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WEAK CONVERGENCE OF THE WEIGHTED EMPIRICAL QUANTILE PROCESS IN $L^2(0, 1)$

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Sufficient conditions are developed for various versions of the weighted empirical quantile process to converge weakly in $L^2(0, 1)$ to a weighted Brownian bridge. The results are directly applicable to the derivation of the asymptotic distribution of goodness of fit tests based on the sample quantiles that can be written as a functional defined on $L^2(0, 1)$ continuous in the norm topology. In the process, tight bounds for the moments of transformed uniform order statistics are derived that are likely to have applications elsewhere.

1. Introduction. Let $X_1, \cdots, X_n$ be independent identically distributed random variables with common distribution function F, and $X_{1,n} \leq \cdots \leq X_{n,n}$ denote their order statistics. Q will denote the quantile function of F defined on $(0, 1)$ ($Q(u) = \inf\{x: F(x) \geq u\}$ for $u \in (0, 1)$). It will be assumed throughout this paper that

(A) F has a continuous density quantile function $f(Q(u))$ defined on $(0, 1)$. (f denotes the density of F.) We will write $h(u) = 1/f(Q(u))$; h is sometimes called the quantile density function. (See Parzen, 1979.)

Define the empirical quantile function on $(0, 1)$ to be $Q_n(u) = X_{i,n}$ whenever $(i-1)/n \leq u < i/n$ for some $1 \leq i \leq n$. We will consider the following three versions of the "weighted empirical quantile process": Let w be any measurable real valued function defined on $(0, 1)$. Set

(I) $r_n^w(u) = n^{1/2}w(u)(Q_n(u) - Q(u))$ for $u \in (1/(n+1), n/(n+1))$ and equal to zero elsewhere; set

(II) $q_n^w(u) = n^{1/2}w(([nu]+1)/(n+1))\{Q_n(u) - Q(([nu]+1)/(n+1))\}$ for $u \in (0, 1)$ ($[x]$ denotes the greatest integer $\leq x$.); and whenever

(B) there exist positive integers k_1 and k_2 such that $EX_{i,n}^2 < \infty$ for every $k_1 \leq i \leq n + 1 - k_2$ for all n sufficiently large, set

(III) $p_n^w(u) = n^{1/2}w(([nu]+1)/(n+1))(Q_n(u) - EQ_n(u))$ for $(k_1 - 1)/n < u < (n - k_2 + 1)/n$ and equal to zero elsewhere.

((B) holds if and only if $E|X|^\delta < \infty$ for some $\delta > 0$, see Mason (1982) and Anderson (1982).)

p_n, q_n, and r_n will denote the processes with the particular choice $w(u) = f(Q(u))$.

Recently there has been considerable interest in determining conditions on w and h under which the process r_n^w converges weakly to a continuous process whB

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defined on [0, 1], where B denotes a Brownian bridge on [0, 1], usually with w specified to be $f(Q(u))$. Refer to Shorack (1972, 1982), Stute (1982), M. Csörgő and Révész (1978), S. Csörgő (1982), and M. Csörgő, S. Csörgő, Horváth, and Révész (1982). We will be concerned here with developing general conditions on w and h that will insure that p_n^w, q_n^w, and r_n^w converge weakly in $L^2(0, 1)$ to the process whB, where w and h are now chosen so that $whB \in L^2(0, 1)$ with probability 1.

One approach is as follows: Let $\mathscr{Q}$ denote the class of positive continuous functions defined on (0, 1), symmetric about ½, nondecreasing on (0, ½) and nonincreasing on (½, 1) such that

$$\lim_{u \downarrow 0} q(u)/\psi(u) = \infty,$$

where $\psi(u) = (u \ln\ln(1/u))^{1/2}$. Suppose $f(Q(u))$ is such that for some $q \in \mathscr{Q}$

$$\text{(K)} \qquad \sup\{|r_n(u) - B(u)|/q(u) : 1/(n+1) \le u \le n/(n+1)\} \to_p 0$$

for the Skorohod (1956) construction. If w is chosen so that

$$\text{(L)} \qquad \int_0^1 w^2(u) h^2(u) q^2(u)\, du < \infty,$$

an elementary argument shows that

$$\int_0^1 (r_n^w(u) - w(u)h(u)B(u))^2\, du \to_P 0,$$

from which we immediately conclude that r_n^w converges weakly to whB in $L^2(0, 1)$. Shorack (1982) and M. Csörgő et al. (1982) have developed conditions on h in relation to given choices of q that insure (K). (M. Csörgő et al. (1982) actually consider the analogue of (K) based on the Komlós, Major, and Tusnády (1975) construction.)

In this paper, we employ an alternate approach to weak convergence of versions of the weighted empirical quantile process in $L^2(0, 1)$ *independent* of the special constructions of the type stated in (K). In the process, we will arrive at a different and perhaps more easily verifiable set of sufficient conditions for L^2 convergence than the conditions derived by Shorack (1982) and M. Csörgő et al. (1982) for (K).

2. Weak convergence of the processes p_n^w, q_n^w and r_n^w to whB in $L^2(0, 1)$. We will first consider the process p_n^w.

Let $\mathscr{A}$ denote the Hilbert space $L^2(0, 1)$, where the inner product $\langle \cdot, \cdot \rangle$ is defined as usual to be

$$\langle f, g \rangle = \int_0^1 f(u)\, g(u)\, du \quad \text{for } f \text{ and } g \in \mathscr{A}.$$

$\mathscr{B}$ will denote the Borel sets generated by the norm topology on $\mathscr{A}$, where the norm is defined to be

$$\| f \| = (\langle f, f \rangle)^{1/2} \text{ for } f \in \mathscr{A}.$$

For any $0 \le \varepsilon < 1/2$, let $i_\varepsilon(u) = 1$ or 0 accordingly as $u \in [\varepsilon, 1-\varepsilon]$ or not, and set $\bar{\imath} = 1 - i_\varepsilon$. In particular, notice that $\langle i_0, p_n^w \rangle$ equals the normalized linear combination of order statistics

$$\textstyle\sum_{j=k_1}^{n+1-k_2} w(j/(n+1))(X_{j,n} - EX_{j,n})/\sqrt{n}.$$

Associated with a weight function w and a quantile density function h, we define a kernel $K^{h,w}$ as follows:

$$K^{h,w}(u, v) = (u \wedge v - uv)h(u)h(v)w(u)w(v) \quad \text{for} \quad (u, v) \in (0, 1) \times (0, 1).$$

(When h and w are understood, the superscripts will be deleted.)

Under assumption (A), and

(C) w is chosen so that

$$\int_0^1 u(1-u)w^2(u)h^2(u)\,du < \infty,$$

$K^{h,w}$ defines a symmetric, positive semidefinite linear operator on $\mathscr{A}$ with finite trace, that is, for f and $g \in \mathscr{A}$:

$$(1) \quad Kf(u) = \int_0^1 (u \wedge v - uv)h(u)h(v)w(u)w(v)f(v)\,dv \quad \text{for} \quad u \in (0, 1).$$

$$(2) \qquad \langle Kf, g \rangle = \langle Kg, f \rangle,$$

$$(3) \qquad \langle Kf, f \rangle \ge 0,$$

and

$$(4) \qquad \text{Trace } K = \int_0^1 K(u, u)\,du < \infty.$$

Choose any orthonormal basis $\{e_m\}_{m=1}^\infty$ for $\mathscr{A}$. It is easy to show that

$$(5) \qquad \textstyle\sum_{m=1}^\infty K(e_m, e_m) = \text{Trace } K.$$

μ_n will denote the probability measure induced by the process p_n^w on $(\mathscr{A}, \mathscr{B})$. μ_n is completely determined by the characteristic functional defined for $f \in \mathscr{A}$ to be

$$\hat{\mu}_n(f) = E \exp(i \langle f, p_n^w \rangle).$$

(Refer to pages 339–340 of Gihman and Skorohod, 1974.)

The following theorem establishes that under very general conditions the sequence of measures μ_n converges weakly to the probability measure μ on $(\mathscr{A}, \mathscr{B})$ induced by the process whB.

THEOREM 1. *In addition to assumptions* (A), (B), *and* (C), *assume that* $w \ge 0$;

(D) *w is continuous almost everywhere on* $(0, 1)$ *and for every* $0 < \delta < 1/2$, *w is bounded on* $(\delta, 1-\delta)$;

(E) $E\|p_n^w\|^2 \to \text{Trace } K$; *and*

(F) $E\langle i_0, p_n^w\rangle^2 \to \langle K^{h,w} i_0, i_0\rangle < \infty$.
Then μ_n *converges weakly to the probability measure* μ ($p_n^w \Rightarrow whB$).

Proof. We will first show that the sequence of measures $\{\mu_n\}$ is tight. It is enough to establish that for any orthonormal basis $\{e_m\}_{m=1}^{\infty}$ of $\mathscr{A}$

$$\lim_{k\to\infty}\sup_{n\geq 1} \sum_{j=k}^{\infty} E\langle e_j, p_n^w\rangle^2 = 0. \tag{6}$$

(Refer, for instance, to page 154 of Parthasarathy, 1967.) We require the following lemmas.

Lemma 1. *Let* J *and* H *be real valued measurable functions defined on* [0, 1] *that satisfy* (D). *Also assume that* (B) *holds and* Q *has a quantile density function* h. *Suppose that* $H \geq 0$;

$$|J(u)| \leq H(u) \quad \textit{for all} \quad u \in (0, 1); \quad \textit{and} \tag{7}$$

$$E\langle i_0, p_n^H\rangle^2 \to \langle K^{h,H} i_0, i_0\rangle < \infty, \quad \textit{then} \tag{8}$$

$$E\langle i_0, p_n^J\rangle^2 \to \langle K^{h,J} i_0, i_0\rangle < \infty, \quad \textit{and} \tag{9}$$

$$\langle i_0, p_n^J\rangle \to_d N(0, \langle K^{h,J} i_0, i_0\rangle). \tag{10}$$

The proof of this lemma follows almost directly from the techniques of the proof for Theorem 2 of Mason (1984).

Lemma 2. *Under the assumptions of Theorem* 1 *for every* $f \in \mathscr{A}$

$$E\langle f, p_n^w\rangle^2 \to \langle K^{h,w} f, f\rangle < \infty, \quad \textit{and} \tag{11}$$

$$\langle f, p_n^w\rangle \to_d N(0, \langle K^{h,w} f, f\rangle). \tag{12}$$

Proof. Observe that whenever $f \in \mathscr{A}$, by Schwarz's inequality and (C) $\langle Kf, f\rangle < \infty$. First assume that f is a polynomial defined on (0, 1). Let

$$f_n(u) = \begin{cases} f(i/(n+1)) & \text{for} \quad (i-1)/n \leq u < i/n, \quad i = 1, \cdots, n \\ 0 & \text{elsewhere.} \end{cases}$$

Notice that by Schwarz's inequality

$$E(\langle f, p_n^w\rangle - \langle f_n, p_n^w\rangle)^2 \leq \|f - f_n\|^2 E\|p_n^w\|^2. \tag{13}$$

Since f has a bounded first derivative, it is easy to see using (E) that the right side of (13) converges to zero. Observe that

$$|f(u)w(u)| \leq Mw(u),$$

where $M = \sup\{|f(u)|: u \in (0, 1)\} < \infty$. Hence Lemma 1 in combination with (F) implies (11) and (12).

Now choose any $f \in \mathscr{A}$ and any sequence of polynomials $\{g_m\}$ such that $\|f - g_m\| \to 0$ (the polynomials are dense in $L^2(0, 1)$). For each m

$$E(\langle f, p_n^w\rangle - \langle g_m, p_n^w\rangle)^2 \leq \|f - g_m\|^2 E\|p_n^w\|^2.$$

Elementary arguments complete the proof. □

We are now in a position to establish (6). Observe that for each integer $k \geq 2$, we have by Parseval's identity that

$$\sum_{j=k}^{\infty} E\langle e_j, p_n^w \rangle^2 = E\| p_n^w \|^2 - \sum_{j=1}^{k-1} E\langle e_j, p_n^w \rangle^2.$$

Choose any $\varepsilon > 0$ and integer $k_0 \geq 2$ such that for all $k \geq k_0$

$$\text{Trace } K - \sum_{j=1}^{k-1} \langle Ke_j, e_j \rangle < \varepsilon. \tag{14}$$

Lemma 2, (E) and (14) imply that there exists an n_0 such that

$$E\| p_n^w \|^2 - \sum_{j=1}^{k-1} E\langle e_j, p_n^w \rangle^2 < \varepsilon$$

for all $n \geq n_0$ and $k \geq k_0$.

Now choose $k \geq k_0$ such that

$$E\| p_n^w \|^2 - \sum_{j=1}^{k-1} E\langle e_j, p_n^w \rangle^2 < \varepsilon$$

for all $1 \leq n \leq n_0 - 1$.

This completes the proof of (6).

To complete the proof of Theorem 1 we must show that $\hat{\mu}_n(f) \to \hat{\mu}(f)$ for each $f \in \mathscr{A}$. (See Lemma 2.1 on page 153 of Parthasarathy, 1967.)

By Lemma 2, for each $f \in \mathscr{A}$

$$\hat{\mu}_n(f) \to \exp(-\langle Kf, f \rangle/2);$$

but since for each $f \in \mathscr{A}$

$$\langle f, whB \rangle \sim_d N(0, \langle Kf, f \rangle)$$

the proof of Theorem 1 is complete. □

The following corollary will be useful later on.

COROLLARY 1. *If assumption* (F) *in Theorem* 1 *is replaced by*

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} E\| \bar{\imath}_\varepsilon p_n^w \|^2 = 0, \tag{G}$$

then the conclusion of Theorem 1 *holds.*

PROOF. Choose any sequence $\{\delta_m\}_{m=1}^{\infty}$ such that for each $m \geq 1$, $0 < \delta_m < ½$, both δ_m and $1 - \delta_m$ are continuity points of w, and $\delta_m \downarrow 0$. By Theorem 5 of Stigler (1974) for each $m \geq 1$,

$$E\langle i_{\delta_m}, p_n^w \rangle^2 \to \langle Ki_{\delta_m}, i_{\delta_m} \rangle.$$

It is simple to show by applying Schwarz's inequality that

$$E\langle \bar{\imath}_{\delta_m}, p_n^w \rangle^2 \leq E\| \bar{\imath}_{\delta_m} p_n^w \|^2.$$

Since by the monotone convergence theorem

$$\langle Ki_{\delta_m}, i_{\delta_m} \rangle \to \langle Ki_0, i_0 \rangle,$$

a routine argument shows that (G) implies (F). □

For the special case of Uniform (0, 1) random variables, we have:

COROLLARY 2. *Let* $X_1, \cdots, X_n$ *be* i.i.d. *Uniform* (0, 1) *random variables. Assume* $w \geq 0$, (C) *and* (D). *If in addition for each* $0 \leq \varepsilon < 1/2$

$$\sum_{i=[n\varepsilon]+1}^{n+1-[n\varepsilon]} w^2(i/(n+1))(i/(n+1)(1 - i/(n+1)))/n \rightarrow \int_\varepsilon^{1-\varepsilon} w^2(u)u(1-u)\,du, \tag{15}$$

then $p_n^w \Rightarrow wB$ *with* $k_1 = k_2 = 1$.

PROOF. For each $0 \leq \varepsilon < 1/2$

$E\| {}_\varepsilon p_n^w \|^2 = \sum_{i=[n\varepsilon]+1}^{n+1-[n\varepsilon]} w^2(i/(n+1))(i/(n+1)(1 - i/(n+1)))/(n+2)$. □

In many practical situations, the following corollary is applicable.

COROLLARY 3. *Assume* (A), (D) *and* $w \geq 0$. *In addition, assume that*

(i) *for some* $-\infty < \nu_1, \nu_2 < \infty$ *and* $0 < M < \infty$

$$h(u) \leq M u^{\nu_1}(1-u)^{\nu_2} \quad \text{for all} \quad u \in (0, 1); \quad \text{and}$$

(ii) *there exists a nonnegative function g that is monotone on* $(0, 1/2]$ *and* $[1/2, 1)$ *such that* $w(u) \leq g(u)$ *for all* $u \in (0, 1)$ *and*

$$\int_0^1 g^2(u)u^{2\nu_1+1}(1-u)^{2\nu_2+1}\,du < \infty;$$

then $p_n^w \Rightarrow whB$ *with* $k_1 > -2\nu_1 - 2$ *and* $k_2 > -2\nu_2 - 2$.

PROOF. Let $f_n(u) = w^2(([nu]+1)/(n+1))\, E(\sqrt{n}(Q_n(u) - EQ_n(u)))^2$ for $(k_1 - 1)/n < u < 1 - (k_2 - 1)/n$ and equal to zero elsewhere. The choice of k_1 and k_2 insures that f_n is finite for all n sufficiently large, that is, (B) holds. (See Proposition 2 of Section 3.)

The following proposition due to Anderson (1982) along with (D) implies that

$$f_n(u) \rightarrow w^2(u)h^2(u)u(1-u) \quad \text{a.e. on } (0, 1). \tag{16}$$

PROPOSITION 1. *Assume* (A). *Then for each* $u \in (0, 1)$ *and positive integer r* $E(\sqrt{n}(Q_n(u) - Q(u)))^r \rightarrow EN^r$, *where N is a normal random variable with mean zero and variance* $h^2(u)u(1-u)$ *if and only if* $E|X|^\delta < \infty$ *for some* $\delta > 0$.

Proposition 2 and (ii) imply that for some constant $0 < K < \infty$

$$f_n(u) \leq Kg^2(([nu]+1)/(n+1)) (([nu]+1)/(n+1))^{2\nu_1+1}(1 - ([nu]+1)/(n+1))^{2\nu_2+1} \tag{17}$$

for all $u \in (0, 1)$ and all n sufficiently large. Also, by the conditions given on g

for each $0 \le \varepsilon < 1/2$

$$\sum_{i=[n\varepsilon]+1}^{n+1-[n\varepsilon]} g^2(i/(n+1))(i/(n+1))^{2\nu_1+1}(1-i/(n+1))^{2\nu_2+1}/n \tag{18}$$
$$\to \int_{\varepsilon}^{1-\varepsilon} g^2(u)u^{2\nu_1+1}(1-u)^{2\nu_2+1}\,du.$$

(17) and (18) show that the sequence of functions $\{f_n\}$ is uniformly integrable. Hence by (16), we conclude (E) and (G). ((C) holds by assumption (ii).) □

Relatively straightforward arguments based on Propositions 1 and 2 show that under the assumptions of Corollary 3 that

$$\| q_n^w - p_n^w \|^2 \to_p 0 \quad \text{and,} \tag{19}$$

$$\| r_n^w - q_n^w \|^2 \to_p 0. \tag{20}$$

(The details are omitted.) (19) and (20) along with Theorem 4.2 of Billingsley (1968) give the following sufficient conditions for the weak convergence of q_n^w and r_n^w in $L^2(0, 1)$:

THEOREM 2. *Under the conditions of Corollary* 3, $q_n^w \Rightarrow whB$.

THEOREM 3. *Under the conditions of Corollary* 3, $r_n^w \Rightarrow whB$.

REMARK 1. The above theorems and corollaries remain true without the assumption that $w \ge 0$, if in the statements of assumptions (F), and (ii) of Corollary 3 w is replaced by $|w|$.

These results are immediately applicable to the derivation of the asymptotic distribution of goodness of fit tests based on sample quantiles that can be written as a continuous functional defined on $L^2(0, 1)$. For example, consider the weighted Cramér-von Mises type statistic

$$T_n = \sum_{i=1}^n w^2(i/(n+1))(X_{i,n} - Q(i/(n+1)))^2 = \| q_n^w \|^2.$$

Under the conditions of Theorem 3,

$$T_n \to_d \int_0^1 B^2(u)w^2(u)h^2(u)\,du.$$

For some other applications of these results, the reader is referred to LaRiccia and Mason (1983) for a study of the weak convergence in $L^2(0, 1)$ of the estimated weighted empirical quantile process.

Some remarks must be made concerning the relationship between the conditions of Theorem 3 and the sufficient conditions obtainable by the approach discussed in the introduction for the conclusion of Theorem 3 to hold. A sufficient condition for (K) due to Shorack (1982) is the following: Assume that $f(Q(u))$ is strictly positive and continuous on (0, 1), increasing near 0, decreasing near 1, and satisfies the following "uniform tail condition": For a given $g \in \mathscr{Q}$

(S) $\lim_{u\downarrow 0}(q(u)h(u))/(\psi(u)h(bu))$

$$= \lim_{u\downarrow 0}(q(u)h(1-u))/(\psi(u)h(1-bu)) = \infty$$

for all $b > 0$ sufficiently close to zero then (K) holds. (M. Csörgő et al. (1982) have refined this condition.)

The exact connection between condition (S) and (i) and (ii) above is not clear to us. The following examples are instructive.

EXAMPLE 1. Let $h(u) = u^{-\nu}\sin^2(1/u) + u^{-\nu-1/2}\cos^2(1/u)$ with $\nu > 0$. Set $b_m = 2^{-m}$ for any positive integer $m \geq 1$. h is not decreasing near zero, and for each $m \geq 1$ it is easy to show that there exists a sequence $\{u_n\}_{n=1}^{\infty}$ with $u_n \downarrow 0$ such that

$$\lim_{n\to\infty} h(u_n)/(\psi(u_n)h(u_nb_m)) \leq \limsup_{n\to\infty} h(u_n)/(u_n^{1/2}h(u_nb_m)) < \infty,$$

so that (S) does not hold. Yet there exists an $0 < M < \infty$ such that

$$h(u) \leq Mu^{-\nu-1/2} \quad \text{and} \quad f(Q(u)) \leq Mu^{\nu}$$

for all $u \in (0, 1)$. Hence conditions (i) and (ii) are satisfied and by Theorem 3, $r_n \Rightarrow B$.

EXAMPLE 2. Let $h(u) = u^{-\nu}$ with $\nu > 0$. Condition (S) is satisfied for any $q \in Q$ and (i) holds trivially. Consider the weight function

$$w(u) = \begin{cases} u^{\nu-1}(\ln\ln(1/u))^{-\beta}(\ln(1/u))^{-1/2} & \text{for } u \in (0, e^{-2}) \\ 0 & \text{elsewhere,} \end{cases}$$

where $-2 < \beta < -1$. Notice that for any $q \in \mathscr{Q}$,

$$\int_0^1 w^2(u)h^2(u)q^2(u)\,du$$

$$\geq \int_0^1 w^2(u)h^2(u)\psi^2(u)\,du$$

$$= \int_0^{e^{-2}} u^{-1}\left(\ln\ln\left(\frac{1}{u}\right)\right)^{-\beta+1}\left(\ln\left(\frac{1}{u}\right)\right)^{-1} = \infty.$$

Hence the "q-metric technique" does not work in this case, condition (L) above does not hold, yet

$$\int_0^1 w^2(u)u^{-2\nu+1}\,du = \int_0^{e^{-2}} u^{-1}(\ln\ln(1/u))^{-\beta}(\ln(1/u))^{-1}\,du < \infty,$$

so that (ii) holds and $r_n^w \Rightarrow whB$.

EXAMPLE 3. Let $h(u) = u^{-\ln\ln(1/u)}$ for $0 < u \leq e^{-2}$ and equal 4 for $e^{-2} < u < 1$. h satisfies conditions (S) and (L) for the choice $q(u) = (u(1-u))^{1/4}$ and $w(u)$

$= f(Q(u))$. Thus by the "q-metric" technique $r_n \Rightarrow B$, whereas condition (i) does not hold.

3. Bounds for moments of transformed uniform order statistics. Let $U_1, \cdots, U_n$ be independent Uniform (0, 1) random variables and denote by $U_{1,n} \le \cdots \le U_{n,n}$ their order statistics. In this section, we establish the bounds for transformed uniform order statistics that were used in several of the proofs of the foregoing results. The following material is likely to have applications elsewhere.

PROPOSITION 2. *Suppose there exist* $-\infty < \nu_1, \nu_2 < \infty$ *and* $0 < M < \infty$ *such that*

$$h(u) \le Mu^{\nu_1}(1-u)^{\nu_2} \quad \text{for} \quad u \in (0, 1), \tag{21}$$

where h is a nonnegative measurable function defined on (0, 1), *then for every* $r \ge 1$ *there exists a finite positive constant L such that*

$$\begin{aligned}\Delta_n(i, r) \equiv E\left|\int_{i/(n+1)}^{U_{i,n}} h(u)\,du\right|^r \\ \le L(i/(n+1))^{r\nu_1 + r/2}(1 - i/(n+1))^{r\nu_2 + r/2}/n^{r/2}\end{aligned} \tag{22}$$

for all $k_1 \le i \le n + 1 - k_2$, *and all n sufficiently large where k_1 and k_2 are fixed positive integers such that* $k_1 > -r\nu_1 - r$ *and* $k_2 > -r\nu_2 - r$.

REMARK 2. If $-r\nu_1 - r \ge 1$ and $i = -r\nu_1 - r$ then $\Delta_n(i, r)$ may be infinite for all $n \ge i$. The analogous statement holds for $\Delta_n(n + 1 - i, r)$.

PROOF. The proof will follow from several lemmas.

LEMMA a. *For every real β there exists a finite positive constant K_β such that*

$$EU_{i,n}^{\beta} \le K_\beta(i/(n+1))^\beta \tag{23}$$

for every $i > -\beta$ *and all* $n \ge i$. (The proof is elementary, thus the details are omitted.)

LEMMA b. *Under the conditions of Proposition 2 there exists a finite positive constant L such that statement (22) is true for all* $k_1 \le i \le n + 1 - k_2$ *and all n sufficiently large where k_1 and k_2 are fixed positive integers such that* $k_1 > -r\nu_1$ *and* $k_2 > -r\nu_2$.

PROOF. Choose any $k_1 \le i \le n + 1 - k_2$.

CASE I. ν_1 and ν_2 are the same sign. In this case, for all $0 < a < u < b < 1$

$$u^{\nu_1}(1-u)^{\nu_2} \le a^{\nu_1}(1-b)^{\nu_2} + b^{\nu_1}(1-a)^{\nu_2}. \tag{24}$$

Hence, by (21), (24) and the c_r inequality, $\Delta_n(i, r)$ is less than or equal to

$$K_1E\{U_{i,n}^{\nu_1}(1 - i/(n + 1))^{\nu_2}|\, U_{i,n} - i/(n + 1)\,|\}^r$$
$$+ K_1E\{(i/(n + 1))^{\nu_1}(1 - U_{i,n})^{\nu_2}|\, U_{i,n} - i/(n + 1)\,|\}^r \equiv A_{i,n} + B_{i,n},$$

for some constant K_1 independent of n and i.

Choose any $\delta > 0$ such that both $k_1 > -r\nu_1(1 + \delta)$ and $k_2 > -r\nu_2(1 + \delta)$. By Hölder's inequality, we have

$$A_{i,n} \le K_1\{EU_{i,n}^{\nu_1 r(1+\delta)}\}^{1/(1+\delta)}\{E(U_{i,n} - i/(n + 1))^{r(1+\delta)/\delta}\}^{\delta/1+\delta}(1 - i/(n + 1))^{r\nu_2},$$

which by Lemma a and Lemma 2 of Wellner (1977) is

$$\le K_2(i/(n + 1))^{r\nu_1+r/2}(1 - i/(n + 1))^{r\nu_2+r/2}n^{-r/2}$$

for some constant K_2 independent of n and i. $B_{i,n}$ is bounded in the same way.

CASE II. ν_1 and ν_2 are of different signs. In this case for all $0 < a < u < b < 1$

(25) $$u^{\nu_1}(1 - u)^{\nu_2} \le a^{\nu_1}(1 - a)^{\nu_2} + b^{\nu_1}(1 - b)^{\nu_2}.$$

Hence by (21), (25), and the c_r-inequality $\Delta_n(i, r)$ is less than or equal to

$$K_3E(U_{i,n}^{\nu_1}(1 - U_{i,n})^{\nu_2}|\, U_{i,n} - i/(n + 1)\,|)^r$$
$$+ K_3E(|\, U_{i,n} - i/(n + 1)\,|^r)(i/(n + 1))^{\nu_1 r}(1 - i/(n + 1))^{\nu_2 r}.$$

The proof now proceeds much as in Case I, applying the generalized Hölder inequality, Lemma a, and Lemma 2 of Wellner (1977). □

LEMMA c. *Under the conditions of Proposition 2 for any $r \ge 1$, and any choice of fixed positive integers $m_1 \ge k_1 > -r\nu_1 - r$ and $m_2 \ge k_2 > -r\nu_2 - r$ there exists a constant $0 < K < \infty$ such that*

(26) *for all $k_1 \le i \le m_1$ and all n sufficiently large*

$$\Delta_n(i, r) \le K(i/(n + 1))^{r\nu_1+r}; \quad \textit{and}$$

(27) *for all $k_2 \le i \le m_2$, and all n sufficiently large*

$$\Delta_n(n + 1 - i, r) \le K(i/(n + 1))^{r\nu_2+r}.$$

PROOF. First consider (26). It can be shown by the same techniques as used in the proof of Lemma 1.1 of Bjerve (1977) that there exists a constant $c > 0$ such that

(28) $$P(U_{i,n} > 1/2) < e^{-cn}$$

for all $k_1 \le i \le m_1$ and n sufficiently large.

Now for any $k_1 \le i \le m_1$, we have by (21) that

$$\Delta_n(i, r) \le M^r E\left(I\left(U_{i,n} \le \frac{1}{2}\right) \left| \int_{i/(n+1}^{U_{i,n}} u^{\nu_1}(1-u)^{\nu_2}\, du \right|^r\right)$$

$$+ M^r E\left(I\left(U_{i,n} > \frac{1}{2}\right) \left| \int_{i/(n+1)}^{U_{i,n}} u^{\nu_1}(1-u)^{\nu_2}\, du \right|^r\right)$$

$$\le M_1 E\left(I\left(U_{i,n} \le \frac{1}{2}\right) \left| \int_{i/(n+1)}^{U_{i,n}} u^{\nu_1}\, du \right|^r\right)$$

$$+ M_2 E\left(I\left(U_{i,n} > \frac{1}{2}\right) \left| \int_{i/(n+1)}^{U_{i,n}} (1-u)^{\nu_2}\, du \right|^r\right) \equiv C_{i,n} + D_{i,n},$$

for some constants M_1 and M_2 independent of n and i.

Consider $C_{i,n}$. First assume $\nu_1 \ne -1$. By the c_r-inequality and integration

$$C_{i,n} \le M_3 E(U_{i,n})^{r\nu_1 + r} + M_4(i/(n+1))^{r\nu_1 + r}$$

for some constants M_3 and M_4 independent of n and i, which by Lemma a is

$$\le K_1(i/(n+1))^{r\nu_1 + r}$$

for some constant K_1 independent of $k_1 \le i \le m_1$ for all n sufficiently large.

Now assume $\nu_1 = -1$. In this case

$$C_{i,n} \le M_1 E\,|\ln U_{i,n} - \ln(i/(n+1))|^r.$$

Since $-\ln U_{i,n} =_d E_{n+1-i,n}$, the $(n+1-i)$th order statistic of n independent exponential random variables with mean 1, $E_1, \cdots, E_n$, and since

$$\ln((n+1)/i) - E(E_{n+1-i,n}) = \ln((n+1)/i) - \textstyle\sum_{k=i}^{n} k^{-1}$$

is bounded by a universal constant for $k_1 \le i \le m_1$ and all $n \ge m_1$, it is easy to see by the c_r-inequality that to complete the proof in this case it is sufficient to show that

$$E\,|E_{n+1-i,n} - E(E_{n+1-i,n})|^r$$

is uniformly bounded for $k_1 \le i \le m_1$ and all n sufficiently large. It is well known that

$$E_{n+1-i,n} - E(E_{n+1-i,n}) =_d \textstyle\sum_{j=1}^{n} (E_j - 1)/j.$$

Let $s = [r] + 1$. By Liaponov's inequality

$$E\,|\textstyle\sum_{j=i}^{n} (E_j - 1)/j\,|^r \le (E(\sum_{j=1}^{n} (E_j - 1)/j)^{2s})^{r/2s}.$$

Notice that by the c_r-inequality

$$E\,|\textstyle\sum_{j=i}^{n} (E_j - 1)/j\,|^{2s} \le C(s)E(\sum_{j=1}^{n} (j^{-1} - \beta_i)(E_j - 1))^{2s}$$

$$+ C(s)\beta_i^{2s} E(\textstyle\sum_{j=i}^{n} (E_j - 1))^{2s},$$

with

$$\beta_i = (n + 1 - i)^{-1}\textstyle\sum_{j=i}^{n} j^{-1};$$

which by application of the Marcinkiewicz-Zygmund inequality and the inequality given in Lemma 2.1 of Mason (1981) is

$$\leq B(s)(\textstyle\sum_{j=1}^{n} j^{-2})^2 E\,|\,E_1 - 1\,|^{2s}.$$

for some constant $B(s)$ dependent only on s. Hence there exists a constant K_1 such that

$$C_{i,n} \leq K_1 \quad \text{for all} \quad 1 \leq i \leq n.$$

Now consider $D_{i,n}$. By Schwarz's inequality

$$D_{i,n} \leq M_2\left(EI\left(U_{i,n} \geq \frac{1}{2}\right)\right)^{1/2}\left(E\left(\int_{i/(n+1)}^{U_{i,n}} (1 - u)^{\nu_2}\,du\right)^{2r}\right)^{1/2}$$

which by Lemma b and (28) is

$$\leq M_5 e^{-nc/2}(1 - i/(n + 1))^{r\nu_2 + r/2}/n^{r/2}$$

for some constant M_5 independent of $k_1 \leq i \leq m_1$ and all n sufficiently large. It is easy to show that this last expression is

$$\leq K_2(i/(n + 1))^{r(\nu_1+1)}$$

for some constant K_2 independent of $k_1 \leq i \leq m_1$ for all n sufficiently large. This completes the proof of (26). (27) follows from (26) by symmetry.

Proposition 2 is now seen to follow from Lemmas b and c. □

In particular, Proposition 2 says that if $X_1, \cdots, X_n$ are i.i.d. F such that its quantile function Q has a quantile density function h that satisfies (21) then there exists a constant $0 < L < \infty$ such that for all n sufficiently large

$$|\,EX_{i,n} - Q(i/(n + 1))\,| \leq L(i/(n + 1))^{\nu_1+1/2}(1 - i/(n + 1))^{\nu_2+1/2}/n^{1/2}$$

for every $-\nu_1 - 1 < i < n + 2 + \nu_2$; and

$$\text{Var } X_{i,n} \leq L(i/(n + 1))^{2\nu_1+1}(1 - i/(n + 1))^{2\nu_2+1}/n$$

for every $-2\nu_1 - 2 < i < n + 3 + 2\nu_1$.

For some closely related work on bounds for transformed uniform order statistics refer to Chernoff and Savage (1958) and Albers, Bickel and van Zwet (1976).

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REFERENCES

ALBERS, W., BICKEL, P. J., and VAN ZWET, W. R. (1976). Asymptotic expansions for the power of distribution free tests in the one-sample problem. *Ann. Statist.* **4** 108–156.

ANDERSON, K. (1982). Moment expansions for robust statistics. Stanford Univ. Dept. of Statistics Tech. Report No. 7, March 12, 1982.

BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

BJERVE, S. (1977). Error bounds for linear combinations of order statistics. *Ann. Statist.* **5** 357–367.

CHERNOFF, H. and SAVAGE, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math. Statist.* **29** 972–994.

CSÖRGŐ, M. and RÉVÉSZ, P. (1978). Strong approximations of the quantile processes. *Ann. Statist.* **6** 882–894.

CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L. and RÉVÉSZ, P. (1982). On weak and strong approximations of the quantile process. To appear in *Proceedings of the Seventh Conference on Probability Theory* (Brasnov, Aug. 29–Sept. 4, 1982).

CSÖRGŐ, S. (1982). On general quantile processes in weighted sup-norm metrics. *Stochastic Process. Appl.* **12** 215–220.

GIHMAN, I. I. and SKOROHOD, A. V. (1974). *The Theory of Stochastic Processes I.* Springer-Verlag, New York.

KOMLÓS, J., MAJOR, P., and TUSNÁDY, G. (1975). An approximation of partial sums of independent R.V.'s and the sample D. F. I. *Z. Wahrsch. verw. Gebiete* **32** 111–131.

LARICCIA, V. and MASON, D. M. (1983). Weak convergence in $L^2(0, 1)$ of the estimated weighted empirical quantile process with application to Cramér-von Mises type statistics based on the sample quantile function. Unpublished manuscript.

MASON, D. M. (1981). On the use of a statistic based on sequential ranks to prove limit theorems for simple linear rank statistics. *Ann. Statist.* **9** 424–436.

MASON, D. M. (1984). A unified approach to the asymptotic distribution of linear statistics. To appear in *Coll. Math. Soc. J. Bolyai* **36**. *Limit Theorems in Probability and Statistics,* Veszprém, Hungary 1982.

PARTHASARATHY, K. R. (1967). *Probability Measures on Metric Spaces.* Academic, New York.

PARZEN, E. (1979). Nonparametric statistical data modeling. *J. Amer. Statist. Assoc.* **74** 105–131.

SHORACK, G. R. (1972). Convergence of quantile and spacings processes with applications. *Ann. Math. Statist.* **43** 1400–1411.

SHORACK, G. R. (1982). Weak convergence of the general quantile process in $\| /q \|$-metrics. I.M.S. Bulletin Abstract 82t-2.

SKOROHOD, A. V. (1956). Limit theorems for stochastic processes. *Theor. Probab. Appl.* **1** 261–290.

STIGLER, S. M. (1974). Linear functions of order statistics with a smooth weight function. *Ann. Statist.* **2** 676–693.

STUTE, W. (1982). The oscillation behavior of empirical processes. *Ann. Probab.* **10** 86–107.

WELLNER, J. A. (1977). A law of the iterated logarithm for functions of order statistics. *Ann. Statist.* **5** 481–494.

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