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A NOTE ON THE LAW OF ITERATED LOGARITHM FOR WEIGHTED SUMS OF RANDOM VARIABLES

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Sufficient conditions for the validity of the upper and lower inequality of the law of iterated logarithm for weighted sums $\sum_{k=1}^{n} a_{nk}X_k$ of i.i.d. random variables (X_i) are given.

1. Introduction and results. In this note we consider i.i.d. random variables (X_i) assuming

$$E(X_1) = 0, \quad E(X_1^2) = 1,$$

and lower triangular real matrices $(a_{nk},\ 1 \le k \le n,\ n = 1, 2, \cdots)$ to form the weighted sums

$$W_n = \textstyle\sum_{k=1}^{n} a_{nk}X_k. \tag{1.1}$$

In case of independent Bernoulli variables (X_i) with values ± 1 and corresponding probabilities $p = q = ½$ and the matrix $a_{nk} = 1 - k/n$, $1 \le k \le n$, $n \ge 1$, Gal [6] (1951) showed the "upper inequality"

$$\limsup_{n\to\infty} \frac{W_n}{\sqrt{⅔\, n \log_2 n}} \le 1 \quad \text{a.s.} \quad (\log_2 n := \log\log n) \tag{1.2a}$$

and Stackelberg [16] (1964) the "lower inequality"

$$\limsup_{n\to\infty} \frac{W_n}{\sqrt{⅔\, n \log_2 n}} \ge 1 \quad \text{a.s.} \tag{1.2b}$$

of the law of iterated logarithm (LIL).

The LIL for Bernoulli variables or in other words for Rademacher functions is also connected with the so called "Borel Property" of summability matrices A. (For the definition and some results see Section 3). Gaposhkin [7] generalized the results (1.2 a, b) to the case of i.i.d. and bounded random variables (X_i) and the matrices $a_{nk} = (1 - k/n)^{\alpha}$, $1 \le k \le n$, for some $\alpha > 0$ (and also some Abel means). He obtained

$$\limsup_{n\to\infty} \frac{W_n}{\sqrt{(2/(2\alpha + 1)) \cdot n \log_2 n}} = 1 \quad \text{a.s.} \tag{1.3}$$

In this case the elements of the matrix are of form: (*) $a_{nk} = f(k/n)$ with $f(t) = (1 - t)^{\alpha}$. Tomkins (1971) [21] considered matrices of type (*) for $f \in C[0, 1]$ and

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independent random variables (X_i) satisfying some exponential inequalities and proved lower and upper inequalities for a LIL. In [23] (1975) he generalized the results to martingale difference sequences (X_i).

A different type of matrices, namely $a_{nk} = c_{n-k}$, $(c_n) \in l_2$, was investigated by Chow and Lai (1973) [2]. In this case a different limit behaviour occurs.

In (1974) again Tomkins [22] proved lower inequalities for the LIL for a class of triangular arrays of r.v's under some more technical conditions. (See also Eicker [5] and results cited there for more results in this direction). In addition he also obtained results for weighted sums W_n. But his conditions do not fit to a situation occurring in nonparametric estimation of regression curves, based on a fixed grid, in which we are interested (see [17]). In other words we consider the model

$$Y_j = m(t_j) + \varepsilon_j, \quad 1 \le j \le N, \quad 0 \le t_1 \le \cdots \le t_N \le 1$$

with i.i.d. error variables (ε_j) and some smooth regression function $m(\cdot)$ on $[0, 1]$, and a general linear estimator

$$\hat{m}_N(t) = \sum_j p_{jn}(t) \sum_{t_k \in I_{jn}} Y_k / (\# t_k \in I_{jn}),$$

where I_{jn} are intervals with $\sum_{j=0}^{n} I_{jn} = [0, 1]$, $n = n(N) \nearrow \infty$ and $\sum_j p_{jn}(t) m(\xi_{jn})$, $\xi_{jn} \in I_{jn}$, is an approximation operator of interpolatory type. Hence $\hat{m}_N(t_0) - E(\hat{m}_N(t_0))$ is of form $\sum_1^N a_{Nk}\varepsilon_k$ and the exact rate of pointwise convergence is connected to the LIL for the sum (1.1).

Most recently Lai and Wei [12] also have investigated weighted sums of independent random variables, but their technique and results are different from ours. For a further comparison of the results, see Section 3.

On the other hand P. Hall [8] has proven a LIL for a class of density estimators, proving a LIL for normally distributed r.v.'s and then using strong approximation results for the empirical distribution function to transfer the result to the estimators. The basic idea can also be used in our case. Observe that W_n is not a functional of the empirical distribution function of $(X_n)_{i=1}^n$, but here strong approximation results for sums of i.i.d. random variables can be used. For (1.1) one can describe the situation roughly as follows: To obtain an upper inequality we need some strong dependence conditions on $W_n, W_{n+1}, \cdots, W_{n+r_n}$, $r_n \to \infty$, $r_n = o(n)$, compare (1.4).

Whereas for the lower inequality we need some kind of "independence condition", compare (1.6), which guarantees that in the step from W_n to W_{n+r_n} the "new" random variables $X_{n+1}, \cdots, X_{n+r_n}$ have a positive weight with respect to W_n. The lower inequality was essentially proven for normally distributed random variables in the paper of Tomkins (Theorem 5 in [22]).

THEOREM 1. *Let $X_1, X_2, \cdots$ be* i.i.d. *random variables with $E(X_1) = 0$, $E(X_1^2) = 1$ and $(a_{nk}, 1 \le k \le n, n = 1, 2, \cdots)$ a lower triangular real matrix with $s_n^2 := \sum_{k=1}^n a_{nk}^2 \nearrow_\infty$ as $n \to \infty$.*

a) *Assume furthermore that $s_n/s_{n+1} \to 1$ as $n \to \infty$,*

$$\text{(1.4)} \qquad \lim_{\rho \to 1+} \liminf_{n\to\infty} \min_{m: 1 \le s_n^2/s_m^2 \le \rho} \sum_{k=1}^m a_{mk} a_{nk} / s_m^2 = 1.$$

and

(1.5a) $$(n \log_2 n)^{1/2}\{\sum_{k=2}^{n} |a_{nk} - a_{n,k-1}| + |a_{nn}|\}/\sqrt{s_n^2 \log_2 s_n^2} = O(1).$$

Then

$$\limsup_{n\to\infty} W_n/\sqrt{2s_n^2 \log_2 s_n^2} \leq 1 \text{ a.s.}$$

b) *If* (1.5a) *and*

(1.4a) $$s_n/s_{n+1} \to 1 \text{ as } n \to \infty$$

hold, and if for any $\varepsilon > 0$ *there exists some* $\lambda_\varepsilon > 1$ *such that with* $n_k := \min_n\{s_n \geq \lambda_\varepsilon^k\}$ *and* k *large enough*

(1.6) $$s_{n_{k-1}}^2 \frac{\varepsilon^2\lambda_\varepsilon^2}{4+\varepsilon} \geq \sum_{l=1}^{n_{k-1}} a_{n_k l}^2,$$

then

$$\limsup_{n\to\infty} W_n/\sqrt{2s_n^2 \log_2 s_n^2} \geq 1 \text{ a.s.}$$

Corollary 1.

a) *Under additional moment conditions,* (1.5a) *can be weakened, so if*

i) $E(|X_1|^p) < \infty$ *for some* $p > 2$, *then*

(1.5b) $$n^{1/p}\{\sum_2^n |a_{nk} - a_{n,k-1}| + |a_{nn}|\}/\sqrt{s_n^2 \log_2 s_n^2} = O(1)$$

ii) $E(e^{t\cdot X_1}) < \infty$ *in a neighborhood of* $t = 0$, *then*

(1.5c) $$\log n\{\sum_2^n |a_{nk} - a_{n,k-1}| + |a_{nn}|\}/\sqrt{s_n^2 \log_2 s_n^2} = o(1)$$

is a sufficient condition.

b) *Assume now, that the* X_i *are only independent and satisfy the moment conditions* $E(X_i) = 0$, $E(X_i^2) = 1$, $E(|X_i|^p) \leq M < \infty$ *for* $i = 1, 2, \cdots$, *and some* $p > 2$, *then* (1.5a) *has to be replaced by*

(1.5d) $$n^{1/2}(\log n)^{-1/4}\{\sum_2^n |a_{nk} - a_{n,k-1}| + |a_{nn}|\}/\sqrt{s_n^2 \log_2 s_n^2} = O(1).$$

Corollary 2. *In case* $s_n^2 \not\to \infty$, *but for some sequence* $(b_n) \nearrow \infty$, $(\tilde{a}_{nk}) = (a_{nk} \cdot b_n)$ *satisfy the conditions of the theorem, we obtain*

$$\limsup_{n\to\infty} W_n/\sqrt{2s_n^2 \log_2 (s_n b_n)^2} \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} 1 \text{ a.s. } \textit{respectively.}$$

Remarks.

i) In case the r.v.'s (X_i) have a joint normal distribution, condition (1.5a) is superfluous.

ii) Even in case the r.v.'s (X_i) are not independent a LIL can be proven. Condition (1.5) has to be replaced by an appropriate condition depending on the strong approximation quality. For results of this kind consult e.g. [1].

iii) If $a_{nk} \le c a_{mk}$ for $1 \le k \le m \le n$ and some $c > 0$, condition (1.6) can be satisfied (see [22]).

iv) In many cases $s_n/s_m \to 1$ is equivalent to $n/m \to 1$, as $n \to \infty$, (see Section 3), but in general this is wrong, e.g. if $s_n = \log n$ or $s_n = \exp((\log n)^\rho)$, $\rho > 1$ (despite $s_{n+1}/s_n \to 1$).

2. Proof. The proof for a) and b) contains two steps:

i) Prove the results for normally distributed r.v.'s

ii) Approximate appropriate normally distributed r.v.'s by versions of the original variables.

a)

i) Assume that $Y_1, Y_2, \cdots$ are i.i.d. $N(0, 1)$-distributed r.v.'s

$$V_n := \textstyle\sum_{k=1}^n a_{nk} Y_k, \quad T_n := \sum_{k=1}^n Y_k. \tag{2.1}$$

For $1 \le m \le n$, (V_n, V_m) has a joint normal distribution with covariance matrix C:

$$C = \begin{pmatrix} \sum_1^n a_{nk}^2 & \sum_1^m a_{mk} a_{nk} \\ & \sum_1^m a_{mk}^2 \end{pmatrix} =: \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}. \tag{2.2}$$

It is well known that

$$P(V_n \le x \mid V_m = z) = \mathcal{N}\left(x; \frac{\sigma_{12}}{\sigma_2^2} z, \quad \sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}\right) \tag{2.3}$$

where $\mathcal{N}(x; \mu, \sigma^2) = \int_{-\infty}^x \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) \frac{dt}{\sqrt{2\pi\sigma^2}}$. Denote $g(n) = (2s_n^2 \log_2 s_n^2)^{1/2}$; by (1.4) we can choose for any $\varepsilon > 0 : \rho(\varepsilon) > 1$ s.t.:

For $n_k := \min\{n, s_n^2 \ge \rho^k\}$ and all $m \in \{n_k + 1, \cdots, n_{k+1}\}$ the following inequalities hold

$$g(m)(1 + 3\varepsilon) \ge (1 + 2\varepsilon) g(n_{k+1}) \tag{2.4}$$

for k large enough.

$$(1 + 2\varepsilon) \frac{\sum_{\nu=1}^m a_{m\nu} a_{n_{k+1}\nu}}{\sum_{l=1}^m a_{ml}^2} \ge 1 + \varepsilon. \tag{2.5}$$

Following an idea of Kiefer [10], page 241, inequality (4.14), see also Hall [8], page 50, we can conclude

$$\begin{aligned} &P(V_{n_{k+1}} \ge (1 + \varepsilon) \cdot g(n_{k+1}) \mid V_m \\ &\qquad \ge (1 + 3\varepsilon) \cdot g(m) \text{ for at least one } m \in \{n_k + 1, \cdots, n_{k+1}\}) \\ &\quad \ge \inf_{m \in \{n_k+1, \cdots, n_{k+1}\}, z \ge (1+3\varepsilon) g(m)} P(V_{n_{k+1}} \ge (1 + \varepsilon) \cdot g(n_{k+1}) \mid V_m = z) \end{aligned}$$

and by (2.3)

$$= \inf_{m \in \{n_k+1, \cdots, n_{k+1}\}, z \ge (1+3\varepsilon) \cdot g(m)} \left\{1 - \mathcal{N}\left(\left[(1 + \varepsilon) g(n_{k+1}) - \frac{\sigma_{12}}{\sigma_2^2} z\right] \Big/ \sqrt{\sigma_1^2 - \sigma_{12}^2/\sigma_2^2}; 0, 1\right)\right\}$$

using (2.4) and (2.5) and the monotonicity in z we obtain

$$\geq 1 - \mathcal{N}([(1+\varepsilon)g(n_{k+1}) - (1+\varepsilon)g(n_{k+1})]/\sqrt{\sigma_1^2 - \sigma_{12}^2/\sigma_2^2};\, 0, 1) = \tfrac{1}{2}.$$

Using $P(B) \leq P(A)/P(A \mid B)$ we have gained the following maximal inequality:

$$P(V_n \geq (1+\varepsilon)g(n) \text{ for at least one } n \in \{n_k + 1, \cdots, n_{k+1}\})$$

$$\leq P(V_{n_{k+1}} \geq (1+\varepsilon)g(n_{k+1}))/P(\, V_{n_{k+1}} \geq (1+\varepsilon)g(n_{k+1}) \mid V_m \geq (1+3\varepsilon)g(m)$$

$$\text{for at least one } m \in \{n_k + 1, \cdots, n_{k+1}\})$$

$$\leq 2P(V_{n_{k+1}} \geq (1+\varepsilon)g(n_{k+1})) = 2(1 - \mathcal{N}((1+\varepsilon)\cdot\sqrt{2\log_2 s^2_{n_{k+1}}};\, 0, 1)) \leq \tilde{c}k^{-(1+\varepsilon)}$$

and by the Borel-Cantelli Lemma we obtain a) for the case i).

b)

i) part b) follows immediately from the proof of Theorem 5 in [22], observing that by (1.6) we obtain with

$$U_k = \textstyle\sum_{\nu=1}^{n_{k-1}} a_{n_k \nu} Y_\nu, \quad u_k^2 = E(U_k^2); \quad t_{n_k}^2 = 2\log_2 s_{n_k}^2; \quad v_k^2 = E((S_{n_k} - U_k)^2):$$

$$P\left(\mid U_k \mid \geq \frac{\varepsilon}{2} s_{n_k} t_{n_k}\right) = P\left(\frac{\mid U_k \mid}{u_k} \geq \varepsilon \cdot \frac{s_{n_{k-1}}}{2u_k} \cdot \frac{s_{n_k}}{s_{n_{k-1}}} t_{n_k}\right)$$

$$\leq P\left(\frac{\mid U_k \mid}{u_k} \geq \left(1 + \frac{\varepsilon}{20}\right) t_{n_k}\right)$$

and

$$\left(\frac{v_k}{s_{n_k}}\right)^2 = 1 - \left(\frac{u_k}{s_{n_k}}\right)^2 \geq 1 - \frac{\varepsilon^2}{4} \text{ for } k \text{ large enough,}$$

which are the key inequalities for the proof in [22].

a) and b)

ii) By Abel's partial summation method we find ($S_k = \sum_{\nu=1}^k X_\nu$)

$$\textstyle\sum_{k=1}^n a_{nk} X_k = \sum_{k=1}^{n-1} (a_{nk} - a_{n,k+1}) S_k + a_{nn} S_n.$$

Using a result of Strassen [19] we obtain for proper versions of the considered random variables (X_i) and (Y_i):

$$\max_{1 \leq k \leq n} \mid S_k - T_k \mid = \text{a.s. } o(\sqrt{n \log_2 n}). \tag{2.6}$$

Hence

$$\textstyle\sum_{k=1}^n a_{nk} X_k = \sum_{k=1}^{n-1} (a_{nk} - a_{n,k+1})(S_k - T_k) + a_{nn}(S_n - T_n)$$

$$+ \textstyle\sum_{k=1}^{n-1} (a_{nk} - a_{n,k+1}) T_k + a_{nn} T_n$$

$$= \textstyle\sum_{k=1}^n a_{nk} Y_k + R_n.$$

By(2.6) and (1.5a) we find that $R_n/\sqrt{s_n^2 \log_2 s_n^2} \overset{\text{a.s.}}{\to} 0$ as $n \to \infty$, which together with part i) proves the theorem.

For the proof of Corollary 1a), use the strong approximation results of Komlós, Major and Tusnády [11]; then we can replace in (2.6) $o(\sqrt{n \log_2 n})$ by $o(n^{1/p})$ or $o(\log n)$ respectively. For Corollary 1b), Theorem 4 in [1] gives an appropriate strong approximation quality.

3. Additional remarks.

i) Looking at the recent results of Lai and Wei [12] we find that their assumptions are related but not quite comparable with ours. They investigate the case of matrices $(a_{nk})_{n,k=1}^{\infty}$, where the l_2 norms of the rows, say s_n, exist for all $n \in \mathbb{N}$ and tend to infinity as $n \to \infty$, and independent normalized random variables with $E(|X_i|^p) \leq M < \infty$, $i = 1, 2, \cdots$, for some $p > 2$. Furthermore it is supposed that:

$$\sup_k a_{nk}^2 = o(s_n^2 (\log s_n^2)^{-\rho}), \quad \text{for all} \quad \rho > 0. \tag{3.1}$$

For the upper inequality of the LIL they demand that there exist constants $c_i \geq 0$, $d \geq 2/p$ s.t.:

$$\begin{aligned} &E((S_n - S_m)^2) \leq (\textstyle\sum_{i=m+1}^n c_i)^d && \text{for all} \quad n > m \geq m_0 \quad \text{and} \\ &(\textstyle\sum_{i=m+1}^n c_i)^d = O(s_n^2) && \text{for all} \quad n > m \geq m_0. \end{aligned} \tag{3.2}$$

We suppose instead of (3.1) and (3.2) that $s_n/s_{n+1} \to 1$ as $n \to \infty$ and that

$$\lim_{\rho \to 1+} \limsup_{n \to \infty} \sup_{1 \leq s_n/s_m \leq \rho} E((S_n - S_m)^2)/s_m^2 = 0. \tag{1.4}$$

Condition (1.4) says that the second moments should vary smoothly whereas (3.2) demands that the changes of the second moments in total should not be too large. We have to assume, in addition, condition (1.5) which in a certain sense is a more technical condition for our proof.

Considering the lower bound of the LIL, we find that condition (1.6) corresponds to (1.13) in [12] with $I_k = \{n_{k-1} + 1, \cdots, n_k\}$. We need (1.5) in addition whereas in [12] the assumptions (1.14–1.15) demand that $s_{n_k}^2$ behaves essentially like c^k for some $c > 1$.

Lai and Wei derive direct estimates to prove their results which leads to a more complicated proof. A further comparison of the results will be found under ii) where we consider special classes of weights.

ii) Special classes of weights.

α) For weighted averages $a_{nk} = a_k$, $1 \leq k \leq n$, the situation is somewhat easier since the n-dependence is separated. This case was extensively investigated, see e.g. Chow, Teicher [3, 4] Teicher [20] or Móricz [15] for results. In case the weights a_k are positive and nondecreasing, our assumptions for the LIL are:

$$s_n/s_{n+1} \to 1 \quad \text{or equivalently} \quad a_n/s_n \to 0 \quad \text{as} \quad n \to \infty \tag{1.4}$$

and

$$(1.5)\quad (n\log_2 n)^{1/2}\frac{a_n}{(s_n^2\log_2 s_n^2)^{1/2}} = O(1) \quad \text{or equivalently} \quad \frac{na_n^2}{s_n^2} = O\left(\frac{\log_2 s_n^2}{\log_2 n}\right)$$

$$(\text{or } na_n^2/s_n^2 = O(n^{1-2/p}\log_2 s_n^2/\log_2 n), \quad \text{if} \quad E(|X_i|^p) < \infty).$$

Condition (1.6) is satisfied trivially (see remark iii). Compare those with the assumptions of Theorem 3 in [20]:

$$(3.3)\quad \left(\frac{a_n}{s_n}\right)^2 = O((\log s_n^2)^{-1}) \quad \text{and} \quad \frac{na_n^2}{s_n^2} = O((\log s_n^2)^{\beta}) \quad \text{for some} \quad \beta < 1.$$

In [12] the condition for LIL is very simple (but more than the second moment has to exist; see [20] for the relation between conditions like (3.3) and moment conditions), namely

$$\left(\frac{a_n}{s_n}\right)^2 = o((\log s_n^2)^{-\rho}) \quad \text{for all} \quad \rho > 0.$$

Considering the basic example $a_k = k^{\alpha}(\log k)^{\beta}$, $2 \le k \le n$, all results contain the cases where (a_k) is nondecreasing. Without further moment conditions our theorem fails for $-\frac{1}{2} \le \alpha < 0$ since then $\Sigma|\Delta a_{nk}|$ is dominated by a_2 but $s_n^2 = o(n^{1/2})$ and condition (1.5) is no longer true.

β) Another interesting class of weights is given by $a_{nk} = f(k/n)$, $f \in C[0, 1]$. Condition (1.4) is true, since

$$s_n^2 \sim n\int_0^1 f^2(t)\,dt = n\|f\|_2^2$$

and since the continuity of f is by far sufficient for:

$$\textstyle\sum_{k=1}^m f\left(\frac{k}{n}\right)\cdot f\left(\frac{k}{m}\right) = \sum_{k=1}^m f\left(\frac{k}{m}\cdot\frac{m}{n}\right)f\left(\frac{k}{m}\right)$$

$$= (1+o(1))\textstyle\sum_{k=1}^m f^2\left(\frac{k}{m}\right) \quad \text{as} \quad n\to\infty \quad \text{and} \quad \frac{n}{m}\to 1.$$

To ensure condition (1.5a) we need in addition that $f \in BV[0, 1]$. (In case $E(|X_1|^p) < \infty$ for some $p > 2$, we have some more freedom to choose f). Hence we obtain for $f \in (C \cap BV)[0, 1]$

$$\limsup_{n\to\infty}\frac{W(n)}{(2n\log_2 n)^{1/2}} \overset{\text{a.s.}}{\le} \|f\|_2.$$

(The same is true for bounded and monotone f). In [21, 23], f is assumed to be at least absolutely continuous. In [12] a sufficient condition is given by $f \in \mathrm{Lip}_{1/2}[0, 1]$, which implies that

$$\frac{1}{m}\textstyle\sum_{k=1}^m \left(f\left(\frac{k}{n}\right) - f\left(\frac{k}{m}\right)\right)^2 \le K\left(1-\frac{m}{n}\right), \quad \text{for} \quad m \ge m_0, \qquad \theta_0 \le \frac{m}{n} < 1.$$

For the lower bound it is sufficient to demand $f^2 \in R[0, 1]$ (Riemann integrable) and $f \in BV[0, 1]$, since in this case conditions (1.5a) and

$$n_{k-1} \int_0^1 f^2(t)\, dt \frac{(\lambda_\varepsilon \cdot \varepsilon)^2}{4+\varepsilon} \geq n_k \int_0^{\lambda_\varepsilon^{-1}} f^2(t)\, dt \tag{1.6}$$

can be satisfied. As specical cases our theorem includes the results of Gál, Stackelberg and Gaposhkin [8, 16, 7] mentioned in the introduction. Using Lai and Wei's result it is sufficient to have $f^2 \in R[0, 1]$ to obtain the lower inequality of the LIL.

iii) In summability theory one says, a summability matrix $A = (a_{nk}, 1 \leq k \leq n, n \geq 1)$ has the "Borel-Property" (BP) (see [9, 13] and the references given there) if:

$$\textstyle\sum_{k=1}^n a_{nk} R_k(x) \overset{\text{a.s.}}{\to} 0 \quad \text{as} \quad n \to \infty$$

where $R_k(x)$ are the Rademacher functions on [0, 1] endowed with Lebesgue measure. This problem can also be treated by our theorem. Observe that in all reasonable cases we have $\sum_1^n a_{nk} = 1$ and hence in most of the cases $s_n^2 \searrow 0$, $n \to \infty$ and we have to use Corollary 2. Furthermore the r.v.'s $R_k(x)$ are bounded and we have only to demand condition (1.5c). Hence the Borel property is true if there exists a suitable sequence $(b_n) \nearrow \infty$ s.t. $\tilde{a}_{nk} = b_n \cdot a_{nk}$ satisfies conditions (1.4) and (1.5c) and

$$s_n^2 \log_2(s_n^2 b_n^2) \to 0 \quad \text{as} \quad n \to \infty.$$

(Compare e.g. with Theorem 2 in [13]). It is easy to see by our results that the Cesàro and Euler methods have the Borel property.

iv) Looking onto the regression problem mentioned in Section 1, we get e.g. for kernel estimators with kernels of bounded support and bounded variation and further conditions on the design (t_j) and the smoothness of $m(\cdot)$ that the exact rate of pointwise convergence is given by (k = kernel, b_n = band width)

$$\left(\frac{nb_n}{2\|k\|_2^2 \log_2 n}\right)^{1/2}$$

where $b_n \sim N^\alpha$ for some $\alpha \in (1/5, 1 - 2/p)$ if $E(|\varepsilon_1|^p) < \infty$ for some $p > 5/2$. The exact result which needs some more technical details will be published elsewhere.

v) Recently at "the conference on limit theorems in Probability and Statistics" in Vesprém (1982), the author has learned that P. Hall and S. Csörgö have investigated upper and lower classes for arrays of r.v.'s of the following type:

$$
\begin{array}{l}
A_1(X_1) \\
A_2(X_1),\, A_2(X_2) \\
A_3(X_1),\, A_3(X_2),\, A_3(X_3) \\
\vdots
\end{array}
$$

where (A_i) are $\mathscr{B}_1$ measurable functions and the (X_i) are i.i.d. r.v.'s. But this situation is different from ours. This work has appeared in *Z. Wahrsch. verw. Gebiete* **62** (1982), 207–233.

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