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A UNIFORM LOWER BOUND FOR HAUSDORFF DIMENSION FOR TRANSIENT SYMMETRIC LÉVY PROCESSES

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For transient symmetric Lévy processes we determine a uniform lower bound for the Hausdorff dimension of the range of a process on various time sets. This complements earlier work which provided a uniform upper bound. An example is provided in which both bounds are attained.

1. Introduction. The object of this paper is to obtain a uniform lower bound for Hausdorff dimension for transient symmetric Lévy processes in R^d. This problem was posed by Hawkes and Pruitt [4], where a uniform upper bound result was established. When combined with Hawkes' [3] uniform lower bound for stable processes, this showed that for strictly stable processes $X(t)$ of stable index $\alpha \le d$

$$P[\dim X(E, \omega) = \alpha \dim E \text{ for all time sets } E] = 1, \tag{1}$$

where $X(E, \omega) = \{x \in R^d : X(t, \omega) = x \text{ for some } t \in E\}$ and $\dim A$ = Hausdorff dimension of the set A.

In his survey, Pruitt [10] establishes some uniform covering principles which when applied to a given Lévy process suffice to give uniform upper and lower bounds upon $\dim X(E)$. His methods give a quite direct proof of (1). He also notes that the missing ingredient for a uniform bound result for general Lévy processes is an estimate upon delayed hitting probabilities of small spheres. We obtain this estimate for transient symmetric Lévy processes and thereby establish a uniform lower bound which agrees (at least for some processes) with one suggested by Pruitt. In the course of the argument we define a new index γ' and show its relation to previously defined indices.

Section 2 provides various definitions. Section 3 gives the delayed hitting probability estimate and Section 4 the uniform dimension result. We conclude in Section 5 by relating γ' to existing indices.

2. Preliminaries. Let $X(t)$, $t \ge 0$, be a transient symmetric R^d-valued Lévy process having characteristic function $\exp(-t\psi(z))$. Note that by symmetric we do not assume $X(t)$ to be radially symmetric; we simply mean that $X(t)$ and $-X(t)$ have the same distribution, so that $\psi(z)$ is real and in fact non-negative. Many of the sample path properties of $X(t)$ can be expressed in terms of various indices, which we now recount. Blumenthal and Getoor [1] defined lower and upper indices, β'' and β which satisfy $0 \le \beta'' \le \beta \le 2$ by: $\beta = \inf\{\theta \ge 0 : \mathrm{Re}\psi(z)/|z|^\theta \to 0 \text{ as } |z| \to \infty\}$; $\beta'' = \sup\{\theta \ge 0 : \mathrm{Re}\psi(z)/|z|^\theta \to \infty \text{ as } |z| \to \infty\}$. In 1969, Pruitt [9] introduced the index γ defined by

$$\gamma = \sup\{\theta \ge 0 : \limsup_{r \to 0} E[T(S_r, 1)]/r^\theta < \infty\} \tag{2}$$

where $T(S_r, 1)$ denotes the sojourn time up to time 1 in a sphere S_r of radius r centered at the origin. He showed that with probability one, $\dim X[0, 1] = \gamma$ and that $\beta'' \le \gamma \le \beta$. A planar stable components process (X_1, X_2) with X_i linear, independent and symmetric stable of index α_i, $i = 1, 2$ and $1 < \alpha_2 < \alpha_1 \le 2$ satisfies $\beta'' = \alpha_2 < \gamma = 1 + \alpha_2 - \alpha_2/\alpha_1 < \alpha_1 = \beta$. Henceforth, we refer to this process as (X_1, X_2). We shall define a new index γ' by:

$$\gamma' = \sup\{\theta \ge 0 : \liminf_{r \to 0} E[T(S_r, 1)]/r^\theta = 0\} \tag{3}$$

and show that $\gamma \le \gamma' \le \beta$, with $\gamma' = \gamma$ for (X_1, X_2).

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The uniform upper bound result of Hawkes and Pruitt states that:

(4) $$P[\dim X(E, \omega) \leq \beta \dim E \text{ for all } E] = 1.$$

These authors also show that uniform lower bound results can fail without some assumptions on the parameters. They raise the question as to whether β'' dim E is a uniform lower bound for dim $X(E)$ when $\beta \leq d$. For transient symmetric processes X with $\beta \leq d$ we show that

(5) $$P[\dim X(E, \omega) \geq \beta''(d - \gamma')(d - \gamma)^{-1}\dim E \text{ for all } E] = 1$$

and shall exhibit fixed time sets E for which the upper bound in (4) is attained for (X_1, X_2) and random time sets for which (X_1, X_2) attains the lower bound in (5). Moreover, $\gamma = \gamma'$ in this example. In the discussion we use constants c_1, c_2, c_3, c_4 which are finite and positive and which remain fixed.

3. Delayed hitting probability estimate. To obtain the desired estimate we first need a result (Lemma 2) from Hendricks [7]:

LEMMA 1. *Let $X(t)$ be a transient symmetric Lévy process in R^d having lower index $\beta'' > 0$, thus guaranteeing the hypotheses in* [7]. *Let μ_L denote Lebesgue measure and* Cap (A) *the capacity of the set A. Then there exists c_1, independent of r, for which*

$$\text{Cap}(S_r) \leq c_1\mu_L(S_r)/E[T(S_r, 1)].$$

(Here we have used the fact that X_t has a density $p(t, x)$ for which, as pointed out in [7], $E[T(S_r, 1)] = \int_{S_r}\int_0^1 p(t, x)\, dt\, dx$.)

We then argue along the same lines as in Lemma 1 of Hendricks [7] to obtain:

LEMMA 2. *Let $X(t)$ be a transient, symmetric Lévy process in R^d having lower index $\beta'' > 0$ and upper index $\beta \leq d$. Assume that $r > 0$, $1 \geq T > 0$, $0 < \alpha < \beta''$, and $0 < \theta < \gamma$. Then there is a constant c_2, independent of T, r, and x for which*

(6) $$P^x[X(t) \in S_r \text{ for some } t \geq T] \leq c_2 \text{Cap}(S_r) T^{(\theta - d)/\alpha}.$$

PROOF. Since $\alpha < \beta''$ we can choose $M > 0$ so that $\psi(z) \geq |z|^\alpha$ if $|z| \geq M$. In addition, the hypothesis of transience guarantees, by page 397 of Port and Stone [8], that $\int_{|z|<M} 1/\psi(z)\, dz < \infty$. Finally, we need Pruitt's [9] characterization of

$$\gamma : \gamma = \sup\left\{\theta < d : \int |z|^{\theta - d}\text{Re}\frac{1 - e^{-\psi(z)}}{\psi(z)}\, dz < \infty\right\}$$

if $\text{Re}\psi(z) \geq 2 \log|z|$ for large $|z|$ to conclude that

(7) $$\int_{|z|>M} |z|^{\theta - d}/\psi(z)\, dz < \infty \quad \text{for} \quad \theta < \gamma.$$

Let v be a capacitory measure on S_r and write, as in Theorem 1 of [7]:

$$P^x[X(t) \in S_r \text{ for some } t \geq T] = \int_{S_r}\int_T^\infty p(t, y - x)\, dt\, v(dy).$$

Estimate the inner integral by using the inversion theorem:

$$\int_T^\infty p(t, y - x)\, dt = (2\pi)^{-d}\int_T^\infty\int_{R^d} e^{-i\langle z, y - x\rangle - t\psi(z)}\, dz\, dt$$
$$\leq \int_T^\infty\int_{R^d} e^{-t\psi(z)}\, dz\, dt = \int_{R^d} 1/(\psi(z)e^{T\psi(z)})\, dz$$

$$\leq \int_{|z|\leq M} 1/\psi(z)\, dz + \int_{|z|>M} 1/(\psi(z)e^{T|z|^\alpha})\, dz.$$

Port and Stone's result assures the convergence of the first of these integrals. Use (7) and the fact that for $u > 0$ we have $1/e^u \leq c_3(1/u)^{(d-\theta)/\alpha}$ to get the desired bound upon the second integral.

We now combine Lemmas 1 and 2, along with our definition of γ' in (3) to obtain

(8) $$P^x[X(t) \in S_r \text{ for some } t \geq T] \leq c_4 r^{(d-\theta')}/T^{(d-\theta)/\alpha} \text{ for all small } r,$$

where $0 < \alpha < \beta''$, $0 < \theta < \gamma$, $\gamma' < \theta'$, $0 < T \leq 1$, and c_4 is independent of r, T and x. This is the key estimate we use in the next section.

4. Uniform dimension theorem. To obtain our uniform lower bound we use a covering lemma of Pruitt [10], referred to by him as covering Principle II, but which we list here as Lemma 3.

LEMMA 3. (Pruitt). *Let $\{\theta_n\}$ be a sequence of positive real numbers with $\Sigma_n \theta_n^p < \infty$ for some $p > 0$, and let C_n be a class consisting of N_n sets in R^d of diameter θ_n where $\log N_n = O(1)|\log \theta_n|$. If $\{t_n\}$ is a sequence of positive real numbers such that for some $\delta > 0$ we have*

$$P\{\inf_{t_n \leq s < \infty} |X_s| \leq \theta_n\} = O(1)\theta_n^\delta,$$

then there exists a positive integer k such that, with probability one, for sufficiently large n, $\{t : X_t \in C\}$ can be covered by k intervals of length t_n whenever C is in C_n.

We now state and prove our principle theorem.

THEOREM. *Let $X(t)$ be a transient symmetric Lévy process in R^d for which $\beta \leq d$. Then*

$$P[\dim X(E) \geq \beta''(d-\gamma')(d-\gamma)^{-1} \dim E \text{ for all time sets } E] = 1.$$

PROOF. The theorem is trivially true for $\beta'' = 0$, so assume $\beta'' > 0$ and choose arbitrary α, θ and θ' for which $0 < \alpha < \beta''$, $0 < \theta < \gamma$, and $\gamma' < \theta'$. The proof now parallels that of Pruitt's [10] Theorem 1, using his covering Principle II. Let $\theta_n = \sqrt{d}\, 2^{-n}$ and $t_n = 2^{-n\lambda}$ where $0 < \lambda < \alpha(d-\theta')/(d-\theta)$. According to (8) we then have

$$P[X(t) \in \text{cube of side } 2^{-n} \text{ for some } t \geq t_n] \leq c_4(2^{-n})^{d-\theta'}(2^{n\lambda})^{(d-\theta)/\alpha} = 2^{-n\delta}$$

where $\delta = d - \theta' - (d-\theta)\lambda/\alpha > 0$. With these conventions, the hypotheses of Lemma 3 are satisfied. Pruitt's argument then leads to $\dim X(E) \geq \lambda \dim E$. Since α, θ and θ' are arbitrary the proof is complete.

We conclude this section with several remarks:

REMARK (1). For processes for which $\gamma' = \gamma$, the uniform lower bound for $\dim X(E)$ is $\beta'' \dim E$. This will occur, for example, if the ratio

$$E[T(S_r, 1)]/r^\gamma$$

is bounded above and below by finite positive numbers as $r \to 0$. For (X_1, X_2), Lemma 5.1 of Pruitt and Taylor [11] guarantees such behavior for the ratio, so that $\gamma = \gamma'$ is indeed possible.

REMARK (2). For (X_1, X_2) our results combine with those of Pruitt and Hawkes to give:

(9) $$P[\beta'' \dim E \leq \dim (X_1, X_2)(E) \leq \beta \dim E \text{ for all } E] = 1.$$

We showed in [5] that dim $(X_1, X_2)(E) = \beta$ dim E for fixed time sets E such that $0 \leq \dim E \leq 1/\alpha_1$. To obtain the lower bound in (9), let $E(\omega) = \{t: X_1(t, \omega)\} = 0$. Since X_1 is recurrent, E is nonempty and in fact $P[\omega: \dim E(\omega) = 1 - 1/\alpha_1] = 1$ by virtue of Theorem A of Blumenthal and Getoor [2]. The uniform upper bound of Hawkes and Pruitt, when applied to X_2 gives:

$$\dim (X_1, X_2)(E(\omega)) = \dim X_2(E(\omega)) \leq \alpha_2 \dim E(\omega).$$

On the other hand, our theorem gives:

$$\dim(X_1, X_2)(E(\omega)) \geq \alpha_2 \dim E(\omega).$$

5. Relation of γ' to other indices.

THEOREM. *With γ defined by* (2) *and γ' by* (3) *we have $\gamma \leq \gamma' \leq \beta$.*

PROOF. The proof of $\gamma \leq \gamma'$ follows immediately from the definitions and is omitted. To prove $\gamma' \leq \beta$, let $\alpha > \beta$ and use the fact that by virtue of Theorem 3.1 of [1] we have $|X(t)| \leq t^{1/\alpha}$ for all small t with probability one. This means that

$$P[T(S_r, 1)/r^\alpha \geq 1 \text{ for all small } r] = 1. \tag{10}$$

Therefore, $P[\liminf_{r\to 0} T(S_r, 1)/r^\alpha \geq 1] = 1$ and

$$1 \leq E[\liminf_{r\to 0} T(S_r, 1)/r^\alpha] \leq \liminf_{r\to 0} \frac{E[T(S_r, 1)]}{r^\alpha}.$$

This shows that $\alpha \geq \gamma'$ and the proof is complete.

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