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PROBABILITY ESTIMATES FOR THE SMALL DEVIATIONS OF d-DIMENSIONAL RANDOM WALK[1]

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Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables taking values in $\mathbb{R}^d$ and $S_n = X_1 + \cdots + X_n$. For a large class of distributions we obtain estimates for the probability that S_n is in a ball centered at the origin. Such an estimate would follow from a local limit theorem if X_1 were in the domain of attraction of a stable law.

1. Introduction. Let $\{X_k\}$ be a sequence of nondegenerate, independent, identically distributed, d-dimensional random variables and $S_n = \sum_{k=1}^n X_k$. The distribution function of X_1 will be denoted by F and its characteristic function by ϕ. X will be another random variable with the same distribution as X_1.

Our aim in this paper is to obtain asymptotic estimates for the probability that S_n is contained in a cube (or ball) centered at the origin. Such an estimate is available when X is in the domain of attraction of a stable law. This is a consequence of the local limit theorems obtained by several authors [1], [2], [7], [8], [9]. We will derive analogous estimates to these but in a more general setting. These new estimates should enable some results, previously known only for random variables in the domain of attraction of a stable law, to be extended to this more general setting. As one example of this, in a forthcoming paper [4] we will show how Takeuchi's integral test for lower envelopes of a symmetric stable random walk, [10], can be generalized to this new setting. (This test was previously unknown even for domains of attraction.)

To describe our results we must first introduce some notation. For $x > 0$ define

$$G(x) = P\{|X| > x\}, \quad K(x) = \frac{1}{x^2}\int_{|y|\le x} |y|^2\, dF(y)$$

$$Q(x) = G(x) + K(x) = E(x^{-1}|X| \wedge 1)^2. \tag{1.1}$$

One readily checks that Q is positive, continuous, decreasing, $Q(x) \to 0$ as $x \to \infty$ and $x^2Q(x)$ increases. Further Q is strictly decreasing on $[x_0, \infty)$ where $x_0 = \sup\{x: P\{0 < |X| \le x\} = 0\}$. As a consequence of this, there is an increasing function a_y defined for $y \ge 1/Q(x_0)$ by

$$Q(a_y) = \frac{1}{y}. \tag{1.2}$$

For convenience we define $a_y = a_{1/Q(x_0)}$ if $y \in [0, 1/Q(x_0)]$.

Our fundamental, underlying assumption will be that

$$\liminf_{x\to\infty} \frac{K(x)}{G(x)} > 0. \tag{1.3}$$

This is clearly much more general than X being in the domain of attraction of a stable law of index α, since in that case

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$$\lim_{x\to\infty}\frac{K(x)}{G(x)}=\frac{\alpha}{2-\alpha}$$

and a_n, defined by (1.2), would then be a correct normalizing sequence for convergence in distribution to the stable law.

We can now state the probability estimate in the symmetric case. Assume that X is spherically symmetric (i.e. the distribution of X is invariant under all rigid rotations about the origin) and (1.3) holds, then there exists λ_0 such that

$$P\{S_n\in C(0,\lambda)\}\approx\left(\frac{\lambda}{a_n}\wedge 1\right)^d \tag{1.4}$$

holds uniformly in λ for $\lambda\ge\lambda_0$, where $C(0,\lambda)$ is the cube of side length 2λ centered at 0 and $\alpha_n\approx\beta_n$ means that there exists $M>1$ such that $M^{-1}\le\alpha_n/\beta_n\le M$ for n sufficiently large. One can easily check that this follows from the local limit theorems in the case that X is in the domain of attraction of a stable law, see Theorem 1 of [9].

The condition that λ be sufficiently large arises essentially because we are not distinguishing between lattice and non-lattice valued random variables. For example, in the case of simple random walk the lower bound can not hold if $\lambda<1$ since $P\{|S_n|<1\}=0$ if n is odd.

The symmetry assumption can be weakened (see Theorem 3.6) but the estimate is not true under just the assumption (1.3), so some further conditions are needed. If one assumes only (1.3) then there appear to be two main reasons why the estimate may fail. The first is that in dimensions two and higher, the spread of S_n may vary greatly in different directions and this causes the estimate to be of the wrong order of magnitude. A good example of this is when X has independent, symmetric stable components, (Example 3.7). In Section 2 we will introduce two geometric conditions which arise naturally when considering domains of attraction of stable laws and which will prevent this from happening. The second reason is that even in one dimension there is a problem with the centering, that is, in order that the estimate hold, one may have to center the cubes at places other than the origin. This is the case with a stable subordinator of index α. Then the cubes need to be centered at $n^{1/\alpha}$ (Example 3.8). In order that the origin be the correct place to center we have to impose a restriction on the characteristic function, namely that $|\operatorname{Im}\phi(t)|=o(1-\operatorname{Re}\phi(t))$ as $t\to 0$. The problem of where to center in one dimension will be completely resolved in a forthcoming paper written in collaboration with Naresh Jain and William Pruitt [5].

Since submitting this paper, the author has received a preprint by Hall [11] in which he obtains bounds for the concentration function under assumption (1.3).

2. Geometric conditions. We will now introduce the two geometric conditions mentioned in the introduction and examine the relationship between them. Both conditions arise naturally when considering domains of attraction of stable laws. These conditions are also of importance to us outside of this context in that they enable us to derive a basic estimate on the characteristic function $\phi(t)$, for t near the origin, in terms of the function Q defined by (1.1), see Theorem 2.10.

DEFINITION. A d-dimensional random variable (distribution) satisfies the direction condition if for every $\sigma\in S^{d-1}$ there exist constants $c>0$ and R_0 such that for all $R\ge R_0$

$$\int_{|x|\le R}(x,\sigma)^2\,dF(x)\ge c\int_{|x|\le R}|x|^2\,dF(x)$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$.

Observe that this condition is always satisfied in one dimension. We will now show, by a compactness argument, that the constants $c>0$ and R_0 can in fact be chosen uniformly in $\sigma\in S^{d-1}$.

LEMMA 2.1. *If X satisfies the direction condition then there exists constants $c > 0$ and R_0 such that for all $R \geq R_0$ and all $\sigma \in S^{d-1}$*

$$\int_{|x| \leq R} (x, \sigma)^2 \, dF(x) \geq c \int_{|x| \leq R} |x|^2 \, dF(x).$$

PROOF. Observe that for $\sigma, \tau \in S^{d-1}$

$$\begin{aligned}\left| \int_{|x| \leq R} (x, \sigma)^2 \, dF(x) - \int_{|x| \leq R} (x, \tau)^2 \, dF(x) \right| &\leq \int_{|x| \leq R} |(x, \sigma)^2 - (x, \tau)^2| \, dF(x) \\ &\leq \int_{|x| \leq R} |(x, \sigma - \tau)| \, |(x, \sigma + \tau)| \, dF(x) \\ &\leq 2 |\sigma - \tau| \int_{|x| \leq R} |x|^2 \, dF(x).\end{aligned}$$

Thus we see that for each $\sigma \in S^{d-1}$ there is a neighbourhood $N(\sigma)$ of σ and constants $c(\sigma) > 0$ and $R(\sigma)$ such that for all $\tau \in N(\sigma)$ and all $R \geq R(\sigma)$

$$\int_{|x| \leq R} (x, \tau)^2 \, dF(x) \geq \frac{c(\sigma)}{2} \int_{|x| \leq R} |x|^2 \, dF(x).$$

Now cover S^{d-1} with the open sets $N(\sigma)$. By compactness we obtain a finite subcover $N(\sigma_1), \cdots, N(\sigma_n)$. Set $R_0 = \max_{1 \leq i \leq n} R(\sigma_i)$ and $c = \min_{1 \leq i \leq n} (c(\sigma_i)/2)$. Then for all $R \geq R_0$

$$\int_{|x| \leq R} (x, \sigma)^2 \, dF(x) \geq c \int_{|x| \leq R} |x|^2 \, dF(x)$$

for every $\sigma \in S^{d-1}$. $\square$

The direction condition enables us to derive the following estimate on the characteristic function.

LEMMA 2.2.
(i) *For every random variable X and for every $t \in \mathbb{R}^d \setminus \{0\}$,*

$$1 - \operatorname{Re} \phi(t) \leq 2Q\left(\frac{1}{|t|}\right).$$

(ii) *If X satisfies the direction condition and* $\liminf_{x \to \infty} K(x)/G(x) > 0$, *then there exist constants $c > 0$ and $t_0 > 0$ such that for all $0 < |t| \leq t_0$*

$$1 - \operatorname{Re} \phi(t) \geq cQ\left(\frac{1}{|t|}\right).$$

PROOF.

$$\begin{aligned}\text{(i)} \qquad 1 - \operatorname{Re} \phi(t) &= \int_{\mathbb{R}^d} [1 - \cos(x, t)] \, dF(x) \\ &\leq \frac{1}{2} \int_{|x|\,|t| \leq 1} (x, t)^2 \, dF(x) + 2 \int_{|x|\,|t| > 1} dF(x) \\ &\leq \frac{|t|^2}{2} \int_{|x|\,|t| \leq 1} |x|^2 \, dF(x) + 2G\left(\frac{1}{|t|}\right) \leq 2Q\left(\frac{1}{|t|}\right).\end{aligned}$$

(ii)
$$1 - \operatorname{Re}\phi(t) \ge \int_{|x|\,|t| \le 1} [1 - \cos(x, t)]\, dF(x)$$
$$\ge c_1 \int_{|x|\,|t| \le 1} (x, t)^2\, dF(x)$$
$$= c_1 |t|^2 \int_{|x|\,|t| \le 1} \left(x, \frac{t}{|t|}\right)^2 dF(x) \ge c_2 |t|^2 \int_{|x|\,|t| \le 1} |x|^2\, dF(x)$$

for $|t|$ sufficiently small by Lemma 2.1. Hence

$$1 - \operatorname{Re}\phi(t) \ge c_2 K\left(\frac{1}{|t|}\right) \ge cQ\left(\frac{1}{|t|}\right)$$

since $\liminf_{x\to\infty} K(x)/G(x) > 0$. □

The direction condition arises naturally when considering random variables in the domain of attraction of a normal law, indeed all such random variables satisfy the direction condition (Lemma 4.15 in [3]).

The second condition that we will consider, and which is satisfied by all random variables in the domain of attraction of a stable law of index $\alpha < 2$ (Lemma 4.14 in [3]), is the cone condition.

DEFINITION. A d-dimensional random variable satisfies the cone condition if for every $\sigma \in S^{d-1}$ there exist constants $c_1 > 0$, $c_2 > 0$, R_0 and $\theta < \pi$ such that for all $R \ge R_0$

$$P\{|X| > R\} \le c_1 P\{|X| > c_2 R, X \in K_\theta(\sigma)\}$$

where

$$K_\theta(\sigma) = \left\{x \in \mathbb{R}^d\colon \cos\frac{\theta}{2} < \frac{|(x, \sigma)|}{|x|}\right\}$$

is the union of two cones of angle θ with axes in the directions of σ and $-\sigma$.

Again this condition is always satisfied in one dimension and a compactness argument shows that all of the constants may be chosen uniformly in $\sigma \in S^{d-1}$.

LEMMA 2.3. *If X satisfies the cone condition then there exist constants $c_1 > 0$, $c_2 > 0$, R_0 and $\theta < \pi$ such that for all $\sigma \in S^{d-1}$ and all $R \ge R_0$*

$$P\{|X| > R\} \le c_1 P\{|X| > c_2 R, X \in K_\theta(\sigma)\}.$$

PROOF. Observe that if $0 < \theta_1 < \theta_2 < \pi$ then $K_{\theta_2}(\tau) \supset K_{\theta_1}(\sigma)$ for all $\tau \in S^{d-1}$ such that $(\sigma, \tau) > \cos \frac{1}{2}(\theta_2 - \theta_1)$.

Thus there is a neighbourhood $N(\sigma)$ of σ and constants $c_1(\sigma) > 0$, $c_2(\sigma) > 0$, $R(\sigma)$ and $\theta_2(\sigma) < \pi$ such that for all $R \ge R(\sigma)$ and all $\tau \in N(\sigma)$

$$P\{|X| > R\} \le c_1(\sigma) P\{|X| > c_2(\sigma) R, X \in K_{\theta_2(\sigma)}(\tau)\}.$$

Now cover S^{d-1} with the open sets $N(\sigma)$. By compactness there is a finite subcover $N(\sigma_1)$, $\cdots$, $N(\sigma_n)$. Set

$$c_1 = \max_{1\le i\le n} c_1(\sigma_i), \quad c_2 = \min_{i\le i\le n} c_2(\sigma_i)$$
$$R_0 = \max_{1\le i\le n} R(\sigma_i), \quad \theta = \max_{1\le i\le n} \theta_2(\sigma_i).$$

Then for all $\sigma \in S^{d-1}$ and all $R \ge R_0$

$$P\{|X| > R\} \le c_1 P\{|X| > c_2 R, X \in K_\theta(\sigma)\}. \;\square$$

DEFINITION. A random variable X is genuinely d-dimensional if the distribution of X is not supported on a (d-1)-dimensional hyperplane (not necessarily through the origin).

The cone condition is more intuitive geometrically than the direction condition. As well as being of interest in its own right, it can be used to verify the direction condition. We will show that if X is a genuinely d-dimensional random variable then the cone condition implies the direction condition under the assumption that $\liminf_{x\to\infty} K(x)/G(x) > 0$. We first need two preliminary lemmas. In these lemmas for $i = 1, 2$ and $x > 0$

$$G_i(x) = P\{Y_i > x\}, \quad K_i(x) = \frac{1}{x^2}\int_{|y|\le x} |y|^2\, dF_i(y), \quad Q_i(x) = G_i(x) + K_i(x)$$

where Y_1 and Y_2 are nonnegative real valued random variables with distribution functions F_1 and F_2 respectively. We will be assuming as always, that $Y_1 \not\equiv 0$ and $Y_2 \not\equiv 0$.

LEMMA 2.4. *Assume that there exist positive constants c_1, c_2, c_3, c_4 and $x_0 \ge 0$ such that $c_1 G_2(c_2 x) \le G_1(x) \le c_3 G_2(c_4 x)$ for all $x \ge x_0$. Then there exist positive constants c_5 and c_6 such that for all $x \ge 0$*

$$c_5 Q_2(x) \le Q_1(x) \le c_6 Q_2(x).$$

PROOF. We first observe that the relationship between G_1 and G_2 still holds if we replace c_2 by $c_2 \vee 1$ and c_4 by $c_4 \wedge 1$. Hence there is no loss of generality in assuming that $c_2 \ge 1$ and $c_4 \le 1$. Now for $x > x_0$

$$\begin{aligned} x^2 Q_1(x) &= \int_0^x 2sG_1(s)\, ds \le \int_0^{x_0} 2sG_1(s)\, ds + c_3 \int_{x_0}^x 2sG_2(c_4 s)\, ds \\ &\le x_0^2 + \frac{c_3}{c_4^2}\int_{c_4 x_0}^{c_4 x} 2sG_2(s)\, ds \le x_0^2 + \frac{c_3}{c_4^2}\int_0^x 2sG_2(s)\, ds \\ &= x_0^2 + \frac{c_3}{c_4^2} x^2 Q_2(x) \le \left(\frac{x_0^2}{x_0^2 Q_2(x_0)} + \frac{c_3}{c_4^2}\right) x^2 Q_2(x) \end{aligned}$$

since $x^2 Q_2(x)\uparrow$. If $0 \le x \le x_0$ then

$$Q_1(x) \le \frac{1}{Q_2(x_0)} Q_2(x)$$

since $Q_1(x) \le 1$ for all $x \ge 0$ and $Q_2(x)$ is nonincreasing for $x \ge 0$. Thus we have shown the existence of a positive constant c_6 such that for all $x \ge 0$

$$Q_1(x) \le c_6 Q_2(x).$$

The other inequality follows similarly by reversing the roles of G_1 and G_2 and the roles of Q_1 and Q_2. □

LEMMA 2.5. *Assume that there exist positive constants c_1, c_2, c_3, c_4 and $x_0 \ge 0$ such that*

$$c_1 G_2(c_2 x) \le G_1(x) \le c_3 G_2(c_4 x)$$

for all $x \ge x_0$ and that

$$\liminf_{x\to\infty} \frac{K_1(x)}{G_1(x)} > 0.$$

Then

$$\liminf_{x\to\infty} \frac{K_2(x)}{G_2(x)} > 0.$$

PROOF. By Lemma 2.4 of [6], there is an $\varepsilon > 0$ such that $x^{\varepsilon}Q_1(x)\downarrow$ for x sufficiently large.

By Lemma 2.4 there exist positive constants c_5 and c_6 such that for all $x \geq 0$

$$c_5 Q_2(x) \leq Q_1(x) \leq c_6 Q_2(x).$$

Pick $b < 1$ small enough that $b^{\varepsilon/2} < c_5/c_6$. For this choice of b pick c so that

$$\frac{2\dfrac{c}{b^2}}{1+\dfrac{c}{b^2}} = \frac{\varepsilon}{2}.$$

We claim that $\liminf_{x\to\infty} K_2(x)/G_2(x) \geq c$. If not there exists arbitrarily large y such that $K_2(y)/G_2(y) \leq c$. We fix such a y large enough that $x^{\varepsilon}Q_1(x)\downarrow$ for $x \geq by$. Since $x^2K_2(x)\downarrow$, if $x \in [by, y]$ we have that

$$\frac{K_2(x)}{G_2(x)} \leq \frac{y^2K_2(y)}{x^2G_2(x)} \leq \frac{1}{b^2}\frac{K_2(y)}{G_2(y)} \leq \frac{c}{b^2}.$$

Thus $x^{\varepsilon/2}Q_2(x)\uparrow$ for $x \in [by, y]$ by Lemma 2.4 of [6]. Hence

$$b^{\varepsilon/2}Q_2(by) \leq Q_2(y).$$

Since $x^{\varepsilon}Q_1(x)\downarrow$ on $[by, y]$ we see that

$$b^{\varepsilon}Q_1(by) \geq Q_1(y).$$

Thus

$$Q_2(by) \leq b^{-\varepsilon/2}Q_2(y) \leq \frac{b^{-\varepsilon/2}}{c_5}Q_1(y) \leq \frac{b^{\varepsilon/2}}{c_5}Q_1(by) < \frac{1}{c_6}Q_1(by) \quad \text{by choice of} \quad b.$$

But $c_6 Q_2(x) \geq Q_1(x)$ for all x which is a contradiction. □

LEMMA 2.6. *Assume that X is genuinely d-dimensional, satisfies the cone condition and* $\liminf_{x\to\infty} K(x)/G(x) > 0$. *Then X satisfies the direction condition.*

PROOF. Fix $\sigma \in S^{d-1}$. By the cone condition there exist constants $c_1 > 0$, $c_2 > 0$, R_0 and $\theta < \pi$, all depending on σ, such that for all $R \geq R_0$

$$P\{|X| > R\} \leq c_1 P\{|X| > c_2 R, X \in K_\theta(\sigma)\}.$$

Let $X^\sigma = X.1\{X \in K_\theta(\sigma)\}$ and $G^\sigma(R) = P\{|X^\sigma| > R\}$. Observe that, by increasing θ if necessary, we may assume that $X^\sigma \not\equiv 0$, since X is not concentrated on a lower dimensional subspace of $\mathbb{R}^d$. Further, if $R \geq R_0$

$$G^\sigma(R) \leq G(R) \leq c_1 G^\sigma(c_2 R).$$

Thus by Lemmas 2.4 and 2.5 there exist positive constants c_3, c_4 and R_1, depending on σ, such that for all $R \geq R_1$

$$K^\sigma(R) \geq c_3 G^\sigma(R)$$

and

$$Q(R) \leq c_4 Q^\sigma(R).$$

Hence for $R \geq R_1$

$$K(R) \leq Q(R) \leq c_4 Q^\sigma(R) \leq c_4\left(\frac{1}{c_3} + 1\right) K^\sigma(R).$$

Thus for $R \geq R_1$

$$\int_{|x|\le R} (x,\sigma)^2\,dF(x) \ge \int_{|x|\le R, x\in K_\theta(\sigma)} (x,\sigma)^2\,dF(x) \ge \cos^2\frac{\theta}{2}\int_{|x|\le R, x\in K_\theta(\sigma)} |x|^2\,dF(x)$$

$$\ge \frac{c_3\cos^2\theta/2}{c_4(1+c_3)}\int_{|x|\le R} |x|^2\,dF(x). \square$$

REMARK 1. The proof shows that we in fact do not need to assume that X is genuinely d-dimensional, but only need that X is not concentrated on a lower dimensional subspace of $\mathbb{R}^d$.

REMARK 2. The only way in which a random variable X can satisfy the cone condition but not be genuinely d-dimensional is if X has bounded range in $\mathbb{R}^d$.

Our aim in the remainder of this section is to derive an estimate similar to Lemma 2.2 involving $|\phi|$ instead of $\operatorname{Re}\phi$.

Let X_s denote the symmetrized random variable $X_1 - X_2$. We will use the notation G_s, K_s and Q_s where for example $G_s(x) = P\{|X_s| > x\}$.

LEMMA 2.7. *For any nondegenerate random variable X there exist positive constants c_1 and c_2 such that for all $x \geq 0$*

$$c_1 Q_s(x) \le Q(x) \le c_2 Q_s(x).$$

PROOF.

$$G_s(x) = P\{|X_s|>x\} = P\{|X_1-X_2|>x\}$$
$$\le P\left\{|X_1|>\frac{x}{2}\right\} + P\left\{|X_2|>\frac{x}{2}\right\} = 2G\left(\frac{x}{2}\right)$$
$$G_s(x) \ge P\{|X_1|\le M, |X_2|>x+M\} \ge \frac{1}{2}G(2x)$$

if M is fixed large enough that $P\{|X|\le M\} \ge ½$ and $x \ge M$. Hence for $x \ge 2M$

$$\frac{1}{2}G_s(2x) \le G(x) \le 2G_s\left(\frac{x}{2}\right).$$

The result now follows from Lemma 2.4 since $X_s \not\equiv 0$. $\square$

LEMMA 2.8. *Assume that X is genuinely d-dimensional, then for any $\sigma \in S^{d-1}$ there exist constants $c > 0$ and R_0 such that for all $R \ge R_0$*

$$\int_{|X_s|\le 2R} (X_s,\sigma)^2\,dP \ge c\int_{|X|\le R} (X,\sigma)^2\,dP.$$

PROOF. Fix $\sigma \in S^{d-1}$. If $E(X,\sigma)^2 < \infty$ then the result is immediate since $E(X_s,\sigma)^2 > 0$, (this is where we use that X is genuinely d-dimensional). Thus we may assume $E(X,\sigma)^2 = \infty$. Now

$$\int_{|X_1-X_2|\le 2R} (X_1-X_2,\sigma)^2\,dP \ge \int_{|X_1|\le R, |X_2|\le R} (X_1-X_2,\sigma)^2\,dP$$
$$= 2(1-G(R))\int_{|X|\le R} (X,\sigma)^2\,dP - 2\left(\int_{|X|\le R} (X,\sigma)\,dP\right)^2.$$

Thus it suffices to show that

$$\left(\int_{|X|\le R} (X,\sigma)\,dP\right)^2 = o\left(\int_{|X|\le R} (X,\sigma)^2\,dP\right) \quad \text{as} \quad R\to\infty.$$

Given $\eta \in (0,1)$ set

$$S = \eta\left(\int_{|X|\le R} (X,\sigma)^2\,dP\right)^{1/2}.$$

Then using Hölder's inequality we have

$$\begin{aligned}\left|\int_{|X|\le R} (X,\sigma)\,dP\right| &= \left|\int_{|X|\le S} (X,\sigma)\,dP + \int_{S<|X|\le R} (X,\sigma)\,dP\right| \\ &\le S + G(S)^{1/2}\left(\int_{|X|\le R} (X,\sigma)^2\,dP\right)^{1/2} \\ &\le \left(\int_{|X|\le R} (X,\sigma)^2\,dP\right)^{1/2} (\eta + G(S)^{1/2}).\end{aligned}$$

This inequality together with the observation that $S \to \infty$ as $R \to \infty$ completes the proof. □

LEMMA 2.9. *Assume that X is nondegenerate and* $\liminf_{x\to\infty} K(x)/G(x) > 0$, *then* $\liminf_{x\to\infty} K_s(x)/G_s(x) > 0$. *If in addition X is genuinely d-dimensional and satisfies the direction condition, then X_s also satisfies the direction condition.*

PROOF. Assume that $\liminf_{x\to\infty} K(x)/G(x) > 0$. In the proof of Lemma 2.7 we saw that for x sufficiently large

$$\frac{1}{2} G_s(2x) \le G(x) \le 2G_s\left(\frac{x}{2}\right).$$

Thus by Lemma 2.5

$$\liminf_{x\to\infty} \frac{K_s(x)}{G_s(x)} > 0.$$

Now assume that in addition X is genuinely d-dimensional and satisfies the direction condition. Fix $\sigma \in S^{d-1}$. Then using Lemmas 2.7 and 2.8, we see that for sufficiently large R

$$\begin{aligned}\int_{|X_1-X_2|\le R} (X_1 - X_2, \sigma)^2\,dP &\ge c_1 \int_{|X|\le R/2} (X,\sigma)^2\,dP \\ &\ge c_2 \int_{|X|\le R/2} |X|^2\,dP \ge c_3 (R/2)^2 Q(R/2) \ge \frac{c_3}{4} R^2 Q(R) \\ &\ge c_4 R^2 Q_s(R) \ge c_4 \int_{|X_1-X_2|\le R} |X_1 - X_2|^2\,dP.\end{aligned}$$

Thus X_s satisfies the direction condition. □

REMARK. In all cases of interest to us, X satisfying the direction condition forces X to be genuinely d-dimensional. In particular if $E|X|^2 = \infty$ and X satisfies the direction condition then X must be genuinely d-dimensional. Of course if $E|X|^2 < \infty$, then X is in the domain of attraction of a normal law and the estimate (1.4) is well known in that case.

THEOREM 2.10. *Assume that X is genuinely d-dimensional, satisfies the direction condition and* $\liminf_{x\to\infty} K(x)/G(x) > 0$. *Then*

$$1 - |\phi(t)| \approx Q\left(\frac{1}{|t|}\right) \quad \text{as} \quad t \to 0.$$

PROOF. By Lemmas 2.2 and 2.9

$$1 - |\phi(t)|^2 \approx Q_s\left(\frac{1}{|t|}\right)$$

since $|\phi|^2$ is the characteristic function of X_s. The result now follows from Lemma 2.7 and the observation that

$$1 - |\phi(t)|^2 = (1 - |\phi(t)|)(1 + |\phi(t)|) \approx 1 - |\phi(t)|. \qquad \square$$

3. The probability estimates. In this section we will prove that under certain regularity conditions (see Theorem 3.6) the fundamental estimate

$$P\{|S_n| \le \lambda\} \approx \left(\frac{\lambda}{a_n} \wedge 1\right)^d$$

holds uniformly in λ for λ sufficiently large.

The proof will involve the use of the following inversion formula,

$$\begin{aligned}\frac{1}{(2\lambda)^d}\int_{\mathbb{R}^d} P\{S_n \in C(x-y,\lambda)\} a^d H(ay)\,dy \\ = \frac{1}{(2\pi)^d}\int_{C(0,a)} e^{-\iota(t,x)}\phi^n(t)k(\lambda t)h(a^{-1}t)\,dt\end{aligned} \tag{3.1}$$

for $a > 0$ and $\lambda > 0$ where

$$H(y) = \textstyle\prod_{i=1}^d \dfrac{1\text{-}\cos y_i}{\pi y_i^2}, \quad y = (y_1, \cdots, y_d)$$

$$h(t) = \textstyle\prod_{i=1}^d (1 - |t_i|)^+, \quad t = (t_1, \cdots, t_d)$$

$$k(t) = \textstyle\prod_{i=1}^d \dfrac{\sin t_i}{t_i}$$

and

$$C(y, a) = \{z \in \mathbb{R}^d: |y_i - z_i| \le a \quad \text{for} \quad i = 1, \cdots, d\}.$$

Since it will be needed in the next lemma, we define

$$B(y, a) = \{z \in \mathbb{R}^d: |y - z| \le a\}.$$

In what follows we will be assuming that $\liminf_{x\to\infty} K(x)/G(x) > 0$. Thus by Lemma 2.4 of [6] there exists $A > 0$ and $\varepsilon > 0$ such that $x^\varepsilon Q(x)$ decreases for $x \ge A$. To put this another way, $r^{-\varepsilon}Q(1/r)$ increases for $r \in (0, A^{-1})$. We will fix such an ε and an $a \in (0, A^{-1}d^{-1/2})$ small enough that the estimate from Lemma 2.10 holds.

LEMMA 3.1. *Assume that X is genuinely d-dimensional, satisfies the direction condition and* $\liminf_{x\to\infty} K(x)/G(x) > 0$. *Then there exist positive constants* c_1 *and* c_2 *such that for every* $p \in (0, 1]$ *there exists an* n_0 *such that for all* $n \ge n_0$ *and all* $\lambda > 0$

$$\int_{C(0,a)\setminus C(0,a_{pn}^{-1})} |\phi^n(t)|\,dt \le c_1 \frac{1}{a_n^d}\int_{(c_2/p)^{1/\varepsilon}}^\infty e^{-r^\varepsilon} r^{d-1}\,dr.$$

PROOF. Fix $p \in (0, 1]$. Choose n_0 large enough that $a_{pn_0}^{-1} < a$. Then for all $n \ge n_0$

$$\begin{aligned}
\int_{C(0,a)\setminus C(0,a_{pn}^{-1})} |\phi^n(t)|\,dt &\le \int_{C(0,a)\setminus C(0,a_{pn}^{-1})} e^{-n(1-|\phi(t)|)}\,dt \\
&\le \int_{C(0,a)\setminus C(0,a_{pn}^{-1})} e^{-c_2 n Q(1/|t|)}\,dt \quad \text{by Lemma 2.10} \\
&\le \int_{B(0,ad^{1/2})\setminus B(0,a_{pn}^{-1})} e^{-c_2 n Q(1/|t|)}\,dt \\
&= c_3 \int_{a_{pn}^{-1}}^{ad^{1/2}} e^{-c_2 n Q(1/r)} r^{d-1}\,dr \quad \text{(spherical coordinates)} \\
&\le c_3 \int_{a_{pn}^{-1}}^{ad^{1/2}} e^{-c_2 n a_{pn}^{\varepsilon} Q(a_{pn}) r^{\varepsilon}} r^{d-1}\,dr \quad \text{since} \quad r^{-\varepsilon} Q\left(\frac{1}{r}\right)\uparrow \\
&= c_3 \int_{a_{pn}^{-1}}^{ad^{1/2}} e^{-c_2 (1/p) a_{pn}^{\varepsilon} r^{\varepsilon}} r^{d-1}\,dr \\
&\le c_1 \left(\frac{p^{1/\varepsilon}}{a_{pn}}\right)^d \int_{(c_2/p)^{1/\varepsilon}}^{\infty} e^{-r^{\varepsilon}} r^{d-1}\,dr \\
&\le \frac{c_1}{a_n^d} \int_{(c_2/p)^{1/\varepsilon}}^{\infty} e^{-r^{\varepsilon}} r^{d-1}\,dr \quad \text{since} \quad x^{\varepsilon} Q(x)\downarrow. \square
\end{aligned}$$

LEMMA 3.2. *Assume that X is genuinely d-dimensional, satisfies the direction condition and* $\liminf_{x\to\infty} K(x)/G(x) > 0$. *Then there exists a positive constant c such that for all n, all* $\lambda > 0$ *and all* $x \in \mathbb{R}^d$

$$\left| \int_{C(0,a)} e^{-\iota(t,x)} \phi^n(t) k(\lambda t) h(a^{-1}t)\,dt \right| \le \frac{c}{a_n^d}.$$

PROOF. By Lemma 3.1 there exist constants c_1 and n_0 such that for all $n \ge n_0$

$$\text{L.H.S.} \le \int_{C(0,a)\setminus C(0,a_n^{-1})} |\phi^n(t)|\,dt + \int_{C(0,a_n^{-1})} dt \le \frac{c_1}{a_n^d} + \frac{2^d}{a_n^d}.$$

For $n < n_0$ just choose c large enough that the estimate holds for each individual n. □

We will now obtain a comparable lower bound for the integral, but first we need,

LEMMA 3.3. *Assume that* $|\mathrm{Im}\,\phi(t)| = o(1 - \mathrm{Re}\,\phi(t))$ *as* $t \to 0$. *Then for all* $p > 0$ *there is an* n_0 *such that for all* $n \ge n_0$ *and all* $t \in C(0, a_{pn}^{-1})$

$$\mathrm{Re}\,\phi^n(t) \ge \tfrac{1}{2}|\phi^n(t)|.$$

PROOF. Write $\phi(t) = |\phi(t)| e^{\iota\theta(t)}$. Then

$$|\mathrm{Im}\,\phi(t)| = |\phi(t)||\sin\theta(t)| \sim |\phi(t)||\theta(t)| \quad \text{as} \quad t \to 0.$$

Thus we see that $|\theta(t)| = o(1 - \mathrm{Re}\,\phi(t))$ and so we may write

$$\theta(t) = \varepsilon(t)(1 - \mathrm{Re}\,\phi(t))$$

where $\varepsilon(t) \to 0$ as $t \to 0$. Now

$$\mathrm{Re}\,\phi^n(t) = |\phi^n(t)|\cos n\theta(t) = |\phi^n(t)|\cos[n\varepsilon(t)(1 - \mathrm{Re}\,\phi(t))].$$

But by Lemma 2.2 for all $t \in C(0, a_{pn}^{-1})$

$$1 - \operatorname{Re} \phi(t) \le 2Q\left(\frac{1}{|t|}\right) \le 2Q(d^{-1/2}a_{pn}) \quad \text{since} \quad Q(x)\downarrow$$
$$\le \frac{2d}{pn} \qquad\qquad \text{since} \quad x^2Q(x)\uparrow.$$

Hence if we choose n_0 large enough that $(2d|\varepsilon(t)|/p) \le (\pi/3)$ whenever $t \in C(0, a_{pn_0}^{-1})$, we see that for all $n \ge n_0$ and all $t \in C(0, a_{pn}^{-1})$

$$\operatorname{Re} \phi^n(t) \ge \tfrac{1}{2}|\phi^n(t)|. \square$$

LEMMA 3.4. *Assume that X is genuinely d-dimensional, satisfies the direction condition,* $\liminf_{x\to\infty} K(x)/G(x) > 0$ *and* $|\operatorname{Im} \phi(t)| = o(1 - \operatorname{Re} \phi(t))$. *Then for every* $p \in (0, 1]$ *there is an* n_0 *such that for all* $n \ge n_0$ *and all* $\lambda \in (0, a_{pn}]$

$$\int_{C(0,a_{pn}^{-1})} \operatorname{Re} \phi^n(t) k(\lambda t) h(a^{-1}t)\, dt \ge \frac{c}{a_n^d}$$

where c is independent of λ, p *and n.*

PROOF. Fix $p \in (0, 1]$ and choose n_0 large enough that for all $n \ge n_0$ and all $t \in C(0, a_{pn}^{-1})$,

$$|\phi(t)| \ge \tfrac{1}{2}, \quad \operatorname{Re} \phi^n(t) \ge \tfrac{1}{2}|\phi^n(t)| \quad \text{and} \quad h(a^{-1}t) \ge \tfrac{1}{2}.$$

This choice is possible by Lemma 3.3 Since $k(\lambda t) \ge (\sin 1)^d$ for $t \in C(0, a_{pn}^{-1})$ if $\lambda \le a_{pn}$, we see that for all $n \ge n_0$

$$\int_{C(0,a_{pn}^{-1})} \operatorname{Re} \phi^n(t)k(\lambda t)h(a^{-1}t)\, dt \ge c_1 \int_{C(0,a_{pn}^{-1})} \operatorname{Re} \phi^n(t)\, dt \ge c_2 \int_{C(0,a_{pn}^{-1})} |\phi^n(t)|\, dt$$
$$\ge c_2 \int_{B(0,a_{pn}^{-1})} e^{-c_3 n(1-|\phi(t)|)}\, dt$$
$$\ge c_2 \int_{B(0,a_{pn}^{-1})} e^{-c_4 n Q(1/|t|)}\, dt$$
$$= c_5 \int_0^{a_{pn}^{-1}} e^{-c_4 n Q(1/r)} r^{d-1}\, dr$$
$$\ge c_5 \int_0^{a_n^{-1}} e^{-c_4 n Q(1/r)} r^{d-1}\, dr$$
$$\ge \frac{c}{a_n^d}$$

since $0 < Q(1/r) \le 1/n$ for $0 < r \le a_n^{-1}$. $\square$

LEMMA 3.5. *Assume that X is genuinely d-dimensional, satisfies the direction condition,* $\liminf_{x\to\infty} K(x)/G(x) > 0$ *and* $|\operatorname{Im} \phi(t)| = o(1 - \operatorname{Re} \phi(t))$. *Then there is a* $p \in (0, 1]$ *and an* n_0 *such that for all* $n \ge n_0$ *and all* $\lambda \in (0, a_{pn}]$

$$\int_{C(0,a)} \phi^n(t)k(\lambda t)h(a^{-1}t)\, dt \ge \frac{c}{a_n^d}$$

where c is independent of λ *and n.*

PROOF. Since the L.H.S. of (3.1) is real, the above integral is also real. Thus by Lemmas 3.1 and 3.4, there is a $p \in (0, 1]$ and an n_0 such that for all $n \geq n_0$ and all $\lambda \in (0, a_{pn}]$

$$\begin{aligned}\int_{C(0,a)} \phi^n(t)k(\lambda t)h(a^{-1}t)\,dt &= \int_{C(0,a)} \operatorname{Re}\phi^n(t)k(\lambda t)h(a^{-1}t)\,dt \\ &\geq \int_{C(0,a_{pn}^{-1})} \operatorname{Re}\phi^n(t)k(\lambda t)h(a^{-1}t)\,dt \\ &\quad - \int_{C(0,a)\setminus C(0,a_{pn}^{-1})} |\phi^n(t)|\,dt \\ &\geq \frac{c_1}{a_n^d} - \frac{c_2}{a_n^d}\int_{(c_3/p)^{1/\varepsilon}}^{\infty} e^{-r^\varepsilon} r^{d-1}\,dr \\ &\geq \frac{c}{a_n^d}\end{aligned}$$

if p is chosen small enough. □

With the estimates from Lemmas 3.2 and 3.5 we are now ready to prove our main result. The hypotheses of the theorem include the direction condition but of course, in view of Lemma 2.6, the result remains valid if this is replaced by the cone condition.

THEOREM 3.6. *Assume that X is genuinely d-dimensional, satisfies the direction condition and* $\liminf_{x\to\infty} K(x)/G(x) > 0$. *Then there exist positive constants* c_1 *and* λ_0 *such that for all* $\lambda \geq \lambda_0$, *all n and all* $x \in \mathbb{R}^d$

$$P\{S_n \in C(x,\lambda)\} \leq c_1\left(\frac{\lambda}{a_n} \wedge 1\right)^d.$$

If in addition $|\operatorname{Im}\phi(t)| = o(1 - \operatorname{Re}\phi(t))$ *then there exist further positive constants* c_2 *and* λ_1 *such that for all* $\lambda \geq \lambda_1$ *and all n*

$$P\{S_n \in C(0,\lambda)\} \geq c_2\left(\frac{\lambda}{a_n} \wedge 1\right)^d.$$

PROOF. We first observe that for $h > 0$

$$\begin{aligned}\int_{\mathbb{R}^d\setminus C(0,h)} a^d H(ay)\,dy &= \left(\frac{a}{\pi}\right)^d \int_{\mathbb{R}^d\setminus C(0,h)} \prod_{i=1}^d \frac{(1-\cos ay_i)}{a^2y_i^2}\,dy \\ &\leq \left(\frac{a}{\pi}\right)^d \left(\int_h^\infty \frac{4}{a^2z^2}\,dz\right)^d \leq \frac{4^d}{(ah\pi)^d}.\end{aligned}$$

Thus by picking h large enough we can make

$$\int_{\mathbb{R}^d\setminus C(0,h)} a^d H(ay)\,dy \leq \frac{1}{2}.$$

For such an h, if $\lambda > h$ then

$$C(x, \lambda - h) \subseteq C(x - y, \lambda)$$

for all $y \in C(0, h)$. Thus for all $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} P\{S_n \in C(x-y, \lambda)\} a^d H(ay)\, dy \geq \int_{C(0,h)} P\{S_n \in C(x, \lambda - h)\} a^d H(ay)\, dy$$

$$\geq \frac{1}{2} P\{S_n \in C(x, \lambda - h)\}.$$

Thus by (3.1) and Lemma 3.2, for all $\lambda > h$ and all n

$$P\{S_n \in C(x, \lambda - h)\} \leq c_3 \left(\frac{\lambda}{a_n}\right)^d.$$

Upon replacing λ by $\lambda' + h$ we see that for all $\lambda' > 0$

$$P\{S_n \in C(x, \lambda')\} \leq c_3 \left(\frac{\lambda' + h}{a_n}\right)^d.$$

Thus if we assume $\lambda' > h$ we have for all n and all $x \in \mathbb{R}^d$

$$P\{S_n \in C(x, \lambda')\} \leq c_2 \left(\frac{\lambda'}{a_n}\right)^d.$$

For the lower bound we observe that for $y \in C(0, h)$ we have $C(x, \lambda + h) \supseteq C(x - y, \lambda)$. Thus

$$\int_{\mathbb{R}^d} P\{S_n \in C(x-y, \lambda)\} a^d H(ay)\, dy \leq \int_{C(0,h)} P\{S_n \in C(x, \lambda + h)\} a^d H(ay)\, dy$$

$$+ \int_{\mathbb{R}^d \setminus C(0,h)} P\{S_n \in C(x-y, \lambda)\} a^d H(ay)\, dy.$$

Now set $x = 0$. By using Lemma 3.5 and the upper bound just derived we see that there exists n_0, λ_1 and $p \in (0, 1)$ such that for all $n \geq n_0$ and all $\lambda \in [\lambda_1 \vee h, a_{pn}]$

$$c_4 \left(\frac{\lambda}{a_n}\right)^d \leq P\{S_n \in C(0, \lambda + h)\} + c_2 \left(\frac{\lambda}{a_n}\right)^d \frac{4^d}{(ah\pi)^d}.$$

By choosing h sufficiently large we see that there is a $\lambda_0 > 0$ and an n_0 such that for all $n \geq n_0$ and all $\lambda \in [\lambda_0, a_{pn}]$

$$P\{S_n \in C(0, \lambda)\} \geq c_5 \left(\frac{\lambda}{a_n}\right)^d.$$

In order to remove the condition $\lambda \leq a_{pn}$ we only need observe that if $\lambda \geq a_{pn}$ then, using the fact that $x^\varepsilon Q(x)\downarrow$ for sufficiently large x, there is an n_1 such that for all $n \geq n_1$

$$P\{S_n \in C(0, \lambda)\} \geq P\{S_n \in C(0, a_{pn})\} \geq c_5 \left(\frac{a_{pn}}{a_n}\right)^d \geq c_5 p^{d/\varepsilon} \approx 1.$$

It is clear that by choosing λ_0 sufficiently large and adjusting the constant we may remove the condition that n be large. □

Theorem 3.6 does not apply to all random variables in the domain of attraction of a stable law, only to those for which $|\operatorname{Im} \phi(t)| = o(1 - \operatorname{Re} \phi(t))$. However it does apply to many random variables which lie outside of the domain of attraction of any stable law and this is the most important aspect of the result.

As we remarked in the introduction, Theorem 3.6 is not necessarily true if we drop either the assumption that X satisfy the direction condition or that $|\operatorname{Im} \phi(t)| = o(1 - \operatorname{Re} \phi(t))$.

EXAMPLE 3.7. Let $\{Y_k\}$ and $\{Z_k\}$ be sequences of i.i.d. symmetric, stable random variables of indices α and β respectively where $\alpha < \beta$. Further assume that the sequences $\{Y_k\}$ and $\{Z_k\}$ are independent of each other. Set $U_n = \sum_{k=1}^n Y_k$ and $V_n = \sum_{k=1}^n Z_k$. Observe that $X_k = (Y_k, Z_k)$ is radically symmetric and hence its characteristic function is real-valued. Set $S_n = \sum_{k=1}^n X_k = (U_n, V_n)$. Then using the well known estimates

$$P\{|U_n| \le \lambda\} \approx \left(\frac{\lambda}{n^{1/\alpha}} \wedge 1\right) \quad \text{and} \quad P\{|V_n| \le \lambda\} \approx \left(\frac{\lambda}{n^{1/\beta}} \wedge 1\right),$$

which follow from Theorem 3.6, we see that

$$P\{S_n \in C(0, \lambda)\} \approx \begin{cases} \dfrac{\lambda^2}{n^{1/\alpha}n^{1/\beta}} & \text{if } 0 \le \lambda \le n^{1/\beta} \\ \dfrac{\lambda}{n^{1/\alpha}} & \text{if } n^{1/\beta} \le \lambda \le n^{1/\alpha} \\ 1 & \text{if } \lambda \ge n^{1/\alpha}. \end{cases}$$

On the other hand it is not too hard to see that $\lim \inf_{x\to\infty} K(x)/G(x) > 0$ and $a_n \approx n^{1/\alpha}$ in this example. Thus the conclusion of Theorem 3.6 fails. The reason for this is that X_1 fails to satisfy the direction condition.

EXAMPLE 3.8. Let X be a one sided stable random variable of index $\alpha \in (0, 1)$. Then it is easy to see that $P\{|S_n| \le \lambda\} \to 0$ with at least a geometric rate since

$$P\{|S_n| \le \lambda\} = P\{0 \le S_n \le \lambda\} \le (P\{0 \le X \le \lambda\})^n.$$

However $\lim_{x\to\infty} K(x)/G(x) = \alpha/(2 - \alpha)$ and $a_n \approx n^{1/\alpha}$. The reason that this example does not violate Theorem 3.6 is that $|\operatorname{Im} \phi(t)| \ne o(1 - \operatorname{Re} \phi(t))$.

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