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AN INTEGRAL TEST FOR THE RATE OF ESCAPE OF d-DIMENSIONAL RANDOM WALK[1]

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Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables taking values in $\mathbb{R}^d$ and $S_n = X_1 + \cdots + X_n$. For a large class of random variables, which includes all of those in the domain of attraction of a type A stable law, an integral test is given which determines whether

$$P\{|S_n| \le \gamma_n \text{ i.o.}\} = 0 \quad \text{or} \quad 1$$

for any increasing sequence $\{\gamma_n\}$. This result generalizes the Dvoretzky-Erdös test for simple random walk and the Takeuchi and Taylor test for stable random walks.

1. Introduction. Let $\{X_k\}$ be a sequence of independent, identically distributed, nondegenerate, d-dimensional random variables and $S_n = \sum_{k=1}^n X_k$. In this paper we will show how the probability estimates derived in [7] can be used to obtain information about the rate of growth of d-dimensional random walk. More precisely, we will obtain an integral test which determines whether or not

$$\liminf_{n\to\infty} \frac{|S_n|}{\gamma_n} \le 1 \quad \text{a.s.} \tag{1.1}$$

for increasing sequences $\{\gamma_n\}$. Of course this problem is only of interest for transient random walks, since (1.1) will hold for all increasing sequences $\{\gamma_n\}$ if the random walk is recurrent.

The corresponding problem for upper envelopes, i.e. that of obtaining an integral test to determine when $\limsup_{n\to\infty} |S_n|/\gamma_n \le 1$ a.s. was, essentially, completely solved by Feller in 1946, [3]. The test here is quite easy and involves only the tail of the distribution of $|X|$.

The first progress made on the lower envelope problem was the famous Dvoretzky-Erdös test for simple random walk in $d \ge 3$ dimensions, [1]. This was generalized to symmetric stable random walks of index $\gamma < d$ by Takeuchi [12] in 1964, who showed that

$$\liminf_{n\to\infty} \frac{|S_n|}{\alpha_n n^{1/\gamma}} = \begin{matrix} 0 \\ \infty \end{matrix} \quad \text{according as} \quad \sum \frac{\alpha_n^{d-\gamma}}{n} \quad \begin{matrix} \text{diverges} \\ \text{converges} \end{matrix} \tag{1.2}$$

for decreasing sequences $\{\alpha_n\}$. The Dvoretzky-Erdös test then corresponds to taking $\gamma = 2$ in (1.2). Taylor [13] in 1967 then showed that the same test holds for all type A (i.e. those with non-vanishing density) stable random walks of index $\gamma < d$. The techniques of Takeuchi and Taylor involved the use of potential theory to calculate the delayed hitting probabilities of balls centered at the origin. In 1970 Kesten [8], while investigating the set of all limit points of $\{S_n/\gamma_n\}$, introduced an alternative approach which was used by Erickson [2] to obtain partial information in the case where X is in the domain of attraction of a type A stable law of index $\gamma < d$.

We will also use Kesten's approach to obtain an integral test analogous to Takeuchi's which will cover the cases left open by Erickson and which will also apply to many random

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variables not in the domain of attraction of any stable law. The method is based on the probability estimates obtained in [7].

In the case that X is in the domain of attraction of the Cauchy distribution in one dimension ($\gamma = d = 1$) or the normal distribution in two dimensions ($\gamma = d = 2$), the situation is far more delicate and we have been unable to obtain an integral test although some partial information is given in [6]. To illustrate some of the difficulties that may arise in this case, and hence presumably in the general case, we will give a somewhat counter-intuitive example. Define the lower index of a transient random walk to be the unique $\delta \geq 0$ such that

$$\liminf_{n\to\infty} \frac{|S_n|}{n^\alpha} = \begin{cases} 0 & \text{a.s. if } \alpha > \delta \\ \infty & \text{a.s. if } \alpha < \delta. \end{cases} \tag{1.3}$$

Then the lower index is not monotone in the tail of the distribution. More precisely, there exist symmetric random variables X and Y, both in the domain of attraction of the one dimensional Cauchy distribution, such that $P\{|X| > x\} \geq P\{|Y| > x\}$ for all $x \geq 0$ but $\delta_Y > \delta_X$ where δ_Y is the lower index of the random walk corresponding to Y and δ_X the lower index corresponding to X.

2. Probability estimates. For convenience we will describe here the main probability estimate in [7] and introduce some notation. Let F denote the distribution function of X_1 and φ its characteristic function. X will be another random variable with the same distribution as X_1. For $x > 0$ define

$$G(x) = P\{|X| > x\}, \quad K(x) = \frac{1}{x^2}\int_{|y|\leq x} |y|^2\, dF(y)$$

$$Q(x) = G(x) + K(x) = E(x^{-1}|X| \wedge 1)^2. \tag{2.1}$$

From (2.1) one readily checks that Q is positive, continuous, decreasing, $Q(x) \to 0$ as $x \to \infty$ and $x^2Q(x)$ increases. Further Q is strictly decreasing on (x_0, ∞) where $x_0 = \sup\{x : P\{0 < |X| \leq x\} = 0\}$. Set $y_0 = 1/Q(x_0)$. Then there is an increasing function a_y defined for $y \geq y_0$ such that

$$Q(a_y) = \frac{1}{y}. \tag{2.2}$$

For convenience we define $a_y = a_{y_0}$ for $y \in (0, y_0)$.

The following result can be found in [7], (Theorem 3.6), where a full discussion of the hypotheses is given. The reader is advised to consult [7] for any definitions not given in the present paper. In particular the direction condition is defined at the beginning of Section 2 in [7].

THEOREM 2.1. *Assume that X is genuinely d-dimensional and satisfies the direction condition,* $\liminf_{x\to\infty} K(x)/G(x) > 0$ *and* $|\mathrm{Im}\varphi(t)| = o(1 - \mathrm{Re}\varphi(t))$. *Then there exist positive constants c_1, c_2 and λ_0 such that for all $\lambda \geq \lambda_0$ and all n*

$$c_1\left(\frac{\lambda}{a_n} \wedge 1\right)^d \leq P\{S_n \in C(0, \lambda)\} \leq c_2\left(\frac{\lambda}{a_n} \wedge 1\right)^d$$

where $C(0, \lambda)$ is the cube of side length 2λ centered at the origin.

As mentioned in [7], the above result does not include all random variables in the domain of attraction of a stable law. We will now give the analogous estimate in this case since it will also be covered by our integral test. Observe that in our notation, X is in the domain of attraction of a stable law means that there exists a centering sequence $\{b_n\}$ such that $(S_n - b_n)/a_n$ converges weakly to a nondegenerate limit. Recall that a stable law is of type A if its density is never zero.

THEOREM 2.2. *Assume that X is in the domain of attraction of a genuinely d-dimensional type* A *stable law. Then there exist positive constants c_1, c_2 and λ_0 such that for all $\lambda \geq \lambda_0$ and all n*

$$c_1\left(\frac{\lambda}{a_n} \wedge 1\right)^d \leq P\{S_n - b_n \in C(0, \lambda)\} \leq c_2\left(\frac{\lambda}{a_n} \wedge 1\right)^d.$$

This result can be found in [6] (Theorem 4.21) or can be derived from Theorem 1 in [11].

We will complete this section by giving a slight variant of a result due to Kesten [8]. Since the proof is similar it will not be given here. Full details can be found in [6].

THEOREM 2.3. *Let $\{\gamma_n\}$ be any increasing sequence such that* $\limsup_{n\to\infty} \gamma_{2n}/\gamma_n = c_1 < \infty$. *Then for every $K > 0$,* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *where*

(i) $P\{S_n/\gamma_n \in C(0, K/2) \text{ i.o.}\} = 1$

(ii) $\sum_{n=1}^{\infty} \dfrac{P\{S_n \in C(0, K\gamma_n)\}}{\sum_{k=0}^{n-1} P\{S_k \in C(0, \gamma_n)\}} = \infty$

(iii) $P\{S_n/\gamma_n \in C(0, 2cK) \text{ i.o.}\} = 1$ *for all* $c > c_1$.

3. Integral tests. We will now show how Theorems 2.1, 2.2 and 2.3 can be used to obtain an integral test. In the theory of random walks it is often encountered that the behaviour in dimensions $d = 1$ and $d = 2$ differs from the behaviour in higher dimensions. For example, all genuinely d-dimensional random walks are transient in $d \geq 3$ dimensions, but it is possible for them to be recurrent in dimensions one and two. This dichotomy is also apparent in our problem. In order to prove the integral test in one and two dimensions we need to add one further hypothesis which can be thought of as ensuring that the random walk is transient.

We begin with a simple lemma which gives some idea as to the order of magnitude of the sequences $\{\gamma_n\}$ that we should be considering.

LEMMA 3.1. *Assume that there exist positive constants c_1, λ_0 and n_0 such that for all $\lambda \geq \lambda_0$ and all $n \geq n_0$*

$$P\{S_n \in C(0, \lambda)\} \geq c_1\left(\frac{\lambda}{a_n} \wedge 1\right)^d. \tag{3.1}$$

Then for all sequences $\{\gamma_n\}$ such that $\limsup_{n\to\infty} \gamma_n/a_n > 0$,

$$\liminf_{n\to\infty} \frac{|S_n|}{\gamma_n} = 0 \quad \text{a.s.} \tag{3.2}$$

PROOF. There exists a constant $c > 0$ and a subsequence $n_k \to \infty$ such $\gamma_{n_k} \geq c\, a_{n_k}$ for all k. Thus for every $\varepsilon > 0$

$$P\{S_{n_k} \in C(0, \varepsilon\gamma_{n_k}) \text{ i.o.}\} \geq P\{S_{n_k} \in C(0, \varepsilon c\, a_{n_k}) \text{ i.o.}\} \geq \limsup_{k\to\infty} P\{S_{n_k} \in C(0, \varepsilon c\, a_{n_k})\} > 0$$

by (3.1). Hence by the Hewitt-Savage 0-1 Law, for every $\varepsilon > 0$

$$P\{S_{n_k} \in C(0, \varepsilon\gamma_{n_k}) \text{ i.o.}\} = 1$$

and so (3.2) holds.

As a notational convenience and for ease of reading, in the remainder of this and in the final section, we will often write

$$\textstyle\sum_{k=a}^{b} c_k \tag{3.3}$$

where a and b are not integers. By this we will mean

$$\sum_{k\in[a,b]\cap\mathbb{Z}^+} c_k$$

where $\mathbb{Z}^+$ denotes the nonnegative integers. Typically a and b will be indexed by n, and we will be interested in the asymptotic behaviour of (3.3) as $n \to \infty$. One further piece of notation that will prove useful is that $\alpha_n \approx \beta_n$ will mean there exists $M > 1$ such that $M^{-1} \le \alpha_n/\beta_n \le M$ for large n.

THEOREM 3.2. *Assume that* $d \ge 3$ *and*

(i) X *is genuinely* d-*dimensional*

(ii) $\lim\inf_{x\to\infty} K(x)/G(x) > 0$

(iii) $|\mathrm{Im}\varphi(t)| = o(1 - \mathrm{Re}\varphi(t))$ as $t \to 0$

(iv) *direction condition holds.*

Then for any sequence $\{\gamma_n\}$ *such that* $\gamma_n \uparrow \infty$ *and* $\lim\sup_{n\to\infty} \gamma_{2n}/\gamma_n < \infty$,

$$\lim\inf_{n\to\infty} \frac{|S_n|}{\gamma_n} = \begin{matrix} 0 \\ \infty \end{matrix} \quad \text{according as} \quad \sum \left(\frac{\gamma_n}{a_n}\right)^d Q(\gamma_n) \quad \begin{matrix} \text{diverges} \\ \text{converges}. \end{matrix}$$

PROOF. Let λ_0 be as in Theorem 2.1. Since $\gamma_n \to \infty$ there is no loss of generality in assuming that $\gamma_n \ge \lambda_0$ for all n and further, by increasing γ_0 if necessary, that (2.2) holds if $a_y \ge \lambda_0$. Set

$$\Delta_n = \sum_{k=0}^{n-1} P\{S_k \in C(0, \gamma_n)\}.$$

Then by Theorem 2.1

$$\Delta_n \approx \sum_{k=0}^{n-1} \left(\frac{\gamma_n}{a_k} \wedge 1\right)^d = \sum_{a_k<\gamma_n} 1 + \sum_{a_k\ge\gamma_n}^{k=n-1} \left(\frac{\gamma_n}{a_k}\right)^d.$$

Now $a_k \ge \gamma_n$ iff $(1/k) = Q(a_k) \le Q(\gamma_n)$ since Q is decreasing. Thus if $\gamma_n < a_{n-1}$

$$\Delta_n \approx \frac{1}{Q(\gamma_n)} + \gamma_n^d \sum_{k=1/Q(\gamma_n)}^{n-1} a_k^{-d}.$$

Now

$$\begin{aligned} \sum_{k=1/Q(\gamma_n)}^{n-1} a_k^{-d} &\approx \sum_{k=1/Q(\gamma_n)}^{n-1} \sum_{m=a_k}^{\infty} m^{-(d+1)} \\ &\approx \sum_{m=\gamma_n}^{a_{n-1}} m^{-(d+1)} \sum_{k=1/Q(\gamma^n)}^{1/Q(m)} 1 + \sum_{m=a_{n-1}}^{\infty} m^{-(d+1)} \sum_{k=1/Q(\gamma_n)}^{n-1} 1 \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Next observe that

$$I \le \sum_{\gamma_n}^{a_{n-1}} \frac{1}{m^{d+1} Q(m)} \le \frac{c_1}{\gamma_n^d Q(\gamma_n)}$$

since $x^2 Q(x) \uparrow$, while

$$\mathrm{II} \le \frac{c_2(n-1)}{a_{n-1}^d} \le \frac{c_2}{\gamma_n^d Q(\gamma_n)}$$

again because $x^2 Q(x) \uparrow$. Thus if $\gamma_n < a_{n-1}$ then

$$\Delta_n \approx \frac{1}{Q(\gamma_n)}.$$

Clearly if $\gamma_n \geq a_{n-1}$ then $\Delta_n \approx n$ and so we conclude that

$$\Delta_n \approx \frac{1}{Q(\gamma_n \wedge a_n)}.$$

Now fix $K > 0$. Then $K\gamma_n \geq \lambda_0$ for n sufficiently large. Thus by Theorem 2.1, there is an n_0 such that for all $n \geq n_0$

$$P\{S_n \in C(0, K\gamma_n)\} \approx \left(\frac{K\gamma_n}{a_n} \wedge 1\right)^d.$$

Hence

$$\sum_n \Delta_n^{-1} P\{S_n \in C(0, K\gamma_n)\} \quad \text{converges iff} \quad \sum_n \left(\frac{K\gamma_n}{a_n} \wedge 1\right)^d Q(\gamma_n \wedge a_n) \quad \text{converges}$$

$$\text{iff} \quad \sum_n \left(\frac{\gamma_n}{a_n} \wedge \frac{1}{K}\right)^d Q(\gamma_n \wedge a_n) \quad \text{converges.}$$

With this we are ready to complete the proof. Assume $\sum (\gamma_n/a_n)^d Q(\gamma_n)$ diverges.

CASE 1. $\gamma_n = o(a_n)$. Then for every $K > 0$ if n is sufficiently large $((\gamma_n/a_n) \wedge (1/K))^d = (\gamma_n/a_n)^d$ and $\gamma_n \wedge a_n = a_n$, thus

$$\sum \Delta_n^{-1} P\{S_n \in C(0, K\gamma_n)\}$$

diverges for every $K > 0$. Hence by Theorem 2.3

$$\liminf_{n\to\infty} \frac{|S_n|}{\gamma_n} = 0 \quad \text{a.s.}$$

CASE 2. $\gamma_n \neq o(a_n)$. Then by Lemma 3.1

$$\liminf_{n\to\infty} \frac{|S_n|}{\gamma_n} = 0 \quad \text{a.s.}$$

Now assume that $\sum (\gamma_n/a_n)^d Q(\gamma_n)$ converges. Then for each $K > 0$, if $\gamma_n \geq a_n/K$

$$(a_n/K)^2 Q(a_n/K) \leq \gamma_n^2 Q(\gamma_n)$$

since $x^2 Q(x) \uparrow$. Hence

$$K^{-d} Q(a_n/K) \leq (\gamma_n/a_n)^d Q(\gamma_n).$$

Now assume $K \geq 1$; then if $\gamma_n \geq a_n/K$

$$\left(\frac{\gamma_n}{a_n} \wedge \frac{1}{K}\right)^d Q(\gamma_n \wedge a_n) \leq \frac{1}{K^d} Q(a_n/K) \leq \left(\frac{\gamma_n}{a_n}\right)^d Q(\gamma_n)$$

while if $\gamma_n \leq a_n/K$

$$\left(\frac{\gamma_n}{a_n} \wedge \frac{1}{K}\right)^d Q(\gamma_n \wedge a_n) = \left(\frac{\gamma_n}{a_n}\right)^d Q(\gamma_n).$$

Thus we see that for all $K \geq 1$

$$\sum \Delta_n^{-1} P\{S_n \in C(0, K\gamma_n)\}$$

converges and so by Theorem 2.3

$$\liminf_{n\to\infty} \frac{|S_n|}{\gamma_n} = \infty \quad \text{a.s.}$$

If $d = 1$ or $d = 2$, then we must add one further hypothesis to ensure that the preceding test works.

THEOREM 3.3. *Assume that $d \le 2$ and that in addition to* (i) – (iv) *of Theorem* 3.2 *holding we have*

(v) *there exist $\varepsilon > 0$ and $x_0 \ge 0$ such that $x^{d-\varepsilon}Q(x)$ increases for $x \ge x_0$.*

Then for any sequence $\{\gamma_n\}$ such that $\gamma_n \uparrow \infty$ and $\limsup_{n\to\infty} \gamma_{2n}/\gamma_n < \infty$,

$$\liminf_{n\to\infty} \frac{|S_n|}{\gamma_n} = \begin{matrix} 0 \\ \infty \end{matrix} \quad \textit{according as} \quad \sum \left(\frac{\gamma_n}{a_n}\right)^d Q(\gamma_n) \quad \begin{matrix} \textit{diverges} \\ \textit{converges}. \end{matrix}$$

PROOF. The proof goes through exactly as before except that we use (v) instead of $x^2Q(x) \uparrow$ to show that for sufficiently large n

$$\sum_{k=1/Q(\gamma_n)}^{n-1} a_k^{-d} \le \frac{c}{\gamma_n^d Q(\gamma_n)}$$

and

$$K^{-d}Q(a_n/K) \le (\gamma_n/a_n)^d Q(\gamma_n)$$

if $\gamma_n \ge a_n/K$.

Under conditions (i)–(iv) condition (v) ensures that the random walk is transient. If we only assume that $x^dQ(x)$ increases then the random walk may be recurrent as can be seen by considering the Cauchy distribution in one dimension and the normal distribution in two dimensions.

Before giving the corresponding test for domains of attraction, we state a well known result concerning the centering terms b_n; see [4] page 580.

LEMMA 3.4. *Assume that X is in the domain of attraction of a stable law of index γ.*

(i) *If $\gamma < 1$ then we may take $b_n = 0$*

(ii) *If $\gamma > 1$ then we may take $b_n = nEX$.*

If $\gamma = 1$ the centering term may well be non-linear and we are unable to use Theorem 2.3. Also, in the case $\gamma > 1$, if $EX \neq 0$ then the Strong Law of Large Numbers gives the asymptotic growth of S_n and so in this case it is natural to assume $EX = 0$ or equivalently investigate the growth of $S_n - nEX$. Bearing these two facts in mind we have:

THEOREM 3.5. *Assume that X is in the domain of attraction of a genuinely d-dimensional type A stable law of index $\gamma < d$. Let $\{\gamma_n\}$ be any sequence that $\gamma_n \uparrow \infty$ and* $\limsup_{n\to\infty} \gamma_{2n}/\gamma_n < \infty$.

(i) *If $\gamma < 1$ or if $\gamma = 1$ and $b_n = 0$, then*

$$\liminf_{n\to\infty} \frac{|S_n|}{\gamma_n} = \begin{matrix} 0 \\ \infty \end{matrix} \quad \textit{according as} \quad \sum \left(\frac{\gamma_n}{a_n}\right)^d Q(\gamma_n) \quad \begin{matrix} \textit{diverges} \\ \textit{converges}. \end{matrix}$$

(ii) *If $\gamma > 1$ then*

$$\liminf_{n\to\infty} \frac{|S_n - nEX|}{\gamma_n} = \begin{matrix} 0 \\ \infty \end{matrix} \quad \textit{according as} \quad \sum \left(\frac{\gamma_n}{a_n}\right)^d Q(\gamma_n) \quad \begin{matrix} \textit{diverges} \\ \textit{converges}. \end{matrix}$$

PROOF. In [4] page 577, it is shown that if X is in the domain of attraction of a stable law of index γ then

$$\lim_{x\to\infty} \frac{K(x)}{G(x)} = \frac{\gamma}{2-\gamma}. \tag{3.4}$$

If $d \ge 2$, then since $\gamma < d$, (3.4) together with Lemma 2.4 of [9] show that there exist $\varepsilon > 0$ and $x_0 \ge 0$ such that $x^{d-\varepsilon}Q(x)$ increases for $x \ge x_0$.

With this the proof proceeds exactly as in Theorems 3.2 and 3.3 by using the estimate from Theorem 2.2.

In the case that X itself is a genuinely d-dimensional type A stable law of index $\gamma < d$, Theorem 3.5 can easily be seen to reduce to Takeuchi and Taylor's result mentioned in the introduction. This is because in that case $Q(x) \sim c/x^\gamma$.

One can also easily check that since $x^\gamma Q(x)$ is slowly varying, the index of the attracting law essentially determines the behaviour of S_n with respect to powers of n. More specifically, for every X in the domain of attraction of a type A stable law of index $\gamma < d$ the lower index δ, defined by (1.3), is given by γ^{-1}. This behaviour was previously noted by Erickson [2].

4. An example. In [5] it was shown that if Y is symmetric with distribution satisfying $P\{|Y| > x\} \sim x^{-1}H(x)$ where

$$H(x) = \frac{\exp\{\gamma(l_2x)(l_3x)\}}{\gamma(lx)^{\gamma-1}(l_3x)} \quad \text{for} \quad \gamma \in (0, \infty)$$

then $\delta_Y = \exp(-\gamma^{-1})$. We are using here the abbreviation lx for $\log x$, l_2x for loglog x etc.

We will now show that if X is symmetric with distribution satisfying $P\{|X| > x\} \sim x^{-1}L(x)$ where

$$L(x) = \exp\{(l_2x)(l_3x)(\gamma + \sin(l_3x))\} \quad \text{for} \quad \gamma \in (\sqrt{2}, \infty)$$

then $\delta_X = \exp\{-(\gamma - \sqrt{2})^{-1}\}$.

PROOF. It is easy to check that L is slowly varying and so X is in the domain of attraction of the Cauchy distribution. Since $\lim_{x\to\infty} K(x)/G(x) = 1$ we see that $Q(x) \sim 2x^{-1}L(x)$ and a simple computation then shows that $a_n \sim 2nL(n)$. Thus by Theorem 2.2, there exists λ_0 such that for all $\lambda \ge \lambda_0$

$$P\{|S_n| \le \lambda\} \approx \frac{\lambda}{nL(n)} \wedge 1. \tag{4.1}$$

For $\alpha \in (0, 1)$ set

$$\Delta_n = \textstyle\sum_{k=0}^{n-1} P\{|S_k| \le n^\alpha\}.$$

Then by (4.1) for sufficiently large n.

$$\sum_{k=n^\alpha/L(n^\alpha)}^{n-1} \left(\frac{n^\alpha}{kL(k)} \wedge 1\right) \le \Delta_n \le \frac{n^\alpha}{L(n^\alpha)} + \sum_{k=n^\alpha/L(n^\alpha)}^{n-1} \left(\frac{n^\alpha}{kL(k)} \wedge 1\right).$$

Since $L(n^\alpha) \sim L(n^\alpha/L(n^\alpha))$ we see that

$$\sum_{k=n^\alpha/L(n^\alpha)}^{n-1} \left(\frac{n^\alpha}{kL(k)} \wedge 1\right) \approx \sum_{k=n^\alpha/L(n^\alpha)}^{n-1} \frac{n^\alpha}{kL(k)}.$$

Let $0 < \beta_1 < \alpha < \beta_2 < 1$. Since $\gamma > \sqrt{2}$, L is increasing for large x and since L is slowly varying, $n^{\beta_1} \le n^\alpha/L(n^\alpha) \le n^{\beta_2}$ for large n. Thus there exist positive constants c_1 and c_2 (depending on β_1 and β_2) such that

$$\frac{c_1 n^\alpha \log n}{L(n^{\beta_2})} \le \sum_{k=n^\alpha/L(n^\alpha)}^{n-1} \frac{n^\alpha}{kL(k)} \le \frac{c_2 n^\alpha \log n}{L(n^{\beta_1})}$$

and since $(L(n^\alpha)/L(n^{\beta_1}))\log n \to \infty$ we have that

$$\frac{c_1 n^\alpha \log n}{L(n^{\beta_2})} \le \Delta_n \le \frac{c_2 n^\alpha \log n}{L(n^{\beta_1})}. \tag{4.2}$$

Next using the Mean Value Theorem one can easily show that for $\beta \in (0, 1)$

$$\frac{L(n)}{L(n^\beta)} \approx \exp\left\{(l_3n)\left(l\frac{1}{\beta}\right)[\gamma + \sin(l_3n) + \cos(l_3n)]\right\}. \tag{4.3}$$

Now let $\alpha < \exp\{-(\gamma - \sqrt{2})^{-1}\}$ and pick $\beta \in (\alpha, \exp\{-(\gamma - \sqrt{2})^{-1}\})$. Then by (4.1), (4.2) and (4.3) for any $K > 0$ there exists a positive constant c such that

$$\Delta_n^{-1} P\{|S_n| \le Kn^\alpha\} \le \frac{c}{n(ln)\exp\left\{(l_3 n)\left(l\frac{1}{\beta}\right)[\gamma + \sin(l_3 n) + \cos(l_3 n)]\right\}}.$$

and this gives rise to a convergent series. Thus by Theorem 2.3

$$\liminf_{n\to\infty} n^{-\alpha} |S_n| = \infty \quad \text{a.s.}$$

Now let $\alpha \in (\exp\{-(\gamma - \sqrt{2})^{-1}\}, 1)$ and pick $\beta \in (\exp\{-(\gamma - \sqrt{2})^{-1}\}, \alpha)$. Then by (4.1), (4.2) and (4.3) for any $K > 0$ there exists a positive constant c such that

$$\Delta_n^{-1} P\{|S_n| \le Kn^\alpha\} \ge \frac{c}{n(ln)\exp\left\{(l_3 n)\left(l\frac{1}{\beta}\right)[\gamma + \sin(l_3 n) + \cos(l_3 n)]\right\}}.$$

Since $\beta > \exp\{-(\gamma - \sqrt{2})^{-1}\}$ there is an $\varepsilon > 0$ such that

$$\left(l\frac{1}{\beta}\right)[\gamma + \sin(l_3 n) + \cos(l_3 n)] \le 1$$

if $l_3 n \in [n_k - \varepsilon, n_k + \varepsilon]$ where $n_k = 5\pi/4 + 2k\pi$ for $k = 0, 1, 2, \cdots$. Hence

$$\sum_n \Delta_n^{-1} P\{|S_n| \le Kn^\alpha\} \ge \sum_k \sum_{l_3 n = n_k - \varepsilon}^{l_3 n = n_k + \varepsilon} \frac{1}{n(ln)(l_2 n)} \approx \sum_k 2\varepsilon$$

which diverges and so by Theorem 2.3

$$\liminf_{n\to\infty} n^{-\alpha} |S_n| = 0 \quad \text{a.s.} \tag{4.4}$$

Clearly if $\alpha \ge 1$ then (4.4) follows immediately from the case just considered and the proof is completed.

If we now take X as in this example with $\gamma = \gamma_0$ and Y as in the example from [5] with $\gamma \in (\gamma_0 - \sqrt{2}, \gamma_0 - 1)$, then we see that $P\{|X| > x\} \ge P\{|Y| > x\}$ for large x, but $\delta_Y > \delta_X$. Further, by putting appropriate amounts of mass at the origin, we could ensure that $P\{|X| > x\} \ge P\{|Y| > x\}$ for all $x \ge 0$.

Shepp [10] showed that a similar phenomenon occurs when considering transience and recurrence of random walks. He exhibited symmetric random variables X and Y such that $P\{|X| > x\} \ge P\{|Y| > x\}$ for all $x \ge 0$ and yet X gives rise to a recurrent random walk while Y gives rise to a transient one. He went on to show that if the distribution of X is convex at infinity, see [10], then this could not happen. However this notion does not seem to be relevant in this problem since both of the distributions in the above example are convex at infinity.

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