x \in Z_d$, $t \geq 0$ define random variables

$$n_t(x) \equiv |\{y \in Z_d : \xi_t^y = x\}|,$$

$$N_t(x) \equiv n_t(\xi_t^x) = |\{y \in Z_d : \xi_t^y = \xi_t^x\}|,$$

so that n_t^x is the size—possibly zero—of the cluster at x, while $N_t(x) \geq 1$ is the size of the cluster containing the particle initially at x. Write $n_t \equiv n_t(0)$ and $N_t \equiv N_t(0)$. Using only the translation invariance of $(\xi_t^x, x \in Z_d)$ for each fixed t, it follows that for $k = 0,1,2,\cdots$,

$$kP(n_t = k) = P(N_t = k). \tag{5}$$

From (5), $En_t = 1$, and $E(n_t^2) = EN_t$. Using Cauchy-Schwartz for $n_t = n_t 1(n_t > 0)$,

$$1 = (En_t)^2 \leq E(n_t^2) P(n_t > 0), \tag{6}$$

$$\text{so} \quad P(0 \in \xi_t) = P(n_t > 0) \geq 1/E(n_t^2) = 1/EN_t.$$

Now EN_t is just the expected number of sites visited by a random walk $S_t = X_1(t) - X_2(t)$, where X_1 and X_2 are independent random walks based on p, starting as 0. In discrete time, for $t = 0, 1, 2, \cdots$,

$$EN_t = \Sigma_{x \in Z_d} P(\xi_t^x = \xi_t^0) = \Sigma_x P(S_m = -x \text{ for some } m \in [0, t]) \leq 1 + t,$$

so by (6), $P(0 \in \xi_t) \geq 1/EN_t \geq 1/(1 + t)$. In continuous time, the random walk $S_t = X_1(t) - X_2(t)$ has jumps at rate 2, so the bound we obtain from the expected number of jumps is $EN_t \le 1 + 2t$ and hence

$$P(0 \in \xi_t) \ge 1/(1 + 2t). \quad \square$$

LEMMA 2. *Let p be an arbitrary random walk on Z_d, and let ξ_t be the corresponding set-valued system of coalescing random walks, in either discrete or continuous time, starting from all sites occupied. Then*

$$P(\limsup_{t\to\infty}(0 \in \xi_t)) = 1.$$

PROOF. By Lemma 1,

$$\int_0^\infty P(0 \in \xi_t)\, dt = \infty. \tag{7}$$

For any $t \in [0, \infty)$, let $\sigma_t = \inf\{s\colon s \ge t, 0 \in \xi_s\} \in [0, \infty]$, so that the conclusion of this lemma is: $\forall t$, $P(\sigma_t < \infty) = 1$. The basic coupling, $\xi_t^A \equiv \{\xi_t^x\colon x \in A\}$ for $A \subset Z_d$, has the property that $A \subset B \subset Z_d$ implies $\xi_t^A \subset \xi_t^B \subset \xi_t$, so the Markov process $(\xi_t^A\colon A \subset Z_d)$, with state space $\{0, 1\}^{Z_d} \equiv \{A\colon A \subset Z_d\}$ ordered by set inclusion, is attractive. Now for $t < u$, the strong Markov property yields

$$\begin{aligned} I(t, u) &\equiv E\left(\int_{s=t}^u 1(0 \in \xi_s)\, ds\right) = \int_{r=t}^u \int_{A\in\{0,1\}^{Z_d}} P(\sigma_t \in dr, \xi_{\sigma_t} \in dA) E\left(\int_{s=0}^{u-r} 1(0 \in \xi_s^A)\, ds\right) \\ &\le \int_{r=t}^u P(\sigma_t \in dr) E\left(\int_{s=0}^u 1(0 \in \xi_s)\, ds\right) \\ &\le P(\sigma_t \le u)(t + I(t, u)). \end{aligned}$$

Thus $P(\sigma_t \le u) \ge I(t, u)/(t + I(t, u))$. For fixed t, taking $u \to \infty$ yields $P(\sigma_t < \infty) \ge 1$, using (7). $\square$

LEMMA 3. *Let p be a random walk on Z_d which is irreducible in the sense that $\forall x$ $\exists n \ni p^n(0, x) + p^n(x, 0) > 0$. For any $e \ne 0 \in Z_d$ it is possible to define random sites X_1, $X_2, \cdots \in Z_d$ and times $0 = T_0 < T_1 < T_2 < \cdots$ such that* a.s. *for $n = 1, 2, \cdots$*

$$\xi(X_n, T_n) = 0, \xi(X_n + e, T_n) \ne 0, \tag{8}$$
$$\forall t \in [0, T_{n-1}], \quad \xi(X_n, t) \ne 0 \quad \text{and} \quad \xi(X_n + e, t) \ne 0.$$

Here $\xi(x, t) = \xi_t^x$ is the position at time t of a particle starting at x, in a system of coalescing random walks based on p, in either discrete or continuous time.

PROOF. For $x \in Z_d$, $t \in [0, \infty)$ let

$$\tau(x) \equiv \inf\{s\colon \xi_s^x = 0\} \in [0, \infty], \quad H(t) = \{x\colon \tau(x) \le t\}, \quad H = \cup_{t>0} H(t).$$

To see that

$$\forall t, \quad P(H_t \text{ is finite}) = 1, \tag{9}$$

consider

$$\begin{aligned} E|H_t| &= \textstyle\sum_{x\in Z_d} P(\tau(x) \le t) \\ &= \textstyle\sum_x P(\xi_s^x = 0 \text{ for some } s \in [0, t]) = \sum P(\xi_s^0 = -x \text{ for some } s \in [0, t]) \\ &= E(\# \text{ of distinct sites visited by } \xi_s^0 \text{ for } s \in [0, t]) \quad \le \quad 1 + t \quad < \quad \infty. \end{aligned}$$

The random walks ξ_t^x are taken to be right-continuous in t, and $H(t)$ is increasing and right-continuous in t. Write $H(t-) \equiv \cup_{x<t} H(s)$, and let (e, x) denote the inner product on Z_d.

In the case that p is recurrent, irreducibility implies that $\forall x\ P(\tau(x) < \infty) = 1$, so $H = Z_d$ a.s. We define the X_n, T_n recursively by: for $n = 1, 2, \cdots$

$$T_n = \inf\{t \geq T_{n-1} + 1:\quad m(H(t)) > m(H(T_{n-1}))\};$$

$$X_n = \text{any element of } H(T_n) \text{ for which } (e, x) \text{ is maximal.}$$

Using (9) and $H = Z_d$ a.s., it is easy to see that a.s. this defines $(X_n, T_n, n = 1, 2, \cdots)$, satisfying (8).

In the case that p is transient, it follows from Lemma 2 and (9) that a.s., $\sup\{t: H_t \neq H_{t-}\} = \infty$. For $x \in Z_d$, let

$$\sigma(x) \equiv 0 \vee \sup\{t \geq 0: \xi_t^x = 0\}$$

so that by transience, $P(\sigma(x) < \infty\ \forall x) = 1$. The X_n, T_n, and auxiliary times Q_n are defined recursively by: $T_0 = Q_0 = 0$, and for $n = 1, 2, \cdots$

$$T_n = \inf\{t \geq Q_{n-1}: H(t) \neq H(t-)\};$$

$$X_n = \text{any element of } \{x: \xi(x, T_n) = 0\} \text{ for which } (e, x) \text{ is maximal;}$$

$$Q_n = 1 + \max\{\sigma(x): x \text{ or } x - e \in H(T_n)\}.$$

A little thought shows that this sequence X_n, T_n a.s. satisfies (8), for any transient p. [A relevant example here is the one-sided random walk $p(0, 1) = 1$ on Z_1, with $e = 1$; note that (e, X_n) decreases as n increases.] □

THEOREM 1. *Let $p(x, y) = p(0, y - x)$ be a random walk on Z_d, $d = 1, 2, \cdots$, which is irreducible in the sense that $\forall x\ \exists n \ni p^n(0, x) + p^n(x, 0) > 0$. Let η_t be the corresponding system of annihilating random walks, with sites $\eta_0 \subset Z_d$ occupied at time* 0. *If*

$$\exists r\ \forall x_1, x_2, \cdots \in Z_d, |x_n| \to \infty \text{ implies } P(\eta_0 \cap B_r(x_n) \neq \phi \text{ i.o.}) = 1,$$

then

$$P(\cap_{x \in Z_d}(\limsup_{t\to\infty}\{x \in \eta_t\})) = 1.$$

Here $B_r(x) \equiv \{y \in Z_d:\ |x - y| < r\}$.

PROOF. Since our hypothesis is translation invariant, it is enough to prove that $P(\limsup\{0 \in \eta_t\}) = 1$. Start with the usual coupling given by a substructure $\mathbf{P} = (\tau_x(n), S_x(n): x \in Z_d, n = 1, 2, \cdots)$ of event times and arrows, independent of the initial configuration η_0. In detail, the random variables $\tau_x(1)$, $\tau_x(n + 1) - \tau_x(n)$ are exponentially distributed with mean 1, the random $S_x(n) \in Z_d$ are distributed according to $P(S_x(n) = y) = p(x, y)$, and all these quantities are mutually independent. At an event time $\tau_x(n)$, a particle at site x jumps to the random site $S_x(n)$. Write $\xi_{s,t}^x \equiv \xi(x, s, t) = y$ to indicate that a particle starting from site x at time s is carried by $\mathbf{P}$ up to site y at time t; write $\xi_t^x \equiv \xi(x, t) \equiv \xi(x, 0, t)$. The coupling is

$$\eta_t = \{y \in Z_d:\ |\{x \in \eta_0: \xi_t^x = y\}| \text{ is odd}\}. \tag{10}$$

The cancellative property of the coupling is relation (4) in the introduction. For an interval $I \subset [0, \infty)$, write $\mathbf{P}_I = (\xi_{s,t}^x: s \leq t;\ s, t \in I;\ x \in Z_d)$ for the substructure restricted to times in I, so that $\mathbf{P}_{[0,1]}$, $\mathbf{P}_{[1,2]}$, and $\mathbf{P}_{[2,\infty)}$ are mutually independent.

Fix r for which the hypothesis on η_0 is satisfied. By the irreducibility hypothesis on p, we can fix $z \in Z_d$ such that $\varepsilon \equiv \min_{y \in B_r(0)} P(\xi_1^y = -z, \xi_1^a \neq -z\ \forall a \neq y) > 0$. Thus

$$\forall x \in Z_d,\quad y \in B_r(x + z),\quad P(\xi_1^y = x) \geq \varepsilon > 0. \tag{11}$$

Fix any $e \in Z_d$ with $e \neq 0$, $p(0, e) > 0$. Depending on p, we can fix a sequence $|e| < r_1$

$< r_2 < \cdots$ increasing so rapidly that for $x, x' \in Z_d$, $y \in B_r(x+z)$, $y' \in B_r(x'+z)$, $n = 1, 2, \cdots$, the events $A(x, y)$ defined by $A(x, y) \equiv \{\xi_1^y = x, \xi_1^a \neq x \ \forall a \neq y\}$ satisfy

(12) $|x - x'| > r_n$ implies $|P(A(x, y) \cap A(x', y')) - P(A(x, y))P(A(x', y'))| < n^{-3}$.

By Lemma 3, shifted to time 2, we can define random $X_1, X_2, \cdots \in Z_d$ and $T_0 = 2 \leq T_1 < T_2 < \cdots$, all $\sigma(\mathbf{P}_{[2,\infty)})$ measurable, so that a.s.

(13) $$\forall m < n \ |X_m - X_n| > r_n$$

and

$$\xi(X_n, 2, T_n) = 0, \quad \xi(X_n, 2, t) \neq 0 \quad \forall t \in [2, T_{n-1}],$$

$$\xi(X_n + e, 2, t) \neq 0 \quad \forall t \in [2, T_{n-1}] \cup \{T_n\}.$$

Condition on $\mathbf{P}_{[2,\infty)}$ so that the values $x_1, x_2, \cdots$ of $X_1, X_2, \cdots$ are determined. By the hypothesis on η_0, for each fixed sequence $x_1, x_2, \cdots$, $P(\eta_0 \cap B_r(x_n + z) \neq \phi \text{ i.o.}) = 1$. Since η_0 is independent of $\mathbf{P}_{[2,\infty)}$, it is possible to select a subsequence of the X_n, T_n (to be again labeled X_n, T_n) and define random $Y_1, Y_2, \cdots \in Z_d$ such that a.s., $\forall n$

(14) $$Y_n \in B_r(X_n + z) \cap \eta_0.$$

These $Y_1, Y_2, \cdots$, and these new $X_1, X_2, \cdots$ and $T_1, T_2, \cdots$ are all $\sigma(\eta_0, \mathbf{P}_{[2,\infty)})$ measurable, and still satisfy (13).

Let $x_1, x_2, \cdots$ and $y_1, y_2, \cdots \in Z_d$ satisfy $\forall m < n \ |x_m - x_n| > r_n$ and $y_n \in B_r(x_n + z)$. Define events $A_n \equiv \{\xi(y_n, 1) = x_n, \xi(a, 1) \neq x_n \ \forall a \neq y_n\} \in \sigma(\mathbf{P}_{[0,1]})$. By (11), $P(A_n) \geq \varepsilon > 0 \ \forall n$, while by (12), $\sum_{m,n} \mathrm{cov}(1(A_m), 1(A_n)) < \infty$. Thus $(\sum_{n \leq N} 1(A_n) / \sum_{n \leq N} P(A_n)) \to 1$ in L_2 as $N \to \infty$, so that $P(A_n \text{ i.o.}) = 1$. Condition on $\sigma(\eta_0, \mathbf{P}_{[2,\infty)})$ so that the values of $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ are determined. It follows that $P((\xi(Y_n, 1) = X_n, \xi(a, 1) \neq X_n \ \forall a \neq Y_n) \text{ i.o.}) = 1$. Thus we can define a subsequence of the X_n, Y_n, T_n (again labeled X_n, Y_n, T_n), now $\sigma(\eta_0, \mathbf{P}_{[0,1]}, \mathbf{P}_{[2,\infty)})$ measurable, so that a.s., $\forall n$

(15) $$\xi(Y_n, 1) = X_n, \quad \xi(y, 1) \neq X_n \ \forall y \neq Y_n$$

and also (13) and (14) are still satisfied.

For each $x \in Z_d$ define an event $E_x \in \sigma(\mathbf{P}_{[1,2]})$ by

$$E_x = \{\omega : \xi_{1,t}^x \in \{x, x+e\} \text{ and } \xi_{1,t}^{x+e} = x + e, \forall t \in [1, 2]\}$$
$$\cap \{\xi_{s,t}^y \neq x \ \forall y \neq x, \forall \ 1 \leq s \leq t \leq 2\}.$$

Let q by the elementary conditional probability $P(\xi_{1,2}^x = x \mid E_x)$. For any $x_1, x_2, \cdots$ such that $|x_n| \to \infty$, as $n \to \infty$, it can be shown that $P(E_{x_n} \text{ i.o.}) = 1$. By conditioning on $\sigma(\eta_0, \mathbf{P}_{[0,1]}, \mathbf{P}_{[2,\infty)})$, which is independent of $\mathbf{P}_{[1,2]}$, it follows that $P(E_{X_n} \text{ i.o.}) = 1$. Thus there is a further subsequence of the X_n, Y_n, T_n (yet again labeled X_n, Y_n, T_n) which satisfies (13), (14), (15), and for which

(16) $$P(\cap_{n \geq 1} E_{X_n}) = 1,$$

so that in particular a.s. $\forall n \ \xi(X_n, 1, 2) \in \{X_n, X_n + e\}$.

Let C_n be the indicator random variable

$$C_n = 1(\xi(X_n, 1, 2) = X_n).$$

Let $\mathbf{F}$ be the σ-field carrying all the information in η_0 and $\mathbf{P}$ *except* for the values of $C_1, C_2, \cdots$; i.e. let $\mathbf{F} \equiv \sigma(\eta_0, \mathbf{P}_{[0,1]}, \mathbf{P}_{[2,\infty)}, \xi(x, s, t) 1(x \notin \{X_1, X_2, \cdots\}) : x \in Z_d, 1 \leq s \leq t \leq 2)$. Even after conditioning on $\mathbf{F}$, the $C_1, C_2, \cdots$ still form an i.i.d. sequence of Bernoulli variables with $q = P(C_n = 1) \in (0, 1)$. Using (13), (14), (15), and (16), we see that a.s. $C_n = 1(Y_n \in \eta_0, \xi(Y_n, T_n) = 0)$ and that the indicator variable B_n defined by

$$B_n \equiv 1(|\{y \in \eta_0 \backslash \{Y_n\} : \ \xi(y, T_n) = 0\}| \text{ is odd})$$

is $\mathbf{F} \vee \sigma(C_1, C_2, \cdots, C_{n-1})$ measurable. By the basic coupling (3), $\forall n$, a.s.

$$\{0 \in \eta_{T_n}\} = \{B_n + C_n = 1\} \in \mathbf{F} \vee \sigma(C_1, C_2, \cdots, C_n).$$

Now a.s. $P(0 \in \eta_{T_n} | \mathbf{F}, C_1, \cdots, C_{n-1}) \geq q \wedge (1-q) > 0$, and it follows, by Levy's conditional form of the Borel-Cantelli lemma (see Freedman, 1973), that

$$P(0 \in \eta_{T_n} \text{ i.o.}) = 1. \square$$

THEOREM 2. *Let p be a random walk on Z_d, $d = 1, 2, \cdots$, which has period ℓ, so that Z_d can be partitioned into ℓ sets $0 \in G_0, G_1, \cdots, G_{\ell-1}$, where $x \in G_i, y \in G_j, p^k(x, y) > 0$ implies that ℓ divides $(i + k - j)$, and ℓ is the largest such integer. Assume that p is irreducible in the sense that for $i = 0$ to $\ell - 1$, $\forall x, y \in G_i, \exists n, z \ni p^n(x, z)p^n(y, z) > 0$. Let $(\eta_k; k = 0, 1, 2, \cdots)$ be the discrete-time system of annihiliating random walks based on p. If the initial configuration $\eta_0 \subset Z_d$ satisfies*

$$\exists i \in \{0, \cdots, \ell-1\} \quad \exists r \, \forall x_1, x_2, \cdots \in Z_d, \tag{17}$$

$$|x_n| \to \infty \textit{ implies } P(\eta_0 \cap B_r(x_n) \cap G_i \neq \phi \text{ i.o.}) = 1,$$

then

$$P(\cap_{x \in Z_d}(\limsup_{k\to\infty}\{x \in \eta_k\})) = 1.$$

PROOF. We focus on the modifications needed to transfer the proof of Theorem 1 to the discrete-time system. The substructure $\mathbf{P}$ now is a family $(S(x, n); x \in Z_d, n = 0, 1, \cdots)$ of independent Z_d-valued random variables with $P(S(x, n) = y) = p(x, y)$. For $m \leq n$, the position $\xi^x_{m,n} \equiv \xi(x, m, n)$ at time n of a particle starting from site x at time m is defined recursively by: $\xi(x, m, m) \equiv x$ and for $n > m$, $\xi(x, m, n) \equiv S(\xi(x, m, n-1), n-1)$. The annihilating system $(\eta_t, t \in Z^+)$ is still defined in terms of the coalescing paths $\xi^x_t \equiv \xi^x_{0,t}$ via the coupling (3). For an interval $I \subset [0, \infty)$, we let $\mathbf{P}_I = (\xi^x_{s,t}: s \leq t; s, t \in I \cap Z, x \in Z_d)$.

The ℓ subsystems $(\xi^x_n: n \in Z^+, x \in G_i)$ for $i = 0, 1, \cdots, \ell - 1$ are mutually independent. Fix $f_1 \neq f_2 \in Z_d$ such that $p(0, f_1)p(0, f_2) > 0$ and set $e = -f_1 + f_2$. The conclusion of Lemma 3 for this choice of e must hold for at least one of the ℓ independent subsystems, so we can fix $j \in \{0, 1, \cdots, \ell-1\}$ such that (8) holds with the additional restriction that $\forall n, X_n \in G_j$.

Fix r and i for which the hypothesis (17) on η_0 is satisfied. By the irreducibility of p, we can choose $z \in Z_d$, $k \in Z^+$ such that

$$\forall x \in G_j, \quad \forall y \in B_r(x + z) \cap G_i, \quad p^k(y, x - f_1) > 0.$$

As in Theorem 1, choose a sequence $r_1 < r_2 < \cdots$ increasing so rapidly that for $n \geq 1$, $x, x' \in Z_d$, $y \in B_r(x + z)$, $y' \in B_r(x' + z)$, the events $A(x, y)$ defined by $A(x, y) = \{\xi^y_k = x - f_1, \xi^a_k \neq x - f_1 \; \forall a \neq y\}$ satisfy (12).

By Lemma 3 shifted to time $k + 1$ we can define random $X_1, X_2, \cdots \in G_j$ and $k + 1 = T_0 < T_1 < \cdots \in Z$, all $\mathbf{P}_{[k+1,\infty)}$ measurable, so that a.s.

$$\forall m < n \, |X_m - X_n| > r_n \quad \text{and} \quad \xi(X_n, k+1, T_n) = 0, \quad \xi(X_n + e, k+1, T_n) \neq 0;$$

$$\forall t \in [k+1, T_{n-1}] \cap Z, \quad \xi(X_n, k+1, t) \neq 0 \quad \text{and} \quad \xi(X_n + e, k+1, t) \neq 0.$$

Since η_0 is independent of $\mathbf{P}_{[k+1,\infty)}$, it is possible to select a subsequence of the X_n, T_n and define random $Y_1, Y_2, \cdots \in Z_d$, now all $\sigma(\eta_0, \mathbf{P}_{[k+1,\infty)})$ measurable, to also satisfy $\forall n$,

$$Y_n \in B_r(X_n + z) \cap G_i \cap \eta_0.$$

Since $\mathbf{P}_{[0,k]}$ is independent of η_0 and $\mathbf{P}_{[k+1,\infty)}$, it is possible to select a further subsequence X_n, Y_n, T_n, now $\sigma(\eta_0, \mathbf{P}_{[0,k]}, \mathbf{P}_{[k+1)})$ measurable, for which $\forall n$

$$\xi(Y_n, k) = X_n - f_1, \qquad \forall y \neq Y_n, \xi(y, k) \neq X_n - f_1.$$

For each $x \in Z_d$ define an event E_x by

$$E_x = \{S(x - f_1, k) \in \{x, x + e\}\}.$$

Since $P(E_x) > 0$ and the E_x, $x \in Z_d$ are mutually independent of each other even after conditioning on $\sigma(\eta_0, \mathbf{P}_{[0,k]}, \mathbf{P}_{[k+1,\infty)})$, it is possible to select a further subsequence for which

$$P(\cap_{n\geq 1} E_{X_n}) = 1.$$

Thus in particular $\forall n\ \xi(X_n - f_1, k, k+1) \in \{X_n, X_n + e\}$.

Let C_n be the indicator random variable

$$C_n = 1(\xi(X_n - f_1, k, k+1) = X_n).$$

Even after conditioning on

$$\mathbf{F} \equiv \sigma(\eta_0, S(x, n)1(n \neq k \text{ or } x + f_1 \notin \{X_1, X_2, \cdots\}) : x \in Z_d, n \in Z^+)$$

which represents all the information in η_0 and P except for the values of $C_1, C_2, \cdots$, the $C_1, C_2, \cdots$ still form an i.i.d. sequence with $P(C_n = 1) = q \in (0, 1)$. Now a.s. $C_n = 1(Y_n \in \eta_0, \xi(Y_n, T_n) = 0)$. The indicator variable $B_n = 1(|\{y \in \eta_0 \backslash \{Y_n\} : \ \xi(y, T_n) = 0\}|$ is odd) is $\mathbf{F} \vee \sigma(C_1, C_2, \cdots, C_{n-1})$ measurable. By the basic coupling (1), $\forall n\ \{0 \in \eta_{T_n}\} = \{B_n + C_n = 1\} \in \mathbf{F} \vee \sigma(C_1, C_2, \cdots, C_n)$. From $P(0 \in \eta_{T_n} | \mathbf{F}, C_1, \cdots, C_{n-1}) \geq q \wedge (1 - q) > 0$, it follows that

$$P(0 \in \eta_{T_n} \text{ i.o.}) = 1. \square$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CALIFORNIA 90089-1113