The Annals of Probability
1983, Vol. 11, No. 3, 593–608

HOW BIG ARE THE INCREMENTS OF THE LOCAL TIME OF A WIENER PROCESS?

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Let $W(t)$ be a standard Wiener process with local time (occupation density) $L(x, t)$. Kesten showed that $L(0, t)$ and $\sup_x L(x, t)$ have the same LIL law as $W(t)$. Exploiting a famous theorem of P. Lévy, namely that $\{\sup_{0\le s\le t} W(s), t \ge 0\} \overset{\mathscr{D}}{=} \{L(0, t), t \ge 0\}$, we study the almost sure behaviour of big increments of $L(0, t)$ in t. The very same increment problems in t of $L(x, t)$ are also studied uniformly in x. The results in the latter case are slightly different from those concerning $L(0, t)$, and they coincide only for Kesten's above mentioned LIL.

Introduction. Let $\{W(t), t \ge 0\}$ be a Wiener process and $0 < a_t \le t$ be a non-decreasing function of t. Consider the process

$$X(t) = a_t^{-1/2} \sup_{0\le s\le t-a_t}(W(s + a_t) - W(s)).$$

Csörgő and Révész (1979) investigated the limit behaviour of $X(t)$ (as $t \to \infty$) and proved

THEOREM A. *Assume that $t^{-1}a_t$ is non-increasing. Then*

$$\begin{aligned}\limsup_{t\to\infty}\beta_t X(t) &= \limsup_{t\to\infty}\beta_t(W(t + a_t) - W(t)) \\ &= \limsup_{t\to\infty}\sup_{0\le s\le a_t}\beta_t(W(t + s) - W(t)) = 1 \quad \text{a.s.}\end{aligned} \tag{1}$$

where

$$\beta_t = (2(\log t a_t^{-1} + \log\log t))^{-1/2}. \tag{2}$$

If we also assume that

$$\lim_{t\to\infty}\frac{\log t a_t^{-1}}{\log\log t} = \infty \tag{3}$$

then

$$\lim_{t\to\infty}\beta_t X(t) = 1 \quad \text{a.s.}$$

The aim of the present paper is to give an analogous result for the local time of the Wiener process W. For any Borel set A of the real line let

$$H(A, t) = \lambda\{s : s \le t, W(s) \in A\}$$

be the occupation time of W, where λ is the Lebesgue measure. It is well-known that $H(A, t)$ is a random measure absolutely continuous with respect to λ. The Radon-Nikodym derivative of H is called the local time of W and will be denoted by L, i.e., $L(x, t)$ is defined by

$$H(A, t) = \int_A L(x, t)\, dx.$$

Received April 1982.

[1] Research partially supported by a NSERC Canada grant at Carleton University, Ottawa.

AMS 1980 *subject classifications.* Primary 60J55, 60J65; secondary 60G17, 60G57.

Key words and phrases. Local time, Wiener process (Brownian motion), big increments and continuity of Brownian local time.

H. Trotter (1958) proved that $L(x, t)$ is continuous in both arguments. Concerning the modulus of continuity he proved

$$\limsup_{h\to 0}\sup_{-\infty<x<\infty}\frac{L(x+h,t)-L(x,t)}{h^{1/2}\log h^{-1}}=0 \quad \text{a.s.} \tag{4}$$

for any $t > 0$ and

$$\limsup_{h\to 0}\sup_{0\le t\le 1-h}\sup_{-\infty<x<\infty}\frac{L(x,t+h)-L(x,t)}{h^{1/3}(\log h^{-1})^{2/3}}=0 \quad \text{a.s.} \tag{5}$$

(4) was improved by H. P. McKean Jr. (1962) and D. B. Ray (1963) as follows:

$$\limsup_{h\to 0}\sup_{-\infty<x<\infty}\frac{L(x+h,t)-L(x,t)}{(h\log h^{-1})^{1/2}}=2(\max_{-\infty<x<\infty}L(x,t))^{1/2} \quad \text{a.s.} \tag{6}$$

They also proved

$$\limsup_{h\to 0}\frac{L(h,t)-L(0,t)}{(h\operatorname{loglog} h^{-1})^{1/2}}=2(L(0,t))^{1/2} \quad \text{a.s.}$$

for any fixed $t > 0$.

As far as the modulus of continuity in t is concerned, the best known results are the following:

$$\lim_{h\searrow 0}\frac{\sup_{0\le t\le 1-h}(L(0,t+h)-L(0,t))}{(h\log h^{-1})^{1/2}}=1 \quad \text{a.s.} \tag{7}$$

(Hawkes, 1971), and

$$\limsup_{h\searrow 0}\frac{\sup_{0\le t\le 1-h}\sup_{-\infty<x<\infty}(L(x,t+h)-L(x,t))}{(2h\log h^{-1})^{1/2}}=1 \quad \text{a.s.} \tag{8}$$

(Perkins, 1981).

Kesten (1965) proved the following law of iterated logarithm:

$$\limsup_{t\to\infty}\frac{L(0,t)}{(2t\operatorname{loglog} t)^{1/2}}=\limsup_{t\to\infty}\frac{\sup_{-\infty<x<\infty}L(x,t)}{(2t\operatorname{loglog} t)^{1/2}}=1 \quad \text{a.s.} \tag{9}$$

Denoting the process $L(0, t)$ by $L(t)$, our main result says:

THEOREM 1. *Let $0 < a_t \le t$ be a non-decreasing function of t. Assume that $t^{-1}a_t$ is non-increasing. Then*

$$\limsup_{t\to\infty}\gamma_t Y(t)=\limsup_{t\to\infty}\gamma_t a_t^{-1/2}(L(t)-L(t-a_t))=1 \quad \text{a.s.} \tag{10}$$

where

$$Y(t)=Y(t,a_t)=a_t^{-1/2}\sup_{0\le s\le t-a_t}(L(s+a_t)-L(s)) \tag{11}$$

and

$$\gamma_t=(\log ta_t^{-1}+2\operatorname{loglog} t)^{-1/2}. \tag{12}$$

If we also assume that (3) *holds true, then*

$$\lim_{t\to\infty}\gamma_t Y(t)=1 \quad \text{a.s.} \tag{13}$$

REMARK 1. In case $a_t = t$, (10) implies the first part of (9), i.e., by (10) we get

$$\limsup_{t\to\infty}\frac{L(0,t)}{(2t\operatorname{loglog} t)^{1/2}}=1 \quad \text{a.s.}$$

Introduce the process

$$M(t) = \sup_{0\le s\le t} W(s).$$

A famous theorem of P. Lévy (cf. Knight, 1981, Theorem 5.3.7) says that the behaviour of $M(t)$ is just the same as that of $L(t)$. In fact we have

THEOREM B.

$$\{M(t), t \ge 0\} \overset{\mathcal{D}}{=} \{L(t), t \ge 0\}. \tag{14}$$

This theorem implies that our Theorem 1 is equivalent to

THEOREM 2. *Let*

$$Z(t) = a_t^{-1/2} \sup_{0\le s\le t-a_t}(M(s + a_t) - M(s)). \tag{15}$$

Then the statements of Theorem 1 *remain true when replacing* $Y(t)$ *by* $Z(t)$ *and* $L(t)$ *by* $M(t)$ *in them.*

Replacing $L(t) = L(0, t)$ by $L(x, t)$ in the definition of $Y(t)$, Theorem 1 clearly remains true for any fixed real number x. However, the investigation of the largest possible increment in t when x is also varying is different, and seems to be also quite interesting when compared to the x fixed case (cf. γ_t of (12) and β_t of (18)). Let

$$U(t) = a_t^{-1/2}\sup_{-\infty<x<+\infty}\sup_{0\le s\le t-a_t}(L(x, s + a_t) - L(x, s)). \tag{16}$$

Then we have:

THEOREM 3. *Let* $0 < a_t \le t$ *be a non-decreasing function of t, and assume also that* $t^{-1}a_t$ *is non-increasing. Then*

$$\limsup_{t\to\infty}\beta_t U(t) = 1 \quad \text{a.s.} \tag{17}$$

where β_t *is defined by* (2). *If we also assume that* (3) *holds true, then*

$$\lim_{t\to\infty}\beta_t U(t) = 1 \quad \text{a.s.} \tag{18}$$

The study of the continuity modulus of $L(\cdot)$ and $M(\cdot)$ is very similar to the problems solved by Theorems 1 and 2. In fact we have

THEOREM 4.

$$\lim_{h\searrow 0}\frac{\sup_{0\le t\le 1-h}(M(t + h) - M(t))}{(h \log h^{-1})^{1/2}} = 1 \quad \text{a.s.} \tag{19}$$

$$\lim_{h\searrow 0}\frac{\sup_{0\le t\le 1-h}(L(t + h) - L(t))}{(h \log h^{-1})^{1/2}} = 1 \quad \text{a.s.} \tag{20}$$

$$\lim_{h\searrow 0}\frac{\sup_{0\le t\le 1-h}\sup_{-\infty<x<+\infty}(L(x, t + h) - L(x, t))}{(2h \log h^{-1})^{1/2}} = 1 \quad \text{a.s.} \tag{21}$$

Here, our relation (20) is not new at all, it is equivalent to the result of Hawkes (1971, cf. (7)). (19) is a consequence of (20). This can be seen by applying the quoted theorem of Lévy (cf. Theorem B). (21) is slightly stronger than the already mentioned theorem of Perkins (1981, cf. (8)).

The proof of Theorem 1 is based on the following three lemmas:

LEMMA 1. *Let* $0 < a_t \le t$ *be a non-decreasing function of t. Assume that* $t^{-1}a_t$ *is non-increasing and Condition* (3) *holds. Then*

(22) $$\liminf_{t\to\infty}\gamma_t Y(t) \geq 1 \quad \text{a.s.}$$

LEMMA 2. *Let $0 < a_t \leq t$ be a non-decreasing function of t and assume that $t^{-1}a_t$ is non-increasing. Then*

(23) $$\limsup_{t\to\infty}\gamma_t Y(t) \leq 1 \quad \text{a.s.}$$

LEMMA 3. *Let $0 < a_t \leq t$ be a non-decreasing function of t. Assume that $t^{-1}a_t$ is non-increasing. Then*

(24) $$\limsup_{t\to\infty}\gamma_t a_t^{-1/2}(L(t) - L(t-a_t)) \geq 1 \quad \text{a.s.}$$

Lemmas 1, 2, and 3 together clearly imply Theorem 1. Their proof will be presented in the next three sections. The main ideas of the proof of Theorem 3 will be given in Section 4. The proof of Theorem 4 can be obtained by repeating the ideas of Lemmas 1 and 2 and Theorem 3 without any major change. Hence the details of the proof of Theorem 4 will be omitted.

We are indebted to the referee for calling our attention to the papers of Hawkes (1971) and Perkins (1981).

1. Proof of Lemma 1. Let $\eta_0 = 0$ and $\xi_1 = \xi_1(a_t)$, which is uniquely defined (with probability 1) by

$$W(\xi_1) = \sup_{0\leq s\leq a_t} W(s).$$

Also let

$$\eta_1 = \eta_1(a_t) = \inf\{s : s > \xi_1,\ W(s) = W(\xi_1)\}, \quad \alpha_1 = \alpha_1(a_t) = \eta_1 - a_t.$$

Further we define $\xi_2 = \xi_2(a_t)$, $\eta_2 = \eta_2(a_t)$ and $\alpha_2 = \alpha_2(a_t)$ by

$$W(\xi_2) = \sup_{\eta_1\leq s\leq \eta_1 + a_t} W(s), \quad \eta_2 = \inf\{s : s > \xi_2,\ W(s) = W(\xi_2)\}$$

and

$$\alpha_2 = \eta_2 - (\eta_1 + a_t).$$

In general, if $\eta_i (i \geq 1)$ is already defined, then we define ξ_{i+1}, η_{i+1}, α_{i+1} by

$$W(\xi_{i+1}) = \sup_{\eta_i\leq s\leq \eta_i + a_t} W(s),$$

$$\eta_{i+1} = \inf\{s : s > \xi_{i+1},\ W(s) = W(\xi_{i+1})\},$$

$$\alpha_{i+1} = \eta_{i+1} - (\eta_i + a_t).$$

The following few lemmas are trivial or well-known.

LEMMA 1.1. $0 = \eta_0 \leq \eta_1 \leq \eta_2 \leq \cdots$ *is a sequence of stopping times (with respect to W).*

LEMMA 1.2. (cf. Knight, (1981), Lemma 2.11).

$$\left\{\frac{\alpha_i}{(W(\xi_i) - W(\eta_{i-1} + a_t))^2};\ i = 1, 2, \cdots\right\}$$

is a sequence of i.i.d. r.v.'*s with*

$$P\left\{\frac{\alpha_i}{(W(\xi_i) - W(\eta_{i-1} + a_t))^2} \leq x \,\middle|\, W(\xi_i) - W(\eta_{i-1} + a_t)\right\}$$

$$= P\left\{\frac{\alpha_i}{(W(\xi_i) - W(\eta_{i-1} + a_t))^2} \leq x\right\} = \frac{1}{\sqrt{2\pi}}\int_0^x v^{-3/2}e^{-1/2v}\,dv := F(x).$$

REMARK 2. $F(x)$ is a stable distribution function with parameters $\alpha = 1/2$, $\beta = 1$, $\gamma = 0$.

Lemma 1.2 clearly implies

LEMMA 1.3.

$$P\left\{\frac{1}{k^2}\sum_{i=1}^k \frac{\alpha_i}{(W(\xi_i) - W(\eta_{i-1} + a_t))^2} \le x\right\} = F(x) \quad (k = 1, 2, \cdots).$$

LEMMA 1.4. *Let*

$$R_i = W(\xi_{i+1}) - W(\eta_i).$$

Then

$$Z(t) \ge a_t^{-1/2} \max_{0\le i\le \nu_t} R_i$$

where $\nu_t = \nu(a_t)$ *is the largest integer for which*

$$\eta_{\nu_t} + a_t = (\nu_t + 1)a_t + \alpha_1 + \alpha_2 + \cdots + \alpha_{\nu_t} \le t.$$

Applying Theorem B one gets:

LEMMA 1.5. $R_1, R_2, \cdots$ *is a sequence of* i.i.d. r.v.'s *with*

$$P\{a_t^{-1/2}R_1 \le x\} = 2\Phi(x) - 1.$$

Hence

$$P\left(R_1 > C\left(a_t \log\frac{t}{a_t}\right)^{1/2}\right) \le \left(\frac{a_t}{t}\right)^2$$

and

$$P\left(\max(R_1, R_2, \cdots, R_k) > C\left(a_t\left(\log\frac{t}{a_t} + \log k\right)\right)^{1/2}\right) \le \left(\frac{a_t}{t}\right)^2,$$

if C is a big enough constant.

Lemmas 1.3 and 1.5 together imply

LEMMA 1.6. *For any* $0 < \varepsilon \le 4$ *and C large enough we have*

$$P\left\{\sum_{i=1}^k \alpha_i > Ck^2a_t\left(\frac{t}{a_t}\right)^\varepsilon\left(\log\frac{t}{a_t} + \log k\right)\right\} \le C\left(\frac{a_t}{t}\right)^{\varepsilon/2}.$$

PROOF. Replacing the denominators by $\max_{1\le i\le k} R_i^2$ in Lemma 1.3 we get

$$P\{\textstyle\sum_{i=1}^k \alpha_i \le k^2[\max(R_1, R_2, \cdots, R_k)]^2x\} \ge F(x).$$

Now applying the third statement of Lemma 1.5 we get

$$P\left\{\sum_{i=1}^k \alpha_i \le c^2k^2\left(a_t\left(\log\frac{t}{a_t} + \log k\right)\right)x\right\} \ge F(x) - \left(\frac{a_t}{t}\right)^2,$$

and letting $x = (t/a_t)^\varepsilon$ $(\varepsilon > 0)$ we get Lemma 1.6. □

LEMMA 1.7. *There exists a small enough* $C_1 > 0$ *and a large enough* $C_2 > 0$ *such that for any* $0 < \varepsilon \le 4$ *we have*

(1.1) $$P\left\{\nu_t < C_1\left(\frac{t}{a_t}\right)^{(1-\varepsilon)/2}\left(\log\frac{t}{a_t}\right)^{-1/2}\right\} \le C_2\left(\frac{a_t}{t}\right)^{\varepsilon/2}.$$

PROOF. Let

$$k = \left[C_1\left(\frac{t}{a_t}\right)^{(1-\varepsilon)/2}\left(\log\frac{t}{a_t}\right)^{-1/2}\right] := f(t)$$

and

$$\psi(t) := C(f(t))^2 a_t\left(\frac{t}{a_t}\right)^{\varepsilon}\left(\log\frac{t}{a_t} + \log f(t)\right),$$

where C is the constant of Lemma 1.6. Choosing C_1 small enough we have

$$\psi(t) < t - a_t(f(t) + 1).$$

Choosing k in Lemma 1.6 to be the just defined $f(t)$ we get

$$C\left(\frac{a_t}{t}\right)^{\varepsilon/2} \ge P\{\textstyle\sum_{i=1}^{f(t)} \alpha_i > \psi(t)\} \ge P\{\sum_{i=1}^{f(t)} \alpha_i > t - a_t(f(t) + 1)\}.$$

Since ν_t is the largest integer for which

$$(\nu_t + 1)a_t + \textstyle\sum_{i=1}^{\nu_t} \alpha_i \le t,$$

it follows that the event $\{\nu_t < f(t)\}$ implies the event $\{\sum_{i=1}^{f(t)} \alpha_i > t - (f(t) + 1)a_t\}$, i.e.,

$$P\{\nu_t < f(t)\} \le P\{\textstyle\sum_{i=1}^{f(t)} \alpha_i > t - (f(t) + 1)a_t\} \le C\left(\frac{a_t}{t}\right)^{\varepsilon/2},$$

and we have (1.1). □

LEMMA 1.8. *Assume that Condition* (3) *holds. Then for any* $0 < \rho \le 1$ *there exists a* r.v. $t_0 = t_0(\rho, \omega) > 0$ *such that*

(1.2) $$Z(t) \ge ((1 - \rho)\log t/a_t)^{1/2} \quad \text{a.s.}$$

for all $t \ge t_0$.

PROOF. By Lemma 1.4

(1.3) $$\begin{aligned} Z(t) &\ge a_t^{-1/2}\max\nolimits_{0\le i\le\nu_t}(W(\xi_{i+1}) - W(\eta_i)) \\ &= a_t^{-1/2}\max\nolimits_{0\le i\le\nu_t}\sup\nolimits_{0\le s\le a_t}(W(\eta_i + s) - W(\eta_i)). \end{aligned}$$

Let

$$A_t(\rho) = \{a_t^{-1/2}\max\nolimits_{0\le i\le\nu_t}\sup\nolimits_{0\le s\le a_t}(W(\eta_i + s) - W(\eta_i)) \le ((1 - \rho)\log t/a_t)^{1/2}\}.$$

Then by Lemma 1.7

$$\begin{aligned} P(A_t(\rho)) &= P\left(A_t(\rho),\ \nu_t < C_1\left(\frac{t}{a_t}\right)^{(1-\varepsilon)/2}\left(\log\frac{t}{a_t}\right)^{-1/2}\right) \\ &\quad + P\left(A_t(\rho),\ \nu_t \ge C_1\left(\frac{1}{a_t}\right)^{(1-\varepsilon)/2}\left(\log\frac{t}{a_t}\right)^{-1/2}\right) \\ &\le C_2\left(\frac{a_t}{t}\right)^{\varepsilon/2} + P\left\{a_t^{-1/2}\max\nolimits_{0\le i\le C_1(t/a_t)^{(1-\varepsilon)/2}(\log t/a_t)^{-1/2}}\right. \end{aligned}$$

$$\sup_{0\le s\le a_t}(W(\eta_i + s) - W(\eta_i)) \le \left((1-\rho)\log\frac{t}{a_t}\right)^{1/2}\Bigg\}$$

$$= C_2\left(\frac{a_t}{t}\right)^{\varepsilon/2} + \left(1 - \left(\frac{a_t}{t}\right)^{(1-\rho)/2}\right)^{[C_1(t/a_t)^{(1-\varepsilon)/2}(\log t/a_t)^{-1/2}]}$$

$$\le C_2\left(\frac{a_t}{t}\right)^{\varepsilon/2} + \exp\left(-\frac{C_1}{2}\left(\frac{t}{a_t}\right)^{(\rho-\varepsilon)/2}\left(\log\frac{t}{a_t}\right)^{-1/2}\right).$$

Letting now $\varepsilon = \rho/2$ and $t_k = \theta_k$ ($\theta > 1$) then (by Condition (3))

$$\textstyle\sum_{k=1}^{\infty} P\{A_{\theta^k}(\rho)\} < \infty.$$

Hence

$$Z_{t_k} > ((1-\rho)\log t_k/a_{t_k})^{1/2} \tag{1.4}$$

for all large enough k with probability 1.

Since $a_t^{1/2}Z_t$ is nondecreasing in t, for $t_{k-1} < t < t_k$ we have

$$a_t^{1/2}Z_t \ge a_{t_{k-1}}^{1/2}Z_{t_{k-1}} \ge ((1-\rho)a_{t_{k-1}}\log(t_{k-1}/a_{t_{k-1}}))^{1/2} \tag{1.5}$$

with probability 1 for all but finitely many k (by (1.4)). Also, on account of $a_t/t \searrow 0$, we have

$$1 \le \frac{a_{t_k}}{a_{t_{k-1}}} = \frac{a_{t_k}}{t_k}\cdot\frac{t_{t-1}}{a_{t_{k-1}}}\cdot\frac{t_k}{t_{k-1}} \le \frac{t_k}{t_{k-1}} = \theta \tag{1.6}$$

and, on account of a_t being nondecreasing,

$$1 \le \frac{t_k}{a_{t_k}}\cdot\frac{a_{t_{k-1}}}{t_{k-1}} = \theta\frac{a_{t_{k-1}}}{a_{t_k}} \le \theta. \tag{1.7}$$

Now choosing θ close enough to one and k large enough, we get

$$\begin{aligned}(1-\rho)a_{t_{k-1}}\log\frac{t_{k-1}}{a_{t_{k-1}}} &> \frac{1-\rho}{\theta}a_{t_k}\log\frac{t_k}{\theta a_{t_k}}\\ &= \frac{1-\rho}{\theta}\left(1 - \frac{\log\theta}{\log t_k/a_{t_k}}\right)a_{t_k}\log\frac{t_k}{a_{t_k}}\\ &\ge (1-2\rho)a_{t_k}\log\frac{t_k}{a_{t_k}} > (1-2\rho)a_t\log\frac{t}{a_t}.\end{aligned} \tag{1.8}$$

Hence (1.5) and (1.8) combined give Lemma 1.8. □

PROOF OF LEMMA 1. Theorem B and Lemma 1.8 together imply Lemma 1.

REMARK 3. It follows from the proof of Lemma 1.8 that

$$\lim_{t\to\infty} P\{A_t(\rho)\} = 0$$

provided $\lim_{t\to\infty} a_t/t = 0$. This implies

$$\limsup_{t\to\infty} \gamma_t a_t^{-1/2} \max_{0\le i\le \nu_t}(W(\xi_{i+1}) - W(\eta_i)) \ge 1 \quad \text{a.s.}$$

which in turn implies

$$\limsup_{t\to\infty} \gamma_t Z(t) \ge 1 \quad \text{a.s.}$$

provided $a_t/t \to 0$ as $t \to \infty$. However a stronger statement will be proved when proving Lemma 3.

2. Proof of Lemma 2.

LEMMA 2.1. *For any positive V, T, and u we have*

$$P\{L(T+V)-L(T)\geq u\sqrt{V}\}\leq 2\sqrt{\frac{V}{T+V}}\exp\left\{-\frac{u^2}{2}\right\}. \tag{2.1}$$

PROOF. Using the exact distribution of the local time due to P. Lévy, the estimation $1-\Phi(x)\leq \exp\{-(x^2/2)\}$ $(x\geq 0)$, and a simple conditioning argument, we get

$$\begin{aligned}
&P\{L(T+V)-L(T)>u\sqrt{V}\}\\
&\quad=\int_{-\infty}^{\infty} P\{L(T+V)-L(T)>u\sqrt{V}\mid W(T)=z\}\,dP\,(W(T)\leq z)\\
&\quad=\int_{-\infty}^{\infty} 2\left(1-\Phi\left(\frac{|z|}{\sqrt{V}}+u\right)\right)d\Phi\left(\frac{z}{\sqrt{T}}\right)\\
&\quad\leq 4\int_0^{\infty}\exp\left\{-\frac{1}{2}\left(\frac{z}{\sqrt{V}}+u\right)^2\right\}\frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{z^2}{2T}\right\}dz \qquad (2.2)\\
&\quad=\frac{4}{\sqrt{2\pi T}}\exp\left\{-\frac{u^2}{2}\right\}\int_0^{\infty}\exp\left\{-\frac{1}{2}\left[z^2\left(\frac{1}{V}+\frac{1}{T}\right)+\frac{2zu}{\sqrt{V}}\right]\right\}dz\\
&\quad\leq\frac{4}{\sqrt{T}}\exp\left\{-\frac{u^2}{2}\right\}\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\left\{-\frac{1}{2}z^2\left(\frac{1}{V}+\frac{1}{T}\right)\right\}dz\\
&\quad=2\sqrt{\frac{V}{T+V}}\exp\left\{-\frac{u^2}{2}\right\}. \qquad \square
\end{aligned}$$

LEMMA 2.2. *For any $\varepsilon>0$, there exists a constant $C=C(\varepsilon)>0$ such that the inequality*

$$P\{\sup_{0\leq T\leq t-V}(L(T+V)-L(T))>u\sqrt{V}\}<C\sqrt{\frac{t}{V}}\exp\left\{-\frac{u^2}{2+\varepsilon}\right\} \tag{2.3}$$

holds for every positive u, and $0\leq V\leq t$.

PROOF. Let $R=2^r$ and choose the integer $K=K(V,R)$ such that

$$\frac{K-1}{R}\leq V<\frac{K}{R} \tag{2.4}$$

should hold, where R will be specified later. For an aribtrary T in $[0,t]$ such that $j/R\leq T<(j+1)/R$, the local time increment $L(T+V)-L(T)$ can be estimated by $L(j/R+(K+1)/R)-L(j/R)$, and hence

$$\sup_{j/R\leq T<(j+1)/R}(L(T+V)-L(T))\leq L\left(\frac{j+K+1}{R}\right)-L\left(\frac{j}{R}\right).$$

Consequently by Lemma 2.1

$$P\{\sup_{0\le T\le t-V}(L(T+V)-L(T))>u\sqrt{V}\}$$

$$\le \sum_{j=0}^{[tR]+1} P\left\{L\left(\frac{j+K+1}{R}\right)-L\left(\frac{j}{R}\right)>u\sqrt{V}\right\}$$

$$(2.5)\qquad = \sum_{j=0}^{[tR]+1} P\left\{L\left(\frac{j+K+1}{R}\right)-L\left(\frac{j}{R}\right)>\frac{u\sqrt{V}\sqrt{R}}{\sqrt{K+1}}\sqrt{\frac{K+1}{R}}\right\}$$

$$\le 2\exp\left\{-\frac{u^2VR}{2(K+1)}\right\}\sum_{j=0}^{[tR]+1}\sqrt{\frac{K+1}{j+K+1}}.$$

Applying (2.4) we get

$$(2.6)\qquad \frac{u^2VR}{2(K+1)}>\frac{u^2(K-1)}{2(K+1)}.$$

Moreover by (2.4) we also have

$$2\sum_{j=0}^{[tR]+1}\sqrt{\frac{K+1}{j+K+1}}\le 2\sqrt{K+1}\sum_{j=1}^{[tR]+2}\sqrt{\frac{1}{j}}$$

$$(2.7)\qquad \le C_1\sqrt{K+1}\sqrt{\frac{tK}{V}+2}\le C_1(K+2)\sqrt{\frac{t}{V}}$$

with an absolute constant C_1.

Choose $K_0=K_0(\varepsilon)$ such that, for the given $\varepsilon>0$, and any $K\ge K_0$

$$(2.8)\qquad \frac{K-1}{2(K+1)}\ge\frac{1}{2+\varepsilon}$$

should hold. Then by (2.5)–(2.8) we have

$$(2.9)\qquad P\{\sup_{0\le T<t-V}L(T+V)-L(T)>u\sqrt{V}\}\le C_1(K+2)\sqrt{\frac{t}{V}}\exp\left\{-\frac{u^2}{2+\varepsilon}\right\}$$

for any $K>K_0$.

Choose R such that $K(V,R)$ (for the given V) should be larger than K_0 (defined by (2.8)) and

$$K_0\le VR<2K_0.$$

Then for K given by (2.4) we have also

$$(2.10)\qquad K_0<K\le 2K_0.$$

Hence from (2.9)

$$P\{\sup_{0\le T\le t-V}(L(T+V)-L(T))>u\sqrt{V}\}\le C(\varepsilon)\sqrt{\frac{t}{V}}\exp\left\{-\frac{u^2}{2+\varepsilon}\right\}$$

where $C(\varepsilon)=C_1(2K_0(\varepsilon)+2)$. □

Proof of Lemma 2. By Lemma 2.2 for any $\varepsilon>0$

$$P\{\gamma_t Y(t)>\sqrt{1+\varepsilon}\}$$

$$=P\{(\log ta_t^{-1}+2\operatorname{loglog} t)^{-1/2}a_t^{-1/2}\sup_{0\le s\le t-a_t}(L(s+a_t)-L(s))>\sqrt{1+\varepsilon}\}$$

$$(2.11)\qquad \le C\sqrt{\frac{t}{a_t}}\exp\left\{-\frac{(1+\varepsilon)(\log ta_t^{-1}+2\operatorname{loglog} t)}{2+\varepsilon}\right\}$$

$$\le C\sqrt{\frac{t}{a_t}}\left(\frac{t}{a_t\log^2 t}\right)^{-(1+\varepsilon)/(2+\varepsilon)}=C\left(\frac{a_t}{t}\right)^{\varepsilon/(2(2+\varepsilon))}\frac{1}{(\log t)^{2(1+\varepsilon)/(2+\varepsilon)}}.$$

Let $t_k = \theta^k$ ($\theta > 1$). Then by (2.11)

$$\sum_{k=1}^{\infty} P\{\gamma_{t_k} Y(t_k) > \sqrt{1+\varepsilon}\} < \infty$$

for every $\varepsilon > 0$, and $\theta > 1$. Consequently

$$\limsup_{k\to\infty} \gamma_{t_k} Y(t_k) \le 1. \tag{2.12}$$

Moreover it is easy to see that if k is big enough then

$$1 \le \frac{\gamma_{t_k} a_{t_k}^{-1/2}}{\gamma_{t_{k+1}} a_{t_{k+1}}^{-1/2}} \le \theta.$$

Hence, if we choose θ near enough to one, the statement of Lemma 2 follows from (2.12), since $a_t^{1/2} Y(t)$ is non-decreasing, and $\gamma_t a_t^{-1/2}$ is non-increasing in t. □

3. Proof of Lemma 3. This proof runs along the same lines as Step 2 in Csörgő and Révész (1981), Theorem 1.2.1. The main difficulty, however, lies in the lack of independence of local time increments over disjoint intervals.

Assume throughout that the conditions of Lemma 3 hold true, i.e. a_t and T/a_T are non-decreasing and moreover $\lim_{T\to\infty} a_T/T < 1$. In the case when $\lim_{T\to\infty} a_T/T = 1$, which holds if and only if $a_T = T$, we refer to the law of iterated logarithm due to Kesten (1965) (cf. (9) in this paper). Hence we assume that $a_T/T \le \rho < 1$ for large enough T with some constant ρ. Let $T_1 = 1$ and define T_{k+1} by

$$T_{k+1} - a_{T_{k+1}} = T_k, \quad k = 1, 2, \cdots. \tag{3.1}$$

(We note that our conditions on a_t imply that a_t is a continuous function of t and that $t - a_t$ is a strictly increasing function if $\rho < 1$). Let the events A_k be defined by

$$A_k = \{L(T_k) - L(T_k - a_{T_k}) \ge a_{T_k}^{1/2} u_k\}, \tag{3.2}$$

where

$$u_k = \left(\log \frac{T_k}{a_{T_k}} + 2(1-\varepsilon)\operatorname{loglog} T_k - 2\log\left(\log\frac{T_k}{a_{T_k}} + 2(1-\varepsilon)\operatorname{loglog} T_k\right)\right)^{1/2}. \tag{3.3}$$

We show that for any $0 < \varepsilon < 1$,

$$P\{A_k \text{ i.o.}\} = 1, \tag{3.4}$$

which implies Lemma 3. The events A_k however are not independent, therefore it is not sufficient to show only that $\sum P(A_k) = \infty$. We apply the following version of Borel-Cantelli lemma, due to Erdős and Rényi (cf. Rényi (1970), page 391).

If $\sum_k P\{A_k\} = \infty$ and

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{\ell=1}^{n} P\{A_k A_\ell\}}{(\sum_{k=1}^{n} P\{A_k\})^2} \le 1 \tag{3.5}$$

then $P\{A_k \text{ i.o.}\} = 1$.

LEMMA 3.1. *For any $V \ge 0$, $T > 0$, $u > 0$ we have*

$$\begin{aligned} \frac{2}{\pi u^2}\left(1-\frac{1}{u^2}\right)\left(1-\frac{2+V/T}{u^2}\right)\sqrt{\frac{V}{T}}\exp\left\{-\frac{u^2}{2}\right\} &\le P\{L(T+V)-L(T) > u\sqrt{V}\} \\ &\le \frac{2}{\pi u^2}\sqrt{\frac{V}{T}}\exp\left\{-\frac{u^2}{2}\right\}. \end{aligned} \tag{3.6}$$

PROOF. Using the estimation

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right) < 1-\Phi(x) \le \frac{1}{x\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right),$$

as in the proof of Lemma 2.1 we obtain

$$\begin{aligned}
&P\{L(T+V)-L(T)>u\sqrt{V}\}\\
&\qquad = 2\int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{|z|}{\sqrt{V}}+u\right)\right)d\Phi\left(\frac{z}{\sqrt{T}}\right)\\
&\qquad \le \frac{2}{\pi u}\int_0^{\infty}\frac{1}{\sqrt{T}}\exp\left\{-\frac{z^2}{2T}-\frac{(z+u\sqrt{V})^2}{2V}\right\}dz\\
&\qquad \le \frac{2}{\pi u}\int_0^{\infty}\frac{1}{\sqrt{T}}\exp\left\{-\frac{u}{\sqrt{V}}z-\frac{u^2}{2}\right\}dz = \frac{2}{\pi u^2}\sqrt{\frac{V}{T}}\exp\left\{-\frac{u^2}{2}\right\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&P\{L(T+V)-L(T)>u\sqrt{V}\}\\
&= 2\int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{|z|}{\sqrt{V}}+u\right)\right)d\Phi\left(\frac{z}{\sqrt{T}}\right)\\
&\ge \frac{2}{\pi}\int_0^{\infty}\frac{1}{\sqrt{T}}\exp\left\{-\frac{z^2}{2T}-\frac{(z+u\sqrt{V})^2}{2V}\right\}\left(\frac{1}{u+\frac{z}{\sqrt{V}}}-\frac{1}{\left(u+\frac{z}{\sqrt{V}}\right)^3}\right)dz\\
&\ge \frac{2}{\pi}\left(1-\frac{1}{u^2}\right)\\
&\quad\cdot\int_0^{\infty}\frac{1}{\sqrt{T}}\frac{1}{u+\frac{z}{\sqrt{V}}}\exp\left\{-\frac{1}{2}\left(z\sqrt{\frac{V+T}{VT}}+u\sqrt{\frac{T}{T+V}}\right)^2-\frac{u^2}{2}+\frac{u^2T}{2(T+V)}\right\}dz\\
&= \frac{2}{\pi}\left(1-\frac{1}{u^2}\right)\exp\left\{-\frac{u^2}{2}+\frac{u^2T}{2(T+V)}\right\}\sqrt{\frac{V}{T}}\\
&\quad\cdot\int_{u\sqrt{T/(T+V)}}^{\infty}\frac{1}{s+\frac{uV}{\sqrt{T(T+V)}}}\exp\left(-\frac{s^2}{2}\right)ds,
\end{aligned}$$

where in the last step we used the substitution

$$s = z\sqrt{\frac{V+T}{VT}}+u\sqrt{\frac{T}{T+V}}.$$

Integration by parts gives

$$\int_{u\sqrt{T/(T+V)}}^{\infty}\frac{1}{s\left(s+\frac{uV}{\sqrt{T(T+V)}}\right)}se^{-(s^2/2)}\,ds$$

$$= \frac{1}{u^2}\exp\left\{-\frac{u^2T}{2(T+V)}\right\} - \int_{u\sqrt{T/(T+V)}}^{\infty} \frac{2s + \dfrac{uV}{\sqrt{T(T+V)}}}{s^2\left(s + \dfrac{uV}{\sqrt{T(T+V)}}\right)^2}\exp\left(-\frac{s^2}{2}\right) ds$$

$$\geq \frac{1}{u^2}\exp\left\{-\frac{u^2T}{2(T+V)}\right\} - \frac{1}{u^4}\int_{u\sqrt{T/(T+V)}}^{\infty} s\left(2 + \frac{uV}{s\sqrt{T(T+V)}}\right)\exp\left(-\frac{s^2}{2}\right) ds$$

$$\geq \frac{1}{u^2}\exp\left\{-\frac{u^2T}{2(T+V)}\right\} - \frac{2 + \dfrac{V}{T}}{u^4}\exp\left\{-\frac{u^2T}{2(T+V)}\right\},$$

from which (3.6) follows. □

Lemma 3.2. *Let A_k be the events defined by* (3.2). *Then*

$$\sum_k P\{A_k\} = \infty. \tag{3.7}$$

Proof. Putting $V = a_{T_k}$, $T = T_k - a_{T_k}$ and $u = u_k$ (given by (3.3)) in (3.6), for large enough k, we obtain

$$P\{A_k\} \geq C\frac{a_{T_k}}{T_k}\frac{1}{(\log T_k)^{1-\varepsilon}} \tag{3.8}$$

with some constant C. But

$$\sum_{k=2}^{n}\frac{a_{T_k}}{T_k}\frac{1}{(\log T_k)^{1-\varepsilon}} \geq \frac{1}{(\log T_n)^{1-\varepsilon}}\sum_{k=2}^{n}\frac{a_{T_k}}{T_k}, \tag{3.9}$$

and because $-\log(1-x) \leq K_\rho x$ for $0 \leq x \leq \rho$, and some $K_\rho > 0$,

$$\log T_n = \sum_{k=2}^{n}\log\frac{T_k}{T_{k-1}} = -\sum_{k=2}^{n}\log\left(1 - \frac{a_{T_k}}{T_k}\right) \leq K_\rho\sum_{k=2}^{n}\frac{a_{T_k}}{T_k}, \tag{3.10}$$

and this combined with (3.9) yields (3.7). □

Lemma 3.3. *Let* $0 < T_1$, $0 < V_1$, $T_1 + V_1 < T_2$, $0 < V_2$, $0 < u_1$, $0 < u_2$. *Then*

$$\begin{aligned} &P\{L(T_1+V_1) - L(T_1) > u_1\sqrt{V_1},\ L(T_2+V_2) - L(T_2) > u_2\sqrt{V_2}\} \\ &\qquad \leq \frac{4}{\pi^2u_1^2u_2^2}\sqrt{\frac{V_1V_2}{T_1(T_2-T_1-V_1)}}\exp\left\{-\frac{u_1^2+u_2^2}{2}\right\}. \end{aligned} \tag{3.11}$$

Proof. The following conditional distribution can be obtained from Lévy (1939):

$$P\{L(T+V) - L(T) > u\sqrt{V} \mid W(T) = x_1,\ W(T+V) = x_2\} = \frac{\phi\left(\dfrac{|x_1| + |x_2|}{\sqrt{V}} + u\right)}{\phi\left(\dfrac{x_2 - x_1}{\sqrt{V}}\right)}.$$

From this it is easy to obtain

$$P\{L(T_1+V_1) - L(T_1) > u_1\sqrt{V_1},\ L(T_2+V_2) - L(T_2) > u_2\sqrt{V_2}\}$$

$$= \iiiint_{-\infty}^{\infty} \frac{1}{\sqrt{T_1}}\phi\left(\frac{x_1}{\sqrt{T_1}}\right)\frac{1}{\sqrt{V_1}}\phi\left(\frac{|x_1| + |x_2|}{\sqrt{V_1}} + u_1\right)$$

$$\times \frac{1}{\sqrt{T_2 - T_1 - V_1}} \phi\left(\frac{x_3 - x_2}{\sqrt{T_2 - T_1 - V_1}}\right) \frac{1}{\sqrt{V_2}} \phi\left(\frac{|x_3| + |x_4|}{\sqrt{V_2}} + u_2\right) dx_1\, dx_2\, dx_3\, dx_4$$

$$\leq \frac{4}{\pi^2} \frac{1}{\sqrt{T_1 V_1 (T_2 - T_1 - V_1) V_2}} \exp\left\{-\frac{u_1^2 + u_2^2}{2}\right\}$$

$$\times \int_0^\infty\!\!\int\!\!\int\!\!\int \exp\left\{-\frac{u_1}{\sqrt{V_1}}(x_1 + x_2) - \frac{u_2}{\sqrt{V_2}}(x_3 + x_4)\right\} dx_1\, dx_2\, dx_3\, dx_4,$$

which, by integrating out, gives (3.11). □

In order to verify (3.5) we prove

LEMMA 3.4. *Let the events A_k be defined by* (3.2). *For any given $\delta > 0$ and positive integer k, define the integer $\ell_0 = \ell_0(k, \delta)$ by*

$$\ell_0 = \max\left(\ell : \frac{T_\ell}{T_k} < \frac{(1+\delta)^2}{(1+\delta)^2 - 1}\right). \tag{3.12}$$

Then for large enough k and $\ell > \ell_0 + 1$ we have

$$P\{A_k A_\ell\} \leq (1+\delta)^2 P\{A_k\} P\{A_\ell\}. \tag{3.13}$$

Moreover

$$\textstyle\sum_{\ell=k}^{\ell_0+1} P\{A_k A_\ell\} \leq C P\{A_k\}, \tag{3.14}$$

where $C = C(\delta)$ is a constant depending only on δ.

PROOF. Comparing (3.6) and (3.11) it is easily seen that for large enough k,

$$P\{A_k A_{\ell+1}\} \leq (1+\delta) \sqrt{\frac{T_\ell}{T_\ell - T_k}} P\{A_k\} P\{A_{\ell+1}\} \quad (k < \ell). \tag{3.15}$$

Now (3.13) follows by the definition of ℓ_0.

We will use the following inequalities:

$$\frac{\log \dfrac{T_\ell}{T_k}}{T_\ell - T_k} \leq \frac{1}{T_k} \tag{3.16}$$

and

$$\frac{\log \dfrac{T_\ell}{T_k}}{\ell - k} \geq \frac{\log \dfrac{T_{\ell_0+1}}{T_k}}{\ell_0 - k + 1}. \tag{3.17}$$

(3.16) follows from the concavity of log function, while (3.17) follows from the fact that

$$\frac{\log \dfrac{T_\ell}{T_k}}{\ell - k} = \frac{\sum_{i=k+1}^{\ell} \log\left(1 - \dfrac{a_{T_i}}{T_i}\right)^{-1}}{\ell - k} \tag{3.18}$$

is an arithmetic mean of nonincreasing terms and hence is nonincreasing itself. Therefore for $k < \ell \leq \ell_0$

$$\frac{T_\ell}{T_\ell - T_k} = \frac{T_\ell}{T_k}\frac{T_k}{T_\ell - T_k} \le \frac{T_\ell}{T_k}\frac{1}{\log \frac{T_\ell}{T_k}} \le \frac{T_{\ell_0}}{T_k}\frac{\ell_0 - k + 1}{\log \frac{T_{\ell_0+1}}{T_k}}\frac{1}{\ell - k}$$

(3.19)

$$\le \frac{(1+\delta)^2}{(2\delta + \delta^2)\log \frac{(1+\delta)^2}{2\delta+\delta^2}}\frac{\ell_0 + 1 - k}{\ell - k} = C_1 \frac{\ell_0 + 1 - k}{\ell - k}.$$

Now

$$\sum_{\ell=k+1}^{\ell_0} \sqrt{\frac{T_\ell}{T_\ell - T_k}} \le \sqrt{C_1}\,\sqrt{\ell_0 + 1 - k}\,\sum_{\ell=k+1}^{\ell_0}\frac{1}{\sqrt{\ell - k}} \le 2\sqrt{C_1}(\ell_0 + 1 - k). \tag{3.20}$$

Since a_T/T is nonincreasing, we get

$$\log\frac{(1+\delta)^2}{2\delta+\delta^2} \ge \log\frac{T_{\ell_0}}{T_k} = \sum_{i=k+1}^{\ell_0}\log\frac{T_i}{T_{i-1}} = \sum_{i=k+1}^{\ell_0}\log\left(1 - \frac{a_{T_i}}{T_i}\right)^{-1}$$

(3.21)

$$\ge \sum_{i=k+1}^{\ell_0}\frac{a_{T_i}}{T_i} \ge (\ell_0 - k)\frac{a_{T_{\ell_0}}}{T_{\ell_0}} \ge (\ell_0 - k)\frac{T_k}{T_{\ell_0}}\frac{a_{T_k}}{T_k}$$

$$\ge (\ell_0 - k)\frac{2\delta+\delta^2}{(1+\delta)^2}\frac{a_{T_k}}{T_k},$$

hence

$$(\ell_0 + 1 - k) \le C_2 \frac{T_k}{a_{T_k}}, \tag{3.22}$$

or

$$\sum_{\ell=k+1}^{\ell_0}\sqrt{\frac{T_\ell}{T_\ell - T_k}} \le 2C_2\sqrt{C_1}\,\frac{T_k}{a_{T_k}}. \tag{3.23}$$

From (3.6) for $k \le \ell$ one obtains

$$P\{A_\ell\} \le C'\frac{a_{T_k}}{T_k(\log T_k)^{1-\varepsilon}}, \tag{3.24}$$

therefore from (3.15), (3.23) and (3.24) it follows that

$$\sum_{\ell=k}^{\ell_0+1} P\{A_k A_\ell\} \le 2P\{A_k\} + \sum_{\ell=k+1}^{\ell_0} P\{A_k A_{\ell+1}\}$$

$$\le 2P\{A_k\} + \sum_{\ell=k+1}^{\ell_0}(1+\delta)\sqrt{\frac{T_\ell}{T_\ell - T_k}}\,P\{A_k\}P\{A_{\ell+1}\}$$

$$\le 2P\{A_k\} + 2(1+\delta)C_2C'\sqrt{C_1}\,P\{A_k\}(\log T_k)^{-1+\varepsilon} \le CP\{A_k\},$$

i.e., we have (3.14). □

Now by Lemmas 3.2 and 3.4 it is easy to verify (3.5), which completes the proof of Lemma 3. □

4. Proof of Theorem 3. This proof is again very similar to that of Theorem 1.2.1 in Csörgő and Révész (1981). The main point is that, contrary to the nonindependence of the increments $L(T_i + V_i) - L(T_i)$ for $T_i + V_i \le T_{i+1}$, $i = 0, 1, 2, \cdots$, the increments $\sup_x(L(x, T_i + V_i) - L(x, T_i))$ for $T_i + V_i \le T_{i+1}$, $i = 0, 1, 2, \cdots$, are clearly independent. Instead of repeating the whole proof of the above mentioned theorem, we will emphasize only two important points. The first is to give an upper bound for $P\{\beta_t U(t) > \sqrt{1+\varepsilon}\}$. In order to solve this problem we give an analogue of Lemma 2.2. Let

$$U^V(T) = \sup_{-\infty<x<+\infty}(L(x, T+V) - L(x, T)).$$

LEMMA 4.1. *For any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that the inequality*

$$P\{\sup_{0\le T\le t-V} U^V(T) \ge u\sqrt{V}\} \le \frac{Ct}{V}\exp\left\{-\frac{u^2}{2+\varepsilon}\right\} \tag{4.1}$$

holds for every positive u and $0 \le T \le t$.

PROOF. Kesten (1965) proved that for any fixed $\delta > 0$ if $x \to \infty$

$$P\left\{\frac{U^V(T)}{\sqrt{V}} \ge x\right\} = o\left(\exp\left\{-\frac{x^2}{2}(1-\delta)\right\}\right).$$

Consequently, for any fixed $\sigma > 0$, there exists a constant $C = C(\delta)$ such that for every $x > 0$

$$P\left\{\frac{U^V(T)}{\sqrt{V}} \ge x\right\} \le C_\delta \exp\left\{-\frac{x^2}{2}(1-\delta)\right\}. \tag{4.2}$$

Let R and K be defined as in Lemma 2.2. By the same argument, from (4.2) we obtain

$$\begin{aligned} P\{\sup_{0\le T\le t-V} U^V(T) \ge u\sqrt{V}\} &\le \sum_{j=0}^{[tR]+1} P\left\{U^{(K+1)/R}\left(\frac{j}{R}\right) > \frac{u\sqrt{VR}}{\sqrt{K+1}}\sqrt{\frac{K+1}{R}}\right\} \\ &\le C_\delta \exp\left\{-\frac{u^2 VR}{2(K+1)}(1-\delta)\right\}([tR]+2). \end{aligned}$$

Now by (2.4)

$$\frac{VR}{K+1} \ge \frac{K-1}{K+1}.$$

Choose a $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{1-\delta}{2} > \frac{1}{2+\varepsilon}$$

should hold. Now choose $K_0 = K_0(\varepsilon)$ such that for $K \ge K_0$

$$\frac{K-1}{2(K+1)}(1-\delta) > \frac{1}{2+\varepsilon}.$$

Then, again by (2.4),

$$\begin{aligned} P\{\sup_{0\le T\le t-V} U^V(T) > u\sqrt{V}\} &\le C_{\delta(\varepsilon)}\exp\left\{-\frac{u^2}{2+\varepsilon}\right\}(tR+2) \\ &= C_{\delta(\varepsilon)}\exp\left\{-\frac{u^2}{2+\varepsilon}\right\}\frac{t}{V}\left(RV+\frac{2V}{t}\right) \\ &\le C_{\delta(\varepsilon)}\exp\left\{-\frac{u}{2+\varepsilon}\right\}\frac{t}{V}(2K_0+2), \end{aligned}$$

since $K < 2K_0$ by (2.10). □

The second point is the observation that for the probability

$$P\{\beta_T a_T^{-1/2}\sup_{-\infty<x<+\infty}(L(x, T) - L(x, T-a_T)) > 1-\varepsilon\}$$

we may give the same lower bound as for the probability containing the corresponding increment of the Wiener process. Namely we have

$$P\{\beta_T a_T^{-1/2}\sup_x(L(x, T) - L(x, T - a_T)) \geq 1 - \varepsilon\}$$
$$= P\{\beta_T a_T^{-1/2}\sup_x L(x, a_T) \geq 1 - \varepsilon\}$$
$$\geq P\{\beta_T a_T^{-1/2} L(0, a_T) \geq 1 - \varepsilon\} = 2\left(1 - \Phi\left(\frac{1-\varepsilon}{\beta_T}\right)\right)$$
$$\geq \frac{\exp\left\{-(1-\varepsilon)^2\left[\log\dfrac{T}{a_T} + \text{loglog}\, T\right]\right\}}{\sqrt{2\pi}\left(2\left(\log\dfrac{T}{a_T} + \text{loglog}\, T\right)\right)^{1/2}} \geq \left(\frac{a_T}{T\log T}\right)^{1-\varepsilon},$$

if T is large enough.

These two observations and Theorem 1.2.1 in Csörgő and Révész (1981) render the proof of Theorem 3 obvious.

Acknowledgments. The authors wish to thank the referee for the careful reading of their manuscript and for calling their attention to the papers of Hawkes (1971) and Perkins (1981).

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