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STRONG LAWS FOR INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES INDEXED BY A SECTOR

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For simplicity, let $d = 2$ and consider the points (n, m) in Z_+^2, with $\theta m \le n \le \theta^{-1}m$, where $0 < \theta < 1$. For i.i.d. random variables with this set as an index set we present a law of the iterated logarithm, strong laws of large numbers and related results. We also observe that (and try to explain why) the martingale proof of the Kolmogorov strong law of large numbers yields a weaker result for this index set than the classical proofs, whereas this is not the case if the index set is all of Z_+^d, $d \ge 1$.

1. Introduction. Let Z_+^d, $d \ge 1$, be the positive integer d-dimensional lattice points with coordinate-wise partial ordering, $<$. Points in Z_+^d are denoted by $\mathbf{m}$, $\mathbf{n}$ etc. Further, $|\mathbf{n}|$ is used for $\prod_{i=1}^d n_i$, $\mathbf{n} \to \infty$ means that $n_i \to \infty$ for all i (cf. Gut, 1978, 1980), the limit superior of $\{a_\mathbf{n}; \mathbf{n} \in Z_+^d\}$, $\limsup_\mathbf{n} a_\mathbf{n}$, is to be interpreted as $\inf_\mathbf{n} \sup_{\mathbf{n}<\mathbf{m}} a_\mathbf{m}$ and similarly for the limit inferior (cf. Gabriel, 1975, 1977).

Let $\{X_\mathbf{n}; \mathbf{n} \in Z_+^d\}$ be independent, identically distributed (i.i.d.) random variables and set $S_\mathbf{n} = \sum_{\mathbf{k}<\mathbf{n}} X_\mathbf{k}$. The law of the iterated logarithm asserts that

$$\limsup_\mathbf{n} (\liminf_\mathbf{n}) \frac{S_\mathbf{n}}{\sqrt{2\sigma^2 |\mathbf{n}| \log\log|\mathbf{n}|}} = \sqrt{d}\ (-\sqrt{d}) \text{ a.s.} \tag{1.1}$$

provided $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$ for $d \ge 1$ and further that

$$EX_1^2 \cdot \frac{(\log^+|X_1|)^{d-1}}{\log^+\log^+|X_1|} < \infty \text{ for } d \ge 2. \tag{1.2}$$

The result in this form is due to Hartman and Wintner (1941) for $d = 1$ and Wichura (1973) for $d \ge 2$. For the converse, which states that, if

$$P\left(\limsup_\mathbf{n} \frac{|S_\mathbf{n}|}{\sqrt{|\mathbf{n}| \log\log|\mathbf{n}|}} < \infty\right) > 0, \tag{1.3}$$

then $EX_1^2 < \infty$ and $EX_1 = 0$ for $d = 1$, see Strassen (1966) and (1.2) holds and $EX_1 = 0$ for $d \ge 2$, see Wichura (1973).

As is pointed out in Wichura (1973), it is interesting to note

a) the discontinuity in the moment requirements for $d = 1$ and $d \ge 2$ and
b) that the converse is immediate when $d \ge 2$ since

$$P(|X_\mathbf{n}| > \varepsilon\sqrt{|\mathbf{n}| \log\log|\mathbf{n}|} \text{ i.o.}) = 0 \text{ for some } \varepsilon > 0$$

is equivalent to (1.2) (which for $d = 1$ does not imply that the variance exists).

Furthermore, Theorem 2 of Wichura (1973) implies that if the summation index tends to infinity along a ray, then the conditions for the law of the iterated logarithm are the same as in the one-dimensional case. (To be precise, the theorems in Wichura (1973) also cover more general cases and they are stated as functional limit theorems.)

The main purpose of this paper is to prove the law of the iterated logarithm in the i.i.d. case when the index set is a sector. This is done in Section 3. For $d = 2$ this index set consists of all points with positive integer coordinates "between" the lines $y = \theta x$ and $y =$

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$\theta^{-1}\cdot x$ for some θ, $0 < \theta < 1$, (a formal definition is given in the next section) and it turns out that the result is the same as for the index set Z_+^1. The proofs are based on estimates of tail probabilities for partial sums. In Section 4 we point out that these techniques can be used to prove a law of the iterated logarithm when the summation index tends to infinity along a ray and we show how the upper class result in (1.1) can be obtained by the same method when $d \geq 2$. Section 5 is devoted to sectorial strong laws of large numbers and Section 6 to convergence rate results. Section 7, finally, contains a remark on why the proof of the Kolmogorov law of large numbers using martingale theory gives a weaker result compared to the classical proofs when the index set is the sector and why this is not the case when the index set is Z_+^d, $d \geq 1$.

We close this section by mentioning that it follows from the computations below that the results for the sector remain valid for any index set consisting of all points with positive integer coordinates in a closed convex cone contained in $R_+^d \cup \{0\}$.

2. Preliminaries. For $d \geq 2$ we define the sector (Gabriel, 1977, or wedge Smythe, 1974) T_θ^d(of Z_+^d) as

$$T_\theta^d = \{\mathbf{n} = (n_1, \cdots, n_d) \in Z_+^d;\ \theta n_i \leq n_j \leq \theta^{-1} n_i \quad \text{for} \quad i \neq j,\ i, j = 1, 2, \cdots, d\},$$

where $0 < \theta < 1$ (cf. Gabriel, 1977, page 891).

The number of points in this index set is defined by

$$M_\theta(\mathbf{n}) = \text{Card}\{\mathbf{k} \in T_\theta^d;\ \mathbf{k} < \mathbf{n}\}. \tag{2.1}$$

Note also that $|\mathbf{n}| = \text{Card}\{\mathbf{k} \in Z_+^d;\ \mathbf{k} < \mathbf{n}\}$.

In the sequel we shall use $\pi(j)$, $j \geq 1$ to denote the points $(j, 1, 1, \cdots, 1)$.

We also need results like Lemma 2.1 of Smythe (1974) (cf. also Lemma 2.1 of Gut, 1978, and Lemma 2.1 of Gut, 1980).

LEMMA 2.1. *For any random variable, X, the following are true.*

$$E|X|^r < \infty \Leftrightarrow \textstyle\sum_{\mathbf{n}\in T_\theta^d} |\mathbf{n}|^{\alpha r-1}\cdot P(|X| > |\mathbf{n}|^\alpha) < \infty \text{ for all } \alpha, r > 0. \tag{2.2}$$

$$EX^2\cdot\frac{(\log^+|X|)^m}{\log^+\log^+|X|} < \infty \Leftrightarrow \textstyle\sum_{\mathbf{n}\in T_\theta^d} (\log|\mathbf{n}|)^m\cdot P(|X| > \sqrt{|\mathbf{n}|\log^+\log^+|\mathbf{n}|}) < \infty, \quad m = 0, 1, 2, \cdots. \tag{2.3}$$

Another tool will be the Lévy-inequalities and an extension of them, the proofs of which are immediate. For Z_+^d, see Paranjape and Park (1973), Theorem 1, Gabriel (1975) and Gut (1980), Lemma 2.3.

LEMMA 2.2. a) *Let* $\{X_\mathbf{k};\ \mathbf{k} \in T_\theta^d\}$ *be independent symmetric random variables and set* $S_\theta(\mathbf{n}) = \sum_{\mathbf{k}\in T_\theta^d, \mathbf{k}<\mathbf{n}} X_\mathbf{k}$. *Then*

$$P(\max_{\mathbf{k}\in T_\theta^d, \mathbf{k}<\mathbf{n}} S_\theta(\mathbf{k}) > \lambda) \leq 2^d\cdot P(S_\theta(\mathbf{n}) > \lambda) \tag{2.4}$$

$$P(\max_{\mathbf{k}\in T_\theta^d, \mathbf{k}<\mathbf{n}} |S_\theta(\mathbf{k})| > \lambda) \leq 2^d\cdot P(|S_\theta(\mathbf{n})| > \lambda). \tag{2.5}$$

b) *Let* $\{X_\mathbf{k};\ \mathbf{k} \in T_\theta^d\}$ *be* i.i.d. *random variables with mean* 0 *and variance* $\sigma^2 < \infty$. *Then*

$$P(\max_{\mathbf{k}\in T_\theta^d, \mathbf{k}<\mathbf{n}} S_\theta(\mathbf{k}) > \lambda) \leq 2^d\cdot P(S_\theta(\mathbf{n}) > \lambda - d\sigma\sqrt{2|\mathbf{n}|}) \tag{2.6}$$

$$P(\max_{\mathbf{k}\in T_\theta^d, \mathbf{k}<\mathbf{n}} |S_\theta(\mathbf{k})| > \lambda) \leq 2^d\cdot P(|S_\theta(\mathbf{n})| > \lambda - d\sigma\sqrt{2|\mathbf{n}|}). \tag{2.7}$$

REMARK. The last term in (2.6) and (2.7) can be improved to $d\sigma\sqrt{2M_\theta(\mathbf{n})}$.

The following result will be our final prerequisite.

LEMMA 2.3. *Let* $\{X_{\mathbf{n}};\ \mathbf{n} \in Z_+^d\}$ *be* i.i.d. *random variables and set* $S_{\mathbf{n}} = \sum_{\mathbf{k}<\mathbf{n}} X_{\mathbf{k}}$ *and* $S(n) = \sum_{j=1}^{n} X_{\pi(j)}$. *Suppose that* $EX_1 = 0$ *and* $EX_1^2 = \sigma^2 < \infty$. *If*

$$EX_1^2 \cdot \frac{(\log^+|X_1|)^{d-1}}{\log^+\log^+|X_1|} < \infty \tag{2.8}$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{-1} \cdot P(|S_{\mathbf{n}}| > \varepsilon\sqrt{|\mathbf{n}| \log^+\log^+|\mathbf{n}|}) < \infty,\ \varepsilon > \sigma\sqrt{2d} \tag{2.9}$$

and, for $c > 1$ *and* $\gamma > 0$,

$$\sum_{j=1}^{\infty} j^{d-1} \cdot P(|S([\gamma c^j])| > \varepsilon\sqrt{c^j \log^+\log^+ c^j}) < \infty, \quad \varepsilon > \sigma\sqrt{2d\gamma}. \tag{2.10}$$

For $d = 1$, (2.9) has been proved in Davis (1968a), Theorem 4 and for $d \geq 2$ in Gut (1980), Theorem 6.2. (2.10) follows by using estimates like in Gut (1978, 1980).

3. The LIL in the sector. In this section we shall prove the law of the iterated logarithm for i.i.d. random variables whose index set is a sector as defined above. For technical reasons, however, we let $\{X_{\mathbf{n}};\ \mathbf{n} \in Z_+^d\}$ be i.i.d. random variables. Further, let $S_\theta(\mathbf{n}) = \sum_{\mathbf{k}\in T_\theta^d, \mathbf{k}<\mathbf{n}} X_{\mathbf{k}}$, $S(n) = \sum_{j=1}^{n} X_{\pi(j)}$ and set $s_\theta^2(\mathbf{n}) = \operatorname{Var} S_\theta(\mathbf{n})$, whenever $\sigma^2 = \operatorname{Var} X_1$ is finite. This means for example in view of (2.1) that

$$S_\theta(\mathbf{n}) \text{ has the same distribution as } S(M_\theta(\mathbf{n})) \tag{3.1}$$

and

$$s_\theta^2(\mathbf{n}) = \sigma^2 \cdot M_\theta(\mathbf{n}) \quad \text{if} \quad \sigma^2 < \infty. \tag{3.2}$$

THEOREM 3.1. *If* $EX_1 = 0$ *and* $\sigma^2 = \operatorname{Var} X_1 < \infty$, *then*

$$\limsup_{\mathbf{n}} \frac{S_\theta(\mathbf{n})}{\sqrt{2\, s_\theta^2(\mathbf{n}) \log\log s_\theta^2(\mathbf{n})}} = 1 \quad \text{a.s.} \tag{3.3}$$

$$\liminf_{\mathbf{n}} \frac{S_\theta(\mathbf{n})}{\sqrt{2\, s_\theta^2(\mathbf{n}) \log\log s_\theta^2(\mathbf{n})}} = -1 \quad \text{a.s.} \tag{3.4}$$

Conversely, if $\limsup_{\mathbf{n}} \frac{|S_\theta(\mathbf{n})|}{\sqrt{M_\theta(\mathbf{n}) \log\log M_\theta(\mathbf{n})}} < \infty$ *with positive probability, then* $EX_1 = 0$ *and* $EX_1^2 < \infty$.

PROOF. To prove the sufficiency, it is clearly enough to prove (3.3). The proof is related to proofs in Davis (1978a, 1978b) and Gut (1978, 1980). To simplify the proof, we treat all indices appearing in $S_\theta(\cdot)$ and $S(\cdot)$ as if they were integers. The technicalities involved can otherwise be overcome as in the above cited papers.

To demonstrate the idea of the proof, we first prove that the limit superior is finite, more precisely, less than a constant (> 1), after which the method is refined to obtain the correct upper class result.

Let $c > 1$ be given and i_0 so large that $c^{di}\theta^{d-1} > c^{d(i-1)}\theta^{d-1} + 1$ for $i \geq i_0$. Define

$$A_\theta(i) = \{\mathbf{k} \in T_\theta^d;\ c^{d(i-1)}\theta^{d-1} \leq |\mathbf{k}| < c^{di}\theta^{d-1}\}, \quad i = 1, 2, \cdots. \tag{3.5}$$

The sets $\{A_\theta(i)\}_{i=1}^{\infty}$ divide the index set into disjoint "curved slices" (cf. Gut, 1980, page 304). The finiteness of the limit superior in (3.3) follows from the Borel-Cantelli lemma once we have established that

$$\sum_{i=i_0}^{\infty} P(\sup_{\mathbf{k}\in A_\theta(i)} |S_\theta(\mathbf{k})/\sqrt{2\, s_\theta^2(\mathbf{k}) \log\log s_\theta^2(\mathbf{k})}| > \varepsilon) < \infty \tag{3.6}$$

for some $\varepsilon > 0$.

If $\mathbf{k} \in T_\theta^d$ and $|\mathbf{k}| = c^{d(i-1)}\theta^{d-1}$, then, by homogeneity, $M_\theta(\mathbf{k}) \geq g(\theta)\cdot c^{d(i-1)}$, where $g(\theta)$ is a positive function of θ, $0 < \theta < 1$, which is independent of c. Thus,

$$P(\sup_{\mathbf{k}\in A_\theta(i)} | S_\theta(\mathbf{k})/\sqrt{2\, s_\theta^2(\mathbf{k}) \log\log s_\theta^2(\mathbf{k})} | > \varepsilon)$$
$$\leq P(\sup_{\mathbf{k}\in A_\theta(i)} | S_\theta(\mathbf{k}) | > \varepsilon\sqrt{2\,\sigma^2 g(\theta)c^{d(i-1)}\log\log \sigma^2 g(\theta) c^{d(i-1)}})$$
$$\leq P(\sup_{\mathbf{k}\in A_\theta(i)} | S_\theta(\mathbf{k}) | > \varepsilon\, c^{-1}\sqrt{2\,\sigma^2 g(\theta)c^{d(i-1)}\log\log c^{d(i-1)}})\ .$$

for $i \geq i_1 \geq i_0$.

At this point we need, as in Gut (1978, 1980), a "dominating point," i.e. some $\mathbf{m} \in T_\theta^d$, such that $\mathbf{k} < \mathbf{m}$ for all $\mathbf{k} \in A_\theta(i)$. One such point is $(c^i, c^i, \cdots, c^i)$ and by using this fact together with the Lévy-inequality (2.7), the last probability above is majorized by

$$2^d\cdot P(| S_\theta(c^i, \cdots, c^i) | > \varepsilon c^{-1}\sqrt{2\sigma^2 g(\theta) c^{d(i-1)}\log\log c^{d(i-1)}} - d\sigma\sqrt{2c^{di}}). \tag{3.7}$$

Since

$$x^{-d}M_\theta(x, \cdots, x) \to V(\theta), \quad \text{say}, \quad \text{as } x \to \infty, \tag{3.8}$$

we can choose $i_2 \geq i_1$, such that

$$c^{d(i-1)}\cdot V(\theta) \leq M_\theta(c^i, \cdots, c^i) \leq c^{d(i+1)}\cdot V(\theta) \quad \text{for } i \geq i_2. \tag{3.9}$$

Therefore, in view of (3.1), the probability in (3.7) is majorized by

$$P(\max_{c^{d(i-1)}\cdot V(\theta) \leq n \leq c^{d(i+1)}\cdot V(\theta)} | S(n) | > \varepsilon c^{-1}\sqrt{2\sigma^2 g(\theta) c^{d(i-1)}\log\log c^{d(i-1)}} - d\sigma\sqrt{2c^{di}}),$$

which by (the 1-dimensional) extension of the Lévy-inequalities is majorized by

$$2P(| S(c^{d(i+1)}\cdot V(\theta)) | > \varepsilon c^{-1}\sqrt{2\sigma^2 g(\theta) c^{d(i-1)}\log\log c^{d(i-1)}} - d\sigma\sqrt{2c^{di}} - \sigma\sqrt{2c^{d(i+1)}\cdot V(\theta)})$$
$$\leq 2\cdot P(| S(c^{d(i+1)}\cdot V(\theta)) | > \varepsilon c^{-d-2}\sqrt{2\sigma^2 g(\theta) c^{d(i+1)}\log\log c^{d(i+1)}})$$

for $i \geq i_3 \geq i_2$.

We have thus proved that

$$\sum_{i=i_3}^\infty P(\sup_{\mathbf{k}\in A_\theta(i)} | S_\theta(\mathbf{k})/\sqrt{2 s_\theta^2(\mathbf{k}) \log\log s_\theta^2(\mathbf{k})} | > \varepsilon)$$
$$\leq 2^{d+1}\sum_{i=i_3}^\infty P(| S(c^{di}\cdot V(\theta)) | > \varepsilon c^{-d-2}\sqrt{2\sigma^2 g(\theta) c^{di}\log\log c^{di}}),$$

which, in view of (2.10), is finite for $\varepsilon c^{-d-2}\sqrt{g(\theta)/V(\theta)} > 1$.

Since c may be chosen arbitrarily close to 1, this establishes that the limit superior in (3.3) is at most equal to $\sqrt{V(\theta)/g(\theta)} > 1$ (and thus finite).

We now turn to a refinement of the above argument to obtain the correct upper class result. To this end we cut all the slices $A_\theta(i)$ along the hyperplanes $x_i = \theta^{k/m}\cdot x_j$ for all $i, j = 1, \cdots, d$, $i \neq j$ and $k = 0, 1, \cdots, m - 1$, where m is some large integer. Each slice $A_\theta(i)$ is thus divided into a fixed number, N, say, of small pieces $A_\theta(i, j)$ $j = 1, \cdots, N$, such that $M_\theta(\mathbf{n})$ is "approximately constant" for all $\mathbf{n}$ belonging to a given $A_\theta(i, j)$ and approximately equal to $M_\theta(\mathbf{n}^*(i, j))$ where $\mathbf{n}^*(i, j)$ is a point dominating all points in $A_\theta(i, j)$. The sum in (3.6) is then majorized by

$$\sum_{j=1}^N \sum_{i=i_0}^\infty P(\sup_{\mathbf{k}\in A_\theta(i,j)} | S_\theta(\mathbf{k})/\sqrt{2 s_\theta^2(\mathbf{k}) \log\log s_\theta^2(\mathbf{k})} | > \varepsilon) \tag{3.10}$$

and we wish to show that this sum is finite for all $\varepsilon > 1$.

Fix j, $1 \leq j \leq N$. Since $s_\theta^2(\mathbf{k}) = \sigma^2\cdot M_\theta(\mathbf{k})$ it follows from the arguments preceding (3.10) that

$$c^{-d}\theta^{d/m}\cdot M_\theta(\mathbf{n}^*(i, j)) \leq M_\theta(\mathbf{k}) \leq M_\theta(\mathbf{n}^*(i, j)) \tag{3.11}$$

for all $\mathbf{k} \in A_\theta(i, j)$ and $i \geq i_4$.

Thus,

$P(\sup_{\mathbf{k}\in A_\theta(i,j)} |S_\theta(\mathbf{k})/\sqrt{2s_\theta^2(\mathbf{k})\log\log s_\theta^2(\mathbf{k})}| > \varepsilon)$

$$\le P(\sup_{\mathbf{k}\in A_\theta(i,j)} |S_\theta(\mathbf{k})| > \varepsilon c^{-d/2-1}\cdot\theta^{d/2m}\sqrt{2\sigma^2\cdot M_\theta(\mathbf{n}^*(i,j))\log\log \sigma^2 M_\theta(\mathbf{n}^*(i,j))})$$

$$\le P(\sup_{\mathbf{k}\in A_\theta(i,j)} |S_\theta(\mathbf{k})| > \varepsilon c^{-d/2-2}\theta^{d/2m}\sqrt{2\sigma^2\cdot M_\theta(\mathbf{n}^*(i,j))\log\log M_\theta(\mathbf{n}^*(i,j))})$$

$$\le 2^d\cdot P(|S(M_\theta(\mathbf{n}^*(i,j)))| > \varepsilon c^{-d/2-3}\theta^{d/2m}\sqrt{2\sigma^2 M_\theta(\mathbf{n}^*(i,j))\log\log M_\theta(\mathbf{n}^*(i,j))})$$

for $i \ge i_5 \ge \max\{i_3, i_4\}$.

Set $n_i = M_\theta(\mathbf{n}^*(i,j))$. We have thus established that (recall that j is fixed)

$$\sum_{i=i_5}^{\infty} P(\sup_{\mathbf{k}\in A_\theta(i,j)} |S_\theta(\mathbf{k})/\sqrt{2s_\theta^2(\mathbf{k})\log\log s_\theta^2(\mathbf{k})}| > \varepsilon)$$
$$\le 2^d\cdot\sum_{i=i_5}^{\infty} P(|S(n_i)| > \varepsilon\cdot c^{-d/2-3}\theta^{d/2m}\sqrt{2\sigma^2 n_i \log\log n_i})$$

which, by (2.10), is finite for all $\varepsilon > c^{d/2+3}\theta^{-d/2m}$, since, by the homogeneity of the construction, the dominating points lie along a ray and $\{n_i\}_{i=1}^{\infty}$ increases geometrically.

Since we have convergence for each j it follows that the sum in (3.10) converges for all $\varepsilon > c^{d/2+3}\theta^{-d/2m}$ and since c can be chosen arbitrarily close to 1 and m arbitrarily large, the desired conclusion follows.

We have thus proved the upper class result, i.e. we have established that the limit superior in (3.3) is ≤ 1.

For the lower class result, i.e. the opposite inequality, we note that

$$\text{(3.12)}\quad \limsup_{\mathbf{n}} \frac{S_\theta(\mathbf{n})}{\sqrt{2s_\theta^2(\mathbf{n})\log\log s_\theta^2(\mathbf{n})}} \ge \limsup_{n\to\infty} \frac{S_\theta(n, n, \cdots, n)}{\sqrt{2s_\theta^2(n, \cdots, n)\log\log s_\theta^2(n \cdots, n)}}.$$

Now, by interpreting $S_\theta(n, \cdots, n)$ as a sum of all X:*es* preceding $(n, \cdots, n)$ which are inside T_θ^d and of zeroes for all points in $Z_+^d\setminus T_\theta^d$ we can apply Wichura (1973), Theorem 1, to conclude that the last limit superior equals 1 and we are done.

To prove the necessity we observe that the assumption in particular implies that

$$\text{(3.13)}\quad P\left(\limsup_{n\to\infty} \frac{|S_\theta(2^n, \cdots, 2^n)|}{\sqrt{M_\theta(2^n, \cdots, 2^n)\log\log M_\theta(2^n, \cdots, 2^n)}} < \infty\right) > 0.$$

Set $Y_\theta(n) = S_\theta(2^n, \cdots, 2^n) - S_\theta(2^{n-1}, \cdots, 2^{n-1})$, $n = 1, 2, \cdots$. It follows that

$$P\left(\limsup_{n\to\infty} \frac{|Y_\theta(n)|}{\sqrt{M_\theta(2^n, \cdots, 2^n)\log\log M_\theta(2^n, \cdots, 2^n)}} < \infty\right) > 0,$$

and hence, by independence and Borel-Cantelli, that

$$\sum_{n=1}^{\infty} P(|Y_\theta(n)| > \varepsilon\sqrt{M_\theta(2^n, \cdots, 2^n)\log\log M_\theta(2^n, \cdots, 2^n)}) < \infty \text{ for some } \varepsilon > 0$$

and finally, since $M_\theta(2^n, \cdots, 2^n) \le 2^{nd}$, that

$$\text{(3.14)}\quad \sum_{n=1}^{\infty} P(|Y_\theta(n)| > \varepsilon\sqrt{2^{nd}\log\log 2^{nd}}) < \infty \text{ for some } \varepsilon > 0.$$

Now, suppose that the variables have a symmetric distribution. It follows from (3.8) that

$$\text{(3.15)}\quad 2^{-nd}(M_\theta(2^n, \cdots, 2^n) - M_\theta(2^{n-1}, \cdots, 2^{n-1})) \to V(\theta)\cdot(1 - 2^{-d}) \text{ as } n \to \infty.$$

This, together with the i.i.d. assumption, the symmetry and Lévy's inequality, imply, for large n, that

$$P(|Y_\theta(n)| > \varepsilon\sqrt{2^{nd}\log\log 2^{nd}})$$
$$= P(|S(M_\theta(2^n, \cdots, 2^n) - M_\theta(2^{n-1}, \cdots, 2^{n-1}))| > \varepsilon\sqrt{2^{nd}\log\log 2^{nd}})$$
$$\ge \tfrac{1}{2} P(\sup_{k\le M_\theta(2^n,\cdots,2^n)-M_\theta(2^{n-1},\cdots,2^{n-1})} |S(k)| > \varepsilon\sqrt{2^{nd}\log\log 2^{nd}})$$

$$\geq \tfrac{1}{2} P(\sup_{k \leq (1/2) V(\theta) \cdot (1-2^{-d}) 2^{nd}} |S(k)| > \varepsilon \sqrt{2^{nd} \log \log 2^{nd}})$$

$$\geq \tfrac{1}{2} P(\sup_{k \leq (3/8) V(\theta) 2^{nd}} |S(k)| > \varepsilon \sqrt{2^{nd} \log \log 2^{nd}})$$

$$\geq \tfrac{1}{2} P(\sup_{k \in A(n)} |S(k)| > \varepsilon \sqrt{2^{nd} \log \log 2^{nd}})$$

$$\geq \tfrac{1}{2} P(\sup_{k \in A(n)} |S(k)/\sqrt{k \log \log k}| > \varepsilon_1),$$

where $A(n) = \{k;\ 3/8 \cdot V(\theta) \cdot 2^{(n-1)d} \leq k < 3/8 \cdot V(\theta) \cdot 2^{nd}\}$.

In view of (3.14) we therefore conclude that

$$\sum_{n=1}^{\infty} P(\sup_{k \in A(n)} |S(k)/\sqrt{k \log \log k}| > \varepsilon_1) < \infty \text{ for some } \varepsilon_1 > 0 \tag{3.16}$$

and thus, by the Borel-Cantelli lemma that

$$P(|S(k)| > \varepsilon_1 \sqrt{k \log \log k} \text{ i.o.}) = 0, \tag{3.17}$$

which implies that $EX_{\pi(1)} = 0$ and $EX_{\pi(1)}^2 < \infty$ (and hence that $EX_1 = 0$ and $EX_1^2 < \infty$) by the converse to the law of the iterated logarithm (Strassen, 1966) applied to $\{S_{\pi(j)}\}_{j=1}^{\infty}$.

This terminates the proof of the necessity in the symmetric case. The desymmetrization is standard (cf. Gut, 1978, 1980) and is omitted.

4. Two remarks on the LIL. One consequence of Wichura (1973), Theorem 1 (see also Pyke, 1972, Theorem 5) is a law of the iterated logarithm when $\mathbf{n}$ tends to infinity along a ray, i.e. when we only consider points (in Z_+^d or in T_θ^d) of the form

$$(kn_1, kn_2, \cdots, kn_d),\ k = 1, 2, \cdots \text{ with } n_1, \cdots, n_d \text{ fixed.}$$

(To be precise, only the sufficiency parts are included in Wichura, 1973.)

The upper class results can alternatively be proved by the methods of the previous section. Note also that the limit superior along a ray in T_θ^d is at most equal to the limit superior in (3.3).

The necessities follow just as in the previous section. Note that, in fact, the necessity in Theorem 3.1 was proved via the "diagonal ray".

Next, we shall indicate how the upper class result in (1.1) can be deduced from the methods above and estimates made in Gut (1980).

Thus, $\{X_\mathbf{n};\ \mathbf{n} \in Z_+^d\}$ are i.i.d. random variables with mean 0 and variance $\sigma^2 < \infty$, $S_\mathbf{n} = \sum_{\mathbf{k} \leq \mathbf{n}} X_\mathbf{k}$ and $\text{Var } S_\mathbf{n} = |\mathbf{n}| \cdot \sigma^2$.

Let $c > 1$ be given and define i_0 so large that $c^{i+1} > c^i + 1$ for $i \geq i_0$. As in Gut (1980), page 303–304, we define

$$E_j = E_j(c) = \{\mathbf{k};\ \text{exactly } j \text{ of the components are} \leq c^{i_0}\},\ j = 0, 1, \cdots, d \tag{4.1}$$

and

$$A(i) = \{\mathbf{k};\ c^{i-1} \leq |\mathbf{k}| < c^i\} \text{ for } i = 1, 2, \cdots. \tag{4.2}$$

Then (cf. Gut, 1980, page 305)

$$\sum_{i=i_0}^{\infty} P(\sup_{\mathbf{k} \in A(i) \cap E_0} |S_\mathbf{k}/\sqrt{|\mathbf{k}| \log \log |\mathbf{k}|}| > \varepsilon)$$

$$\leq \sum_{i=i_0}^{\infty} P(\sup_{\mathbf{k} \in A(i) \cap E_0} |S_\mathbf{k}| > \varepsilon \sqrt{c^{i-1} \log \log c^{i-1}})$$

$$\leq 2^{d+1} \sum_{i=i_0}^{\infty} i^{d-1} \cdot P(|S([c^{i+2d}])| > \varepsilon \cdot c^{-d-3} \sqrt{c^{i+2d} \log \log c^{i+2d}}),$$

which, under the conditions of the theorem, is finite for $\varepsilon > c^{d+3} \sigma \sqrt{2d}$ in view of (2.10).

By treating the sums corresponding to $E_1, \cdots, E_d$ similarly one obtains sums of the same kind but with lower powers of c (since the problem essentially has been reduced to fewer dimensions), cf. Gut (1980), page 306.

Thus all relevant sums converge for $\varepsilon > c^{d+3} \sigma \sqrt{2d}$ and, since c may be chosen arbitrarily close to 1 it follows that

$$P(S_{\mathbf{n}} > \varepsilon\sqrt{|\mathbf{n}|\log\log|\mathbf{n}|}\text{ i.o.}) = 0 \text{ for } \varepsilon > \sigma\sqrt{2d} \tag{4.3}$$

and the upper class result has been established.

5. Strong laws of large numbers. In this section we shall present a sectorial Marcinkiewicz strong law of large numbers. For the sectorial Kolmogorov strong law, see Gabriel (1974, 1975, 1977) and Smythe (1974) and for corresponding results in Z_+^d, see Smythe (1973), Section 2 ($r = 1$) and Gut (1978), Theorem 3.2, ($0 < r < 2$, $r \neq 1$). The proof given below is related to the proof in Section 3 and to Gut (1978), Section 5, and is therefore only sketched.

THEOREM 5.1. *Let* $\{X_{\mathbf{n}};\ \mathbf{n} \in T_\theta^d\}$ *be* i.i.d. *random variables with* $E|X_1|^r < \infty$, $0 < r < 2$ *and* $EX_1 = 0$ *if* $1 \le r < 2$. *Then*

$$|\mathbf{n}|^{-1/r}\cdot S_\theta(\mathbf{n}) \to 0 \text{ a.s. as } \mathbf{n} \to \infty. \tag{5.1}$$

Conversely, (5.1) *implies that* $E|X_1|^r < \infty$.

REMARK. In analogy with Z_+^d, $d \ge 1$, the normalization in (5.1) should be $(M_\theta(\mathbf{n}))^{1/r}$, but because of the homogeneity of the index set (cf. Section 3) (5.1) is equivalent to $(M_\theta(\mathbf{n}))^{-1/r}\cdot S_\theta(\mathbf{n}) \to 0$ a.s. as $\mathbf{n} \to \infty$.

PROOF. For the sufficiency we assume without restriction that $\{X_{\mathbf{n}};\ \mathbf{n} \in Z_+^d\}$ are i.i.d. random variables satisfying the given assumptions.

Suppose first that the variables have a symmetric distribution and define

$$B_\theta(n) = \{\mathbf{k} \in T_\theta^d;\ 2^{d(n-1)}\theta^{d-1} \le |\mathbf{k}| < 2^{dn}\theta^{d-1}\}, \quad n = 1, 2, \cdots. \tag{5.2}$$

The conclusion follows from the Borel-Cantelli lemma once we have established that

$$\textstyle\sum_\theta = \sum_{n=1}^\infty P(\sup_{\mathbf{k}\in B_\theta(n)}|S_\theta(\mathbf{k})/|\mathbf{k}|^{1/r}| > \varepsilon) < \infty \text{ for all } \varepsilon > 0. \tag{5.3}$$

Computations like those of Section 3 and Gut (1978), Section 5, yield

$$\begin{aligned}\textstyle\sum_\theta &\le 2^{d+1}\textstyle\sum_{n=1}^\infty P(|S(2^{nd})| > \varepsilon_1(2^{nd})^{1/r})\\ &\le 2^{d+1}\textstyle\sum_{j=0}^\infty \frac{1}{2^{dj}(2^d-1)}\sum_{i=2^{dj}+1}^{2^{d(j+1)}} 2P(|S(i)| > \varepsilon_1\cdot 2^{-d/r}\cdot(2^{d(j+1)})^{1/r})\\ &= 4^{d+1}\textstyle\sum_{n=1}^\infty \frac{1}{n}P(|S(n)| > \varepsilon_2\cdot n^{1/r}),\end{aligned}$$

where $\varepsilon_2 = \varepsilon_1\cdot 2^{-d/r} = \varepsilon(\theta^{d-1}\cdot 4^{-d})^{1/r}$.

Now, under the conditions of the theorem, the last sum is finite for all $\varepsilon_2 > 0$ by Theorem 3 of Baum and Katz (1965), which proves the theorem in the symmetric case. The desymmetrization is standard (cf. Gut (1978), Theorem 3.2).

PROOF. (Necessity). Exactly as in Gut (1978), Theorem 3.2, (5.1) implies that $\sum_{\mathbf{n}\in T_\theta^d} P(|X_1| > |\mathbf{n}|^{1/r}) < \infty$, which by Lemma 2.1.a is equivalent to $E|X_1|^r < \infty$.

6. Convergence rates. Several results on convergence rates in the law of large numbers and the law of the iterated logarithm have been extended to the index set Z_+^d, $d \ge 2$, in Gut (1978), Sections 4–6 and in Gut (1979, 1980). All the results given there (except for Theorem 3.5 of Gut, 1980) also hold if the index set is T_θ^d, $d \ge 2$, and in all cases without "extra logarithms". In this section we state one of these extensions.

Throughout this section, $\{X_{\mathbf{n}};\ \mathbf{n} \in T_\theta^d\}$ are i.i.d. random variables.

THEOREM 6.1. *Suppose that* $EX_1 = 0$ *and* $EX_1^2 = \sigma^2 < \infty$. *If*

$$EX_1^2 \cdot \frac{\log^+|X_1|}{\log^+\log^+|X_1|} < \infty \tag{6.1}$$

then

$$\sum_{\mathbf{n}\in T_\theta^d} |\mathbf{n}|^{-1}\cdot \log|\mathbf{n}|\cdot P(|S_\theta(\mathbf{n})| > \varepsilon\sqrt{M_\theta(\mathbf{n})\log\log M_\theta(\mathbf{n})}) < \infty \text{ for } \varepsilon > 2\sigma \tag{6.2}$$

$$\sum_{\mathbf{n}\in T_\theta^d} |\mathbf{n}|^{-1}\cdot \log|\mathbf{n}|\cdot P(\max_{\mathbf{k}\in T_\theta^d, \mathbf{k}<\mathbf{n}} |S_\theta(\mathbf{k})| > \varepsilon\sqrt{M_\theta(\mathbf{n})\log\log M_\theta(\mathbf{n})}) < \infty \text{ for } \varepsilon > 2\sigma \tag{6.3}$$

$$\sum_{j=1}^\infty j^{-1}\cdot P(\sup_{\mathbf{k}\in T_\theta^d, j\le|\mathbf{k}|} |S_\theta(\mathbf{k})/\sqrt{M_\theta(\mathbf{k})\log\log M_\theta(\mathbf{k})}| > \varepsilon) < \infty \text{ for } \varepsilon > 2\sigma. \tag{6.4}$$

Conversely, if one of the sums is finite for some $\varepsilon > 0$, *then so are the others*, $EX_1 = 0$ *and* (6.1) *holds.*

Proofs of such theorems are rather straightforward modifications of the proofs given in Gut (1978, 1980). However, since no extra logarithms are involved in the moment assumptions, one can sometimes proceed as in Baum and Katz (1965) for the converses (which makes that part easier).

We also remark that one can also reduce some of the proofs to the one-dimensional case.

It is also possible to obtain results on the number of boundary crossings and last exit times, which for the case of the law of the iterated logarithm are

$$N_d = \sum_{\mathbf{n}\in T_\theta^d} I\{|S_\mathbf{n}| > \varepsilon\sqrt{|\mathbf{n}|\log\log|\mathbf{n}|}\} \quad \text{and}$$
$$L_d = \sup_{\mathbf{n}\in T_\theta^d}\{|\mathbf{n}|;\, |S_\mathbf{n}| > \varepsilon\sqrt{|\mathbf{n}|\log\log|\mathbf{n}|}\}. \tag{6.5}$$

Corresponding results for Z_+^d, $d \ge 2$, are obtained in Gut (1980), Section 8, see also Gut (1979), Section 6.

As a final result we mention another convergence rate result, sometimes called a dominated ergodic theorem.

THEOREM 6.2. *Let* $r \ge 2$. *For* $EX_1 = 0$ *the following statements are equivalent:*

$$EX_1^2 \cdot \frac{\log^+|X_1|}{\log^+\log^+|X_1|} < \infty \text{ if } r = 2 \text{ and } E|X_1|^r < \infty \text{ if } r > 2 \tag{6.6}$$

$$E\sup_{\mathbf{n}\in T_\theta^d}\left|\frac{X_\mathbf{n}}{\sqrt{|\mathbf{n}|\log\log|\mathbf{n}|}}\right|^r < \infty \tag{6.7}$$

$$E\sup_{\mathbf{n}\in T_\theta^d}\left|\frac{S_\mathbf{n}}{\sqrt{|\mathbf{n}|\log\log|\mathbf{n}|}}\right|^r < \infty \tag{6.8}$$

Note that the method of proof requires Theorem 3.1 for the step (6.7) $\Rightarrow$ (6.8) (cf. Gut (1979), where the case Z_+^d is proved).

A similar result can be obtained for the strong laws of large numbers For $r = 1$ and positive summands, this has been done in Theorem 6 of Gabriel (1977). For Z_+^d, see Gut (1979), Theorem 3.2.

7. Strong laws via martingales. A "modern" way of proving the strong law of large numbers for i.i.d. random variables is to use martingale theory—the point being that the sequence of arithmetic means constitutes a reversed martingale. This method also works in Z_+^d, $d \ge 2$ (see Smythe, 1973, and Gut, 1976). For the sector the method works too, but one does not obtain the best result. In this final section we shall try to see why this is the case.

When the index set is Z_+^d, $d \ge 1$, the necessary and sufficient integrability condition for the strong law of large numbers is $L(\log L)^{d-1}$ (see Smythe, 1973). This is also the correct integrability condition for (reversed) martingales, see Cairoli (1970), Gut (1976). Now, for the strong law of large numbers and for (reversed) martingales in the sector, the correct integrability conditions are L^1 and $L(\log L)^{d-1}$ respectively; see Gabriel (1977), Theorem 3 (i.e. Theorem 5.1 above with $r = 1$) and Gabriel (1977), Section 5. Therefore, one has to use a condition which is stronger than necessary to prove the law of large numbers in the sector via martingale theory.

In Smythe (1974) the strong law of large numbers is proved for more general index sets and the integrability condition there is $EM(|X|) < \infty$, which for the index set Z_+^d yields $L(\log L)^{d-1}$ and for the index set T_θ^d yields L^1. The integrability condition is here linked to the *size* of the index set.

The proofs of convergence results for martingales depend on a maximal inequality (see Cairoli, 1970, Theorem 1), which is proved by induction on the dimension, thus yielding one logarithmic factor for each of the $d - 1$ iterations. For Z_+^d as well as for T_θ^d this yields $L(\log L)^{d-1}$. Here, however, the integrability condition is linked to the *dimension* of the index set.

It is thus "a coincidence" that the martingale proof of the strong law of large numbers yields the correct result when the index set is Z_+^d, $d \geq 1$ and "quite natural" that this is not the case when one considers sectorial laws.

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