The Annals of Probability
1984, Vol. 12, No. 3, 876–881

APPROXIMATIONS TO OPTIMAL STOPPING RULES FOR EXPONENTIAL RANDOM VARIABLES[1]

BY ADAM T. MARTINSEK

University of Illinois at Urbana-Champaign

For $X_1, X_2, \cdots$ i.i.d. with finite mean and $Y_n = \max(X_1, \cdots, X_n) - cn$, c positive, a number of authors have considered the problem of determining an optimal stopping rule for the reward sequence Y_n. The optimal stopping rule can be given explicitly in this case; however, in general its use requires complete knowledge of the distribution of the X_i. This paper examines the problem of approximating the optimal expected reward when only partial information about the distribution is available. Specifically, if the X_i are known to be exponentially distributed with unknown mean, stopping rules designed to approximate the optimal rule (which can be used only when the mean is known) are proposed. Under certain conditions the difference between the expected reward using the proposed stopping rules and the optimal expected reward vanishes as c approaches zero.

1. Introduction. The following problem and variations on it have been considered by MacQueen and Miller (1960), Derman and Sacks (1960), Sakaguchi (1961), Chow and Robbins (1961, 1963), Yahav (1966), Cohn (1967), and DeGroot (1968). Let $X_1, X_2, \cdots$ be independent and identically distributed (i.i.d.) with $E|X_1| < \infty$. For $n \geq 1$, define the reward sequence

$$Y_n = \max_{1 \leq i \leq n} X_i - cn, \quad c > 0; \tag{1.1}$$

the problem is to find a stopping rule which maximizes the expected reward.

The optimal stopping rule for this problem, i.e., the rule which maximizes $E(Y_\tau)$ over all stopping rules τ with $E(Y_\tau^-) < \infty$, is

$$\tau_c^* = \inf\{n \geq 1 : X_n \geq \gamma\}, \tag{1.2}$$

where $E(X_1 - \gamma)^+ = c$ (for a proof of this result, see Chow, Robbins and Siegmund 1971, pages 56–58). However, in order to use the stopping rule τ_c^* it is necessary to know γ, which in turn requires knowledge of the distribution of the X_i. If only partial information about the distribution is available, it may not be possible to compute γ, and in such cases it would be desirable to approximate the optimal rule τ_c^* and (one hopes) the optimal reward $E(Y_{\tau_c^*})$ as well. The purpose of the present paper is to consider this approximation problem for the special case of the exponential distribution with unknown mean, and to prove a result which suggests that a certain approximation to the optimal rule performs well, at least asymptotically.

Received July 1983; revised January 1984.
[1] Revision supported by the National Science Foundation under Grant MCS 83-01834.
AMS 1980 *subject classifications.* Primary 62L15; secondary 60G40.
Key words and phrases. Optimal stopping, uniform integrability, last times.

Assume throughout the rest of this section and the next that the distribution of the X_i is exponential with mean μ. An easy computation shows that $\gamma = -\mu \log(c/\mu)$, and this suggests that when μ is unknown (but the distribution of the X_i is known to be exponential), the stopping rule

$$\hat{\tau}_c = \inf\{n \geq 1: X_n \geq -\bar{X}_n \log(c/\bar{X}_n)\}, \tag{1.3}$$

or more generally

$$\hat{\tau}_c = \inf\{n \geq n_c: X_n \geq -\bar{X}_n \log(c/\bar{X}_n)\}, \tag{1.4}$$

where $\bar{X}_n = n^{-1}S_n = n^{-1}\sum_1^n X_i$, and n_c is a positive integer depending on c, may approximate the optimal rule τ_c^* in the sense that $E(Y_{\hat{\tau}_c})$ is close to $E(Y_{\tau_c^*})$. In the next section it is proved that if $\delta c^{-\alpha} \leq n_c = o(c^{-1})$ as $c \to 0$, for some $\delta > 0$ and $0 < \alpha < 1$, then $E(Y_{\tau_c^*}) - E(Y_{\hat{\tau}_c}) \to 0$ as $c \to 0$.

This type of approximation problem has been considered previously by Bramblett (1965), who showed that for certain cases involving unknown location parameters, the ratio of the expected reward using an approximating stopping rule to the optimal expected reward approaches one as c goes to zero. In other words, he showed that certain approximating stopping rules are asymptotically optimal in the sense of Kiefer and Sacks (1963) and Bickel and Yahav (1967, 1968). Bramblett also obtained asymptotic optimality of a truncated version of the present stopping time for exponential X_i, but he was unable to get results about the vanishing of the difference in expected rewards as c approaches zero, for this case or any other (although he describes this property as being very desirable).

2. Performance of $\hat{\tau}_c$.

Unlike τ_c^*, the stopping-rule $\hat{\tau}_c$ defined by (1.4) is not a geometric random variable. However, the key to proving the theorem below is to approximate $\hat{\tau}_c$ by appropriate geometrically distributed random variables, as in the proof of Lemma 1.

LEMMA 1. *Define $\hat{\tau}_c$ by* (1.4) *with $n_c = O(c^{-1})$ as $c \to 0$. Then for every $\alpha \in (0, 1)$ and $0 < \beta < \alpha/2$, as $c \to 0$,*

$$E\hat{\tau}_c \leq [c/(1 + c^{\beta})\mu]^{-(1+c^{\beta})} + n_c + O(c^{-\alpha}). \tag{2.1}$$

Furthermore, if $\delta c^{-\alpha} \leq n_c$ for some $\delta > 0$, then as $c \to 0$,

$$E\hat{\tau}_c \geq [c/(1 - c^{\beta})\mu]^{-(1-c^{\beta})} + o(c^{-q}) \quad \textit{for all} \quad q > 0. \tag{2.2}$$

PROOF. To prove (2.1), define

$$L_{c,\beta} = \sup\{n \geq 1: |S_n - n\mu| \geq c^{\beta} n\mu\},$$

where $\sup(\phi) = 0$. For $x > 0$, put $g(x) = -x\log(c/x)$. Then $g'(x) = -\log(c/x) + 1$ is positive if and only if $x > ce^{-1}$. Choose c_0 small enough so that $(1 - c_0^{\beta})\mu > c_0 e^{-1}$, i.e., g is increasing on $((1 - c_0^{\beta})\mu, \infty)$. Let

$$\tau_c^+ = \inf\{n \geq 1: X_n \geq -(1 + c^{\beta})\mu \log[c/(1 + c^{\beta})\mu]\}.$$

Then for K sufficiently large, $Kc^{-1} > 2n_c$ for all c, and we have for $c \le c_0$,

$$P[\hat{\tau}_c > Kc^{-1}] \le P[L_{c,\beta} > Kc^{-1}/2] + P[L_{c,\beta} \le Kc^{-1}/2,\ \hat{\tau}_c > Kc^{-1}] \tag{2.3}$$
$$\le P[L_{c,\beta} > Kc^{-1}/2] + P[\tau_c^+ \ge Kc^{-1}/2].$$

It is easily checked that

$$\{(c\tau_c^+)^p \colon c \le c_0\}$$

is uniformly integrable for all $p > 0$, and by Theorem 7 of Chow and Lai (1975) the same is true of

$$\{(c^{2\beta}L_{c,\beta})^p \colon c \le c_0\}.$$

In particular, for every $p > 0$

$$E(L_{c,\beta}^p) = O(c^{-2\beta p}) \quad \text{as} \quad c \to 0. \tag{2.4}$$

Hence by (2.3), since $2\beta < 1$,

$$\{(c\hat{\tau}_c)^r \colon c \le c_0\} \quad \text{is uniformly integrable for all} \quad r > 0. \tag{2.5}$$

Let $n_c' = \max(n_c, c^{-\alpha})$ and

$$\hat{\tau}_c' = \inf\{n \ge n_c' \colon X_n \ge -\bar{X}_n \log(c/\bar{X}_n)\}.$$

Then from (2.4) with $p > (\alpha/2 - \beta)^{-1}$, (2.5) with $r = 2$, and the usual expression for the expectation of a geometric random variable, for $c \le c_0$,

$$\begin{aligned} E(\hat{\tau}_c) &\le E(\hat{\tau}_c I_{\{L_{c,\beta} \le n_c'\}}) + E(\hat{\tau}_c' I_{\{L_{c,\beta} > n_c'\}}) \\ &\le E^{1/2}(\hat{\tau}_c^2) P^{1/2}(L_{c,\beta} \ge n_c') \\ &\quad + E(\inf\{n \ge n_c' \colon X_n \ge -(1 + c^\beta)\mu \log[c/(1 + c^\beta)\mu]\}) \\ &\le E^{1/2}(\hat{\tau}_c^2)(n_c')^{-p/2} E^{1/2}(L_{c,\beta}^p) + (n_c' - 1) + E(\tau_c^+) \\ &= O(c^{-1-\beta p + \alpha p/2}) + (n_c' - 1) + E(\tau_c^+) \\ &\le o(1) + n_c + c^{-\alpha} + [c/(1 + c^\beta)\mu]^{-(1+c^\beta)} \\ &= [c/(1 + c^\beta)\mu]^{-(1+c^\beta)} + n_c + O(c^{-\alpha}), \end{aligned} \tag{2.6}$$

as $c \to 0$. This proves (2.1).

To prove (2.2), note that

$$E(\hat{\tau}_c) \ge E[\tau_c^- I_{\{L_{c,\beta} < n_c\}}], \tag{2.7}$$

where

$$\tau_c^- = \inf\{n \ge 1 \colon X_n \ge -(1 - c^\beta)\mu \log[c/(1 - c^\beta)\mu]\}$$

and $L_{c,\beta}$ is defined as above. Now

$$\begin{aligned} E(\tau_c^-) &= [P(X_1 \ge -(1 - c^\beta)\mu \log[c/(1 - c^\beta)\mu])]^{-1} \\ &= [c/(1 - c^\beta)\mu]^{-(1-c^\beta)}, \end{aligned} \tag{2.8}$$

and from Hölder's inequality, the expression for the second moment of τ_c^-, and

(2.4) with $p > (q+1)/(\alpha/2 - \beta)$,

$$E[\tau_c^- I_{\{L_{c,\beta} \ge n_c\}}] \le E^{1/2}[(\tau_c^-)^2] P^{1/2}(L_{c,\beta} \ge n_c) \le O(c^{-1}) c^{\alpha p/2} O(c^{-\beta p})$$

$$= O(c^{-1+p(\alpha/2-\beta)}) = o(c^q) \quad \text{as} \quad c \to 0. \tag{2.9}$$

(2.2) now follows from (2.7), (2.8), and (2.9).

LEMMA 2. *If $n_c \ge \delta c^{-\alpha}$ for some $\delta > 0$ and $0 < \alpha < 1$, then for every $\beta \in (0, \alpha/2)$,*

$$\textstyle\sum_{j=n_c}^{\infty} E[X_j I_{\{|S_j - j\mu| \ge jc^\beta \mu\}}] \to 0,$$

as $c \to 0$.

PROOF. Choosing p in (2.4) large enough so that $\beta < \alpha(p-2)/2p$, we have

$$\textstyle\sum_{j=n_c}^{\infty} E[X_j I_{\{|S_j - j\mu| \ge jc^\beta \mu\}}] \le \sum_{j=n_c}^{\infty} E^{1/2}(X_1^2) P^{1/2}(L_{c,\beta} \ge j)$$

$$\le O(1) \textstyle\sum_{j=n_c}^{\infty} j^{-p/2} E^{1/2}(L_{c,\beta}^p) = O(1) \sum_{j=n_c}^{\infty} j^{-p/2} c^{-\beta p}$$

$$= O(c^{\alpha(p/2-1)-\beta p}) = o(1).$$

THEOREM. *Define $\hat{\tau}_c$ by (1.4). If $\delta c^{-\alpha} \le n_c = o(c^{-1})$ as $c \to 0$, for some $\delta > 0$ and $0 < \alpha < 1$, then as $c \to 0$,*

$$E(Y_{\tau_c^*}) - E(Y_{\hat{\tau}_c}) \to 0.$$

That is, the expected loss due to not knowing μ and using the (suboptimal) approximating rule $\hat{\tau}_c$ vanishes as $c \to 0$.

PROOF. Because τ_c^* is optimal and $E(Y_{\tau_c^*}) = -\mu \log(c/\mu)$ (see Chow, Robbins and Siegmund, 1971, page 57),

$$0 \le E(Y_{\tau_c^*}) - E(Y_{\hat{\tau}_c}) \le -\mu \log(c) + \mu \log(\mu) - E(X_{\hat{\tau}_c}) + cE\hat{\tau}_c. \tag{2.10}$$

By Lemma 2, for $0 < \beta < \alpha/2$ and c small enough so that $(1 - c^\beta)\mu > ce^{-1}$, by independence of the X_i,

$$E(X_{\hat{\tau}_c}) = \textstyle\sum_{j=n_c}^{\infty} E[X_j I_{\{\hat{\tau}_c = j\}}] \ge \sum_{j=n_c}^{\infty} E[X_j I_{\{\hat{\tau}_c = j, |S_j - j\mu| \le jc^\beta \mu\}}]$$

$$\ge \textstyle\sum_{j=n_c}^{\infty} E[X_j I_{\{\hat{\tau}_c \ge j, |S_j - j\mu| \le jc^\beta\mu, X_j \ge -(1+c^\beta)\mu \log[c/(1+c^\beta)\mu]\}}]$$

$$= \textstyle\sum_{j=n_c}^{\infty} E[X_j I_{\{\hat{\tau}_c \ge j, X_j \ge -(1+c^\beta)\mu\log[c/(1+c^\beta)\mu]\}}] + o(1) \tag{2.11}$$

$$= \textstyle\sum_{j=n_c}^{\infty} P(\hat{\tau}_c \ge j) E[X_1 I_{\{X_1 \ge -(1+c^\beta)\mu\log[c/(1+c^\beta)\mu]\}}] + o(1)$$

$$= \{-(1+c^\beta)\mu \log[c/(1+c^\beta)\mu][c/(1+c^\beta)\mu]^{(1+c^\beta)}$$

$$+ \mu[c/(1+c^\beta)\mu]^{(1+c^\beta)}\}(E\hat{\tau}_c) + o(1).$$

From (2.10), (2.11), and (2.1), (2.2) of Lemma 1, since $c/(1+c^\beta)\mu < 1$ for c

sufficiently small, as $c \to 0$,

$$\begin{aligned}
0 &\le E(Y_{\tau_c^*}) - E(Y_{\hat{\tau}_c}) \\
&\le -\mu \log(c) + \mu \log(\mu) \\
&\quad - (E\hat{\tau}_c)\{-(1 + c^{\beta})\mu \log[c/(1 + c^{\beta})\mu][c/(1 + c^{\beta})\mu]^{(1+c^{\beta})} + \mu[c/(1 + c^{\beta})\mu]^{(1+c^{\beta})}\} \\
&\quad + c^{-c^{\beta}}[(1 + c^{\beta})\mu]^{(1+c^{\beta})} + o(1) \\
&\le -\mu \log(c) + [c/(1-c^{\beta})\mu]^{-(1-c^{\beta})}(1 + c^{\beta})\mu \log(c)[c/(1 + c^{\beta})\mu]^{(1+c^{\beta})} \\
&\quad + \mu \log(\mu) - \mu(1 + c^{\beta})\log[(1 + c^{\beta})\mu][c/(1 + c^{\beta})\mu]^{(1+c^{\beta})}[c/(1 - c^{\beta})\mu]^{-(1-c^{\beta})} \\
&\quad - \mu(c/(1 + c^{\beta})\mu)^{(1+c^{\beta})}[c/(1 - c^{\beta})\mu]^{-(1-c^{\beta})} + c^{-c^{\beta}}[(1 + c^{\beta})\mu]^{(1+c^{\beta})} + o(1) \\
&= -\mu \log(c)[1 - c^{2c^{\beta}}(1 + c^{\beta})^{-c^{\beta}}(1 - c^{\beta})^{(1-c^{\beta})}\mu^{-2c^{\beta}}] \\
&\quad + \mu \log(\mu)\{1 - \log[(1 + c^{\beta})\mu]c^{2c^{\beta}}\mu^{-2c^{\beta}}(1 + c^{\beta})^{-c^{\beta}}(1 - c^{\beta})^{(1 - c^{\beta})}/\log(\mu)\} \\
&\quad - \mu[c^{2c^{\beta}}(1 + c^{\beta})^{-(1+c^{\beta})}\mu^{-2c^{\beta}}(1 - c^{\beta})^{(1-c^{\beta})} - c^{-c^{\beta}}(1 + c^{\beta})^{(1+c^{\beta})}\mu^{c^{\beta}}] + o(1) = o(1)
\end{aligned}$$

by repeated application of l'Hôpital's Rule, proving the theorem.

3. Further remarks. The discussion above suggests that for any distribution of the X_i, if $E(X_1 - \gamma)^+ = f_{\theta}(\gamma)$, where θ is an unknown parameter (or perhaps a vector of parameters), and $f_{\hat{\theta}_n}(\hat{\gamma}_n) = c$, where $\hat{\theta}_n$ is an estimator of θ based on the first n observations, a stopping rule of the form

$$\hat{\tau}_c = \inf\{n \ge n_c\colon X_n \ge \hat{\gamma}_n\}$$

might be used to approximate the optimal rule τ_c^*. One would like to know something about the performance of such stopping rules (in particular, whether $E(Y_{\hat{\tau}_c})/E(Y_{\tau_c^*}) \to 1$ or $E(Y_{\tau_c^*}) - E(Y_{\hat{\tau}_c}) \to 0$ as $c \to 0$) for more general distributions than the exponential, or at least in certain other specific cases of interest, e.g., Poisson, normal, and general gamma distributions.

Unfortunately, the function $f_{\theta}(\gamma)$ is in general quite complicated, and it is not possible to give closed-form expressions for γ and $\hat{\gamma}_n$. Therefore, results about the performance of $\hat{\tau}_c$ depend on obtaining nice approximations to γ and $\hat{\gamma}_n$ (and ultimately to $E(\hat{\tau}_c)$ and $E(X_{\hat{\tau}_c})$), based on the properties of $f_{\theta}(\gamma)$.

As mentioned in Section 1, Bramblett (1965) was able to show asymptotic optimality of the approximating stopping times for certain cases involving unknown location parameters. However, results analogous to the theorem above have yet to be derived for those cases (or for any case other than the exponential).

Finally, it should be mentioned that many optimal stopping problems have solutions whose form is not given as explicitly as in (1.2), even when the relevant distributions are known. In such cases the methods of the present paper presumably cannot be used to approximate optimal expected rewards using only partial information, although the question of how well one can do in such situations is still an interesting one.

REFERENCES

BICKEL, P. and YAHAV, J. (1967). Asymptotically pointwise optimal procedures in sequential analysis. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* **1** 401–413.

BICKEL, P. and YAHAV, J. (1968). Asymptotically optimal Bayes and minimax procedures in sequential estimation. *Ann. Math. Statist.* **39** 442–456.

BRAMBLETT, J. E. (1965). Some approximations to optimal stopping procedures. Unpublished Columbia University Ph.D. dissertation.

CHEN, W.-C. and STARR, N. (1980). Optimal stopping in an urn. *Ann. Probab.* **8** 451–464.

CHOW, Y. S. and LAI, T. L. (1975). Some one-sided theorems on the tail distribution of sample sums with applications to the last time and largest excess of boundary crossings. *Trans. Amer. Math. Soc.* **208** 51–72.

CHOW, Y. S. and ROBBINS, H. (1961). A martingale system theorem and applications. *Proc. Fourth Berkeley Symp. Math. Statist. Probab.* **1** 93–104.

CHOW, Y. S. and ROBBINS, H. (1963). On optimal stopping rules. *Z. Wahrsch. verw. Gebiete* **2** 33–49.

CHOW, Y. S., ROBBINS, H. and SIEGMUND, D. (1971). *Great Expectations: The Theory of Optimal Stopping.* Houghton, Boston.

CHOW, Y. S. and TEICHER, H. (1978). *Probability Theory: Independence, Interchangeability, Martingales.* Springer, New York.

COHN, H. (1967). On certain optimal stopping rules. *Rev. Roumaine Math. Pures Appl.* **12** 1173–1177.

DEGROOT, M. (1968). Some problems of optimal stopping. *J. Roy. Statist. Soc. B* **30** 108–122.

DERMAN, C. and SACKS, J. (1960). Replacement of periodically inspected equipment (an optimal stopping rule). *Naval Res. Logist. Quart.* **7** 597–607.

KIEFER, J. and SACKS, J. (1963). Asymptotically optimum sequential inference and design. *Ann. Math. Statist.* **34** 705–750.

MACQUEEN, J. and MILLER, R. G., JR. (1960). Optimal persistence policies. *Operat. Res.* **8** 362–380.

RASMUSSEN, S. and STARR, N. (1979). Optimal and adaptive stopping in the search for new species. *J. Amer. Statist. Assoc.* **74** 661–667.

SAKAGUCHI, M. (1961). Dynamic programming of some sequential sampling designs. *J. Math. Anal. Appl.* **2** 446–466.

STARR, N. (1974). Optimal and adaptive stopping based on capture times. *J. Appl. Probab.* **11** 294–301.

YAHAV, J. (1966). On optimal stopping. *Ann. Math. Statist.* **37** 30–35.

DEPT. OF MATHEMATICS
273 ALTGELD HALL
UNIVERSITY OF ILLINOIS
1409 W. GREEN ST.
URBANA, ILLINOIS 61801