The Annals of Probability
1984, Vol. 12, No. 4, 1181–1193

STRONG LIMIT THEOREMS FOR MAXIMAL SPACINGS FROM A GENERAL UNIVARIATE DISTRIBUTION

BY PAUL DEHEUVELS

Université Paris VI

Let $X_1, X_2, \cdots$ be an i.i.d. sequence of random variables with a continuous density f. We consider in this paper the strong limiting behavior as $n \to \infty$ of the kth largest spacing $M_k^{(n)}$ induced by $X_1, \cdots, X_n$ in the sample range.

In the case where f is bounded away from zero inside a bounded interval and vanishes outside, we characterize the limiting behaviour of $M_k^{(n)}$ in terms of the local behavior of f in the neighborhood of the point where it reaches its minimum. In the case where the support of f is an unbounded interval, we prove that for any $k \geq 1$, $M_k^{(n)} \to 0$ a.s. as $n \to \infty$ if and only if the distribution of X_1 has strongly stable extremes.

1. Introduction. Let $X_1, X_2, \cdots$ be independent random variables with continuous distribution function F. Let

$$X_{1,n} < X_{2,n} < \cdots < X_{n,n}$$

denote the order statistics of $X_1, \cdots, X_n$, and let

$$S_i^{(n)} = X_{i+1,n} - X_{i,n}, \quad i = 1, \cdots, n-1,$$

define the corresponding *spacings*.

Denote the order statistics of $S_1^{(n)}, \cdots, S_{n-1}^{(n)}$ by

$$M_{n-1}^{(n)} < \cdots < M_2^{(n)} < M_1^{(n)}.$$

Our main goal is to characterize for a fixed $k \geq 1$ the limiting strong behavior of the kth *maximal spacing* $M_k^{(n)}$ as $n \to \infty$ under various assumptions on F.

In the case of the uniform distribution on (0, 1), the upper and lower strong classes of $M_k^{(n)}$ are known (Devroye, 1981, 1982a; Deheuvels, 1982, 1983a) and given respectively by

$$\begin{aligned} &P(nM_k^{(n)} > \operatorname{Log} n \\ &\quad + (1/k)(2\operatorname{Log}_2 n + \operatorname{Log}_3 n + \cdots + \operatorname{Log}_{p-1} n + (1+\varepsilon)\operatorname{Log}_p n) \quad \text{i.o.}) \\ &\qquad = P(nM_k^{(n)} < \operatorname{Log} n - \operatorname{Log}_3 n - \operatorname{Log} 2 - \varepsilon \quad \text{i.o.}) = 0 \quad \text{or} \quad 1, \end{aligned}$$

according as $\varepsilon > 0$ or $\varepsilon < 0$. Here, $p \geq 5$ is arbitrary, and Log_j stands for the jth iterated logarithm.

On the other hand, very few results are available in the nonuniform case. We intend to investigate this problem in this paper.

Our motivation for this work comes from the study of the convergence of

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Received April 1983; revised January 1984.

AMS 1980 *subject classification.* Primary 60F15.

Key words and phrases. Laws of the iterated logarithm, order statistics, spacings, strong laws, almost sure convergence, empirical processes, quantile processes.

empirical measures. It is straightforward that the maximal spacing corresponds to the maximal jump of the empirical quantile process. It follows that the study of questions such as the modulus of continuity of empirical processes is closely related to the study of the limiting behavior of $M_k^{(n)}$ as $n \to \infty$.

In particular, it is of a great interest to characterize the distributions for which $M_k^{(n)}$ tends to zero as n increases indefinitely. We shall show in the following that these distributions coincide with the distributions with *stable extremes* (i.e. whose extreme values $Y_n = X_{1,n}$ or $X_{n,n}$ are such that $Y_n - a_n \to 0$, for a nonrandom sequence a_n).

The principal achievement of this paper is to show that, in general, for distributions F with a continuous density f, the major influence on the behavior of maximal spacings is exerted by the behavior of f in the neighborhood of its minimum.

In Section 2, we present the main results, whose proofs will be given in Section 3 (see also Deheuvels, 1983c).

2. Theorems. We shall assume throughout this section that the following hypotheses are satisfied.

(H1) $F(x) = P(X_1 \leq x)$ *has a continuous first derivative* $f(x) > 0$ *on* (A, B) *where*

$$A = \inf\{x; F(x) > 0\} < B = \sup\{x; F(x) < 1\}.$$

We first settle the case of distributions with bounded support.

THEOREM 1. *Let* (H1) *be satisfied. Assume further that:*

(H2) *The distribution* F *has bounded support* $(-\infty < A < B < +\infty)$, *and there exists an* $x_0 \in (A, B)$ *such that, for all* $x \in (A, B)$, $x \neq x_0$, $f(x) > f(x_0) > 0$.

(H3) *There exists an* r, $0 < r +\infty$, *such that*

$$\lim \inf_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0)}{|h|^r} = d_r, \quad 0 < d_r < +\infty.$$

Then, for any $p \geq 5$, $k \geq 1$ *and* $\varepsilon > 0$,

$$P(nM_k^{(n)} f(x_0) > \operatorname{Log} n - (1/r) \operatorname{Log}_2 n + (1/k)(2 \operatorname{Log}_2 n + \operatorname{Log}_3 n + \cdots + \operatorname{Log}_{p-1} n + (1 + \varepsilon) \operatorname{Log}_p n) \text{ } i.o.) = 0.$$

THEOREM 2. *Let* (H1) *and* (H2) *be satisfied. Assume further that*

(H4) *There exists an* r, $0 < r < +\infty$, *such that*

$$\lim \sup_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0)}{|h|^r} = D_r, \quad 0 < D_r < +\infty.$$

Then, for any $k \geq 1$ *and* $\varepsilon > 0$,

$$P\left(nM_k^{(n)} f(x_0) < \operatorname{Log} n - \frac{1}{r} \operatorname{Log}_2 n - \operatorname{Log}_3 n - \frac{1}{r} \operatorname{Log}\left(\frac{reD_r}{\{f(x_0)\}^{r+1}}\right) - \varepsilon \text{ } i.o.\right) = 0.$$

REMARKS.

1. A typical application of Theorem 1 and Theorem 2 is given in the case where f is $2j$ times differentiable in a neighborhood of x_0 with

$$f^{(i)}(x_0) = 0, \quad 1 \le i < 2j, \quad f^{(2j)}(x_0) > 0.$$

It follows then that, almost surely,

$$-\frac{1}{2j} \le \liminf_{n\to\infty} \frac{nM_k^{(n)}f(x_0) - \operatorname{Log} n}{\operatorname{Log}_2 n} \le \limsup_{n\to\infty} \frac{nM_k^{(n)}f(x_0) - \operatorname{Log} n}{\operatorname{Log}_2 n} \le \frac{2}{k} - \frac{1}{2j}.$$

In Theorem 4, we shall show that these bounds are optimal.

2. In the case where f is infinitely differentiable in a neighborhood of x_0 with

$$f^{(i)}(x_0) = 0, \quad \text{for all } i \ge 1,$$

we get similar bounds as in the case of uniformly distributed random variables.

3. As mentioned in the introduction, the local behavior of f in the neighborhood of the point where it reaches its minimum conditions the limiting behavior of $M_k^{(n)}$, $n \to \infty$. In particular, we have:

THEOREM 3. *Let* (H1) *and* (H2) *be satisfied. Then, for any* $k \ge 1$, *and for any* $\delta > 0$, *there exists almost surely an* N *such that, for any* $n \ge N$, *the spacing interval* $(X_{i,n}, X_{i+1,n})$ *such that* $M_n^{(k)} = X_{i+1,n} - X_{i,n}$ *is included in* $(x_0 - \delta, x_0 + \delta)$.

Theorem 3 says that the kth maximal spacing must occur in a neighborhood of the value x_0 where f is minimal, when n increases.

The bounds in Theorem 1 and Theorem 2 are sharp up to the second term, as shown in the following.

THEOREM 4. *Let* (H1), (H2), (H3) *and* (H4) *be satisfied with* d_r *and* D_r *such that*

$$0 < d_r \le D_r < +\infty.$$

Then, we have, almost surely,

$$-\frac{1}{r} = \liminf_{n\to\infty} \frac{nM_k^{(n)}f(x_0) - \operatorname{Log} n}{\operatorname{Log}_2 n} < \limsup_{n\to\infty} \frac{nM_k^{(n)}f(x_0) - \operatorname{Log} n}{\operatorname{Log}_2 n} = \frac{2}{k} - \frac{1}{r}.$$

REMARKS.

1. It is remarkable that the limits are independent of the explicit values of d_r and D_r.

2. The above results do not enable us to handle the case where $f(x_0) = 0$. This will be dealt with elsewhere (see also Deheuvels, 1983c).

3. Throughout, we have assumed that $x_0 \in (A, B)$. In fact, the results remain valid without modification if $x_0 = A$ (resp. $x_0 = B$), with appropriate definitions of d_r and D_r. We must then assume that f is defined on $[A, B[$ (resp. $]A, B]$).

We now consider the case where the distribution has an unbounded support, which implies that

$$\min_{A<x<B} f(x) = 0.$$

It can be seen without great difficulty that if $A = -\infty$, $B = +\infty$, the study of the limiting behavior of the spacings can be done by considering the subsamples of the X's which fall in $(-\infty, 0)$ and $(0, +\infty)$ separately. Hence, without loss of generality, we shall only look at the case where $A = 0$, $B = +\infty$. Our main result is the following:

THEOREM 5. *Let* (H1) *be satisfied with* $A = 0$, $B = +\infty$. *Assume further that*
(H5) *The function* $\{1 - F(x)\}/f(x)$ *is ultimately nonincreasing as* $x \uparrow \infty$.
Let $k \geq 1$ *be fixed. Then,* $M_k^{(n)} \to 0$ *a.s. as* $n \to \infty$ *if and only if* $X_{n,n}$ *is strongly stable, i.e. iff there exists a nonrandom sequence* $\{a_n\}$ *such that*

$$X_{n,n} - a_n \to 0 \text{ a.s.}, \quad \text{as} \quad n \to \infty.$$

REMARKS.

1. Geffroy (1958) has proved that the weak stability of $X_{n,n}$ (i.e. the existence of $\{a_n\}$ such that $X_{n,n} - a_n \to 0 \quad P$ (i.e. in probability)) is equivalent to the condition

$$\lim_{x\uparrow\infty} \frac{1 - F(x)}{f(x)} = 0,$$

when f has a derivative $f'(x)$ ultimately nonincreasing as $x \uparrow \infty$.

He has also proved that the weak stability of $X_{n,n}$ is equivalent to the fact that $S_{n-1,n} = X_{n,n} - X_{n-1,n} \to 0 \quad P$ as $n \to \infty$.

It follows that the condition (H5) does not appear to be severely restrictive.

2. A survey on stability for extreme values is to be found in Galambos (1978), pages 213–231.

3. There is a great flurry of particular cases whose study may be of interest. In order to avoid a long discussion, we have limited ourselves to the essential points.

3. Proofs of the theorems. The proofs are rather lengthy and are given here in a condensed form. Details may be found in Deheuvels (1983c).

PROOF OF THEOREM 1. By (H1), we may assume without loss of generality that $A = 0$ and $B = 1$.

Next, we note that the event $\{M_k^{(n)} > \delta\}$ occurs only if there are at least k disjoint intervals of length δ in (0, 1) left empty by $X_1, \cdots, X_n$. Consider one of these intervals: $I = (a, a + \delta)$. For $n\delta > 3$, there exists an $i: 0 \le i \le n$, such that

$$(i - 1)/n < a \le i/n.$$

This implies that

$$i/n < (i - 2)/n + \delta < (i - 1)/n + \delta < a + \delta \le 1,$$

which in turn implies that $(i/n, (i - 2)/n + \delta)$ is included in I and that

$$i < 1 + n - n\delta < n - 1.$$

We have just proved that (for $n\delta > 3$) $\{M_k^{(n)} > \delta\}$ occurs only if there are at least k intervals of the form $(i/n, \delta + (i - 2)/n)$, $0 \le i < n + 1 - n\delta$, left empty by $X_1, \cdots, X_n$. It follows that, for $n\delta > 3$, if $N = [n + 1 - n\delta] \le n - 2$, we have

$$P(M_k^{(n)} > \delta) \le \sum_{i_1=0}^{N} \cdots \sum_{i_k=0}^{N} \prod_{j=1}^{k} (1 - \{F(\delta + (i_j - 2)/n) - F(i_j/n)\})^n$$

$$\le \sum_{i_1=0}^{N} \cdots \sum_{i_k=0}^{N} \exp(-n \textstyle\sum_{j=1}^{k} \{F(\delta + (i_j - 2)/n) - F(i_j/n)\}).$$

By (H1) and (H3), there exists a $\gamma > 0$ such that, for any $x \in (0, 1)$,

$$f(x) \ge f(x_0) + \min\{\gamma, \tfrac{1}{2} d_r |x - x_0|^r\}.$$

It follows that, for $0 \le x \le x + h \le 1$,

$$F(x + h) - F(x) \ge hf(x_0) + \frac{h}{r + 1} \min\left\{\gamma, \frac{1}{2} d_r |x + h - x_0|^r, \frac{1}{2} d_r |x - x_0|^r\right\},$$

and likewise, for $0 \le i/n \le \delta + (i - 2)/n \le 1$, that

$$F\left(\delta + \frac{i - 2}{n}\right) - F\left(\frac{i}{n}\right)$$

$$\ge \left(\delta - \frac{2}{n}\right)\left(f(x_0) + \frac{1}{r + 1} \min\left\{\gamma, \frac{1}{2} d_r \left|\delta + \frac{i - 2}{n} - x_0\right|^r, \frac{1}{2} d_r \left|\frac{i}{n} - x_0\right|^r\right\}\right).$$

This implies that

$$P(M_k^{(n)} > \delta) \le \exp\left(-nk\left(\delta - \frac{2}{n}\right)f(x_0)\right)\{P_1 + P_2 + P_3\},$$

where

$$P_1 = n^k \exp\left(-nk\left(\delta - \frac{2}{n}\right)\frac{\gamma}{r+1}\right),$$

$$P_2 = \sum_{i_1=0}^{n-2} \cdots \sum_{i_k=0}^{n-2} \exp\left(-n\left(\delta - \frac{2}{n}\right)\frac{d_r}{2(r+1)} \sum_{j=1}^{k} \left| \frac{i_j}{n} - x_0 \right|^r\right),$$

$$P_3 = \sum_{i_1=0}^{n-2} \cdots \sum_{i_k=0}^{n-2} \exp\left(-n\left(\delta - \frac{2}{n}\right)\frac{d_r}{2(r+1)} \sum_{j=1}^{k} \left| \delta + \frac{i_j - 2}{n} - x_0 \right|^r\right).$$

The evaluation of P_2 and P_3 can then be made by using the fact that Riemann sums can be approximated by integrals, which gives

$$P_i = n^k O\left\{\int_{-\infty}^{+\infty} \exp\left(\frac{-n\delta d_r}{2(r+1)} \,|x|^r\right) dx\right\}^k,$$

$$i = 2, 3, \quad \delta \to 0, \quad n\delta \to \infty, \quad n\delta = O(\operatorname{Log} n).$$

It follows that

$$P_i = O\{n^k (n\delta)^{-k/r}\}, \quad i = 1, 2, 3, \; n \to \infty.$$

Combining these results, we see that

$$P(M_k^{(n)} > \delta) = O\{n^k (n\delta)^{-k/r} \exp(-nk\delta f(x_0))\}.$$

Taking

$$nf(x_0)\delta = \operatorname{Log} n - r^{-1}\operatorname{Log}_2 n + k^{-1}\{2\operatorname{Log}_2 n + \operatorname{Log}_3 n + \cdots + (1+\varepsilon)\operatorname{Log}_p n\},$$

we obtain the following lemma.

LEMMA 1. *Under the hypotheses of Theorem* 1,

$$P\left(nM_k^{(n)} f(x_0) > \operatorname{Log} n - \frac{1}{r}\operatorname{Log}_2 n + \frac{1}{k}\{2\operatorname{Log}_2 n + \operatorname{Log}_3 n + \cdots + (1+\varepsilon)\operatorname{Log}_p n\}\right)$$

$$= O\left(\frac{1}{(\operatorname{Log} n)^2 (\operatorname{Log}_2 n) \cdots (\operatorname{Log}_{p-1} n)^{1+\varepsilon}}\right).$$

The proof of Theorem 1 is now complete since (see Deheuvels, 1983b, Lemma 1) it is enough to apply Borel-Cantelli to the events in Lemma 1 for the subsequence $n_j = [\exp(\sqrt{j})]$ and arbitrary $\varepsilon > 0$. This follows from the fact that if $u_\varepsilon(n) = n^{-1}(\operatorname{Log} n - r^{-1}\operatorname{Log}_2 n + k^{-1}\{2\operatorname{Log}_2 n + \operatorname{Log}_3 n + \cdots + (1+\varepsilon)\operatorname{Log}_p n\})$, then

$$u_{\varepsilon/2}(n_j) \geq u_\varepsilon(n_{j+1})$$

for large j.

Proof of Theorem 2. In the proof, we consider the spacings generated in the interval $(x_0 - h_n, x_0 + h_n)$, where $h_n = a(\text{Log } n)^{-1/r}$.

By a result of Hall (1982, Theorem 2), it can be shown that the number N_n of terms among $X_1, \cdots, X_n$ which fall into $(x_0 - h_n, x_0 + h_n)$ is such that $P(B_n^c \text{ i.o.}) = 0$, where

$$B_n = \left\{ \left| \frac{n\{F(x_0 + h_n) - F(x_0 - h_n)\}}{N_n} - 1 \right| < 2\left\{ \frac{\text{Log}_2 n}{nh_n f(x_0)} \right\}^{1/2} \right\}.$$

Our next argument is based on a simple remark. Let $Y_1, \cdots, Y_N$ be independent random variables with a common density $h(\cdot)$ on (a, b). Let $H(x) = \int_a^x h(t)\, dt$. Since $H(Y_1), \cdots, H(Y_N)$ are i.i.d. random variables uniformly distributed on $(0, 1)$, if μ_k stands for the kth maximal spacing generated by $H(Y_1), \cdots, H(Y_N)$ and if Δ_k stands for the kth maximal spacing generated by $Y_1, \cdots, Y_n$ inside $(0, 1)$ and (a, b) respectively, we have

$$\mu_k \le \Delta_k \sup_{a<x<b} h(x).$$

If we consider in particular the spacings generated by N(nonrandom) random variables among $X_1, X_2, \cdots$ falling into $(x_0 - h_n, x_0 + h_n)$, we have

$$h(x) = \frac{f(x)}{F(x_0 + h_n) - F(x_0 - h_n)}.$$

It follows that if $\Delta_k^{(n)}$ denotes the kth largest among these spacings and if $\mu_k^{(N)}$ denotes the kth maximal spacing generated by a sample of N uniformly distributed random variables on $(0, 1)$, then

$$P\left(\Delta_k^{(n)} \le \frac{F(x_0 + h_n) - F(x_0 - h_n)}{\phi_n} \frac{\text{Log } N}{N} (1 - a_N) \,|\, N \right) \le P\left(\mu_k^{(N)} \le \frac{\text{Log } N}{N} (1 - a_N) \right),$$

where

$$\phi_n = \sup\{f(x);\ |x - x_0| \le h_n\}$$

By a result of Devroye (1981, Lemma 3.2), if we take

$$a_N = \frac{\text{Log}_3 N + \lambda + \text{Log } 2}{\text{Log } N},$$

where λ is a constant, we get

$$P\left(\mu_k^{(N)} \le \frac{\text{Log } N}{N} (1 - a_N) \right) \sim \frac{1}{(k-1)!} N^{(k-1)a_N} \exp(-N^{a_N}) = O\{(\text{Log}_2 N)^{k-1} (\text{Log } N)^{-2e^\lambda}\}, \quad N \to \infty.$$

If we assume now that B_n is satisfied, then

$$\text{Log } N_n = \text{Log } N - \frac{1}{r} \text{Log}_2 n + \text{Log}(2af(x_0)) + o(1), \quad n \to \infty.$$

It follows, by some straightforward evaluations, that for $\varepsilon > 0$,

$$P\left(M_k^{(n)} \leq \frac{\text{Log } n - (1/r)\text{ Log}_2 n - \text{Log}_3 n + \text{Log}(2af(x_0)) - \lambda - \text{Log } 2 - \varepsilon}{n\phi_n} \,\Big|\, B_n\right)$$
$$= O\{(\text{Log}_2 n)^{k-1}(\text{Log } n)^{-2e^\lambda}\}, \quad n \to \infty.$$

Here we have used the fact that $M_k^{(n)} \geq \Delta_k^{(n)}$ when $N = N_n$.

It remains to evaluate ϕ_n. By (H4), we have

$$\phi_n \leq f(x_0) + D_r(1 + o(1))h_n^r = f(x_0) + D_r(1 + o(1))a^r(\text{Log } n)^{-1}, \quad n \to \infty.$$

It follows that

$$\frac{\text{Log } n}{n\phi_n} \geq \frac{\text{Log } n - \{D_r a^r/f(x_0)\}(1 + o(1))}{nf(x_0)}, \quad n \to \infty.$$

Therefore we obtain the lemma:

LEMMA 2. *Under the assumptions of Theorem* 2, *we have, for any* $\varepsilon > 0$, *as* $n \to \infty$,

$$P\left(\left\{M_k^{(n)} \leq \frac{\text{Log } n - (1/r)\text{Log}_2 n - \text{Log}_3 n + \text{Log}(af(x_0)) - \{D_r a^r/f(x_0)\} - \lambda - \varepsilon}{nf(x_0)}\right\} \cap B_n\right)$$
$$= O\{(\text{Log}_2 n)^{k-1}(\text{Log } n)^{-2e^\lambda}\}.$$

Noting that B_n^c occurs finitely often with probability one, Theorem 2 is proved if we chose a in Lemma 2 in order to minimize $\text{Log}(af(x_0)) - \{D_r a^r/f(x_0)\}$ (whose minimum is $-r^{-1}\text{Log}(reD_r/\{f(x_0)\}^{r+1})$) and use a Borel-Cantelli argument based on subsequences $M_k^{(n_j)}$ with $n_j = [\exp(j^\alpha)]$, $0 < \alpha < 1/2$.

For this, note that

$$\textstyle\sum_k (\text{Log}_2 n_j)^{k-1}(\text{Log } n_j)^{-2e^\lambda} < \infty \quad \text{if} \quad 2\alpha e^\lambda > 1.$$

Furthermore, if $v_\varepsilon(n) = n^{-1}\{\text{Log } n - r^{-1}\text{Log}_2 n - \text{Log}_3 n - C - \varepsilon\}$, $\varepsilon > 0$, and if $n_j = [\exp(j^\alpha)]$, $0 < \alpha < 1/2$, we have

$$v_\varepsilon(n_j) \geq v_{\varepsilon/2}(n_{j+1})$$

for large j.

Finally $\lambda > 0$, $\varepsilon > 0$ can be chosen arbitrarily small with $(1/2)e^{-\lambda} < \alpha < 1$, hence result.

PROOF OF THEOREM 3. Theorem 3 follows from Theorem 1 and Theorem 2, by considering separately the maximal spacing outside $(x_0 - \delta, x_0 + \delta)$ and the kth maximal spacing inside $(x_0 - \delta/2, x_0 + \delta/2)$, and by letting $\delta \downarrow 0$.

PROOF OF THEOREM 4. The proof is based on the same arguments as in Deheuvels (1982) or Deheuvels (1983a). Hence, details will be omitted.

We consider the stopping times ν_j, where ν_j is the jth value of n where $M_k^{(n)}$ decreases. Let $\Lambda_k^n = (X_{\ell,n}, X_{\ell+1,n})$ be the (unique w.p.1) interval such that

$X_{\ell+1,n} - X_{\ell,n} = M_k^{(n)}$. We prove successively that:

1. Without loss of generality, it is possible to define on the same probability space as $X_1, X_2, \cdots$ an i.i.d. sequence $\omega_1, \omega_2, \cdots$ of exponentially distributed random variables such that

$$\nu_{j+1} - \nu_j = [\omega_j / \{-\mathrm{Log}(1 - P_x(X \in \cup_{i=1}^{k} \Lambda_i^{(\nu_j)}))\}] + 1, \quad j = 1, 2, \cdots,$$

where X is distributed as X_1 and independent of $X_1, X_2, \cdots$.

2. As $j \to \infty$, we have, almost surely

$$\nu_{j+1} = \nu_j \left\{ 1 + \frac{\omega_j(1 + o(1))}{k \,\mathrm{Log}\, \nu_j} \right\}, \quad \mathrm{Log}^2 \nu_{j+1} - \mathrm{Log}^2 \nu_j = \frac{2\omega_j(1 + o(1))}{k},$$

and

$$\mathrm{Log}^2 \nu_j = \frac{2j}{k}(1 + o(1)).$$

3. We have almost surely for infinitely many j's:

$$\omega_j > \mathrm{Log}\, j + \mathrm{Log}_2 j > 2\mathrm{Log}_2 \nu_j.$$

4. Let $\mu_j = M_k^{(\nu_j)}$. Put $\nu_j^\star = \nu_{j+1} - 1$. By Theorem 2, we have almost surely for $\varepsilon > 0$ and $j \to \infty$:

$$\mu_j > \frac{\mathrm{Log}\, \nu_j - ((1/r) + \varepsilon)\mathrm{Log}_2 \nu_j}{\nu_j f(x_0)}.$$

If now j is such that $\omega_j > 2\mathrm{Log}_2 \nu_j$, we have then (for large j)

$$M_k^{(\nu_j^\star)} = \mu_j > \frac{\mathrm{Log}\, \nu_j^\star + ((2/k) - (1/r) - \varepsilon + o(1))\mathrm{Log}_2 \nu_j^\star}{\nu_j^\star f(x_0)}.$$

This proves that the upper bound of Theorem 4 is reached. A similar argument can be used for the lower bound.

PROOF OF THEOREM 5. We shall make use of the function

$$G(u) = (1 - F)^{-1}(u) = \sup\{x;\, 1 - F(x) > u\}, \quad 0 < u < 1.$$

Without loss of generality, it is possible to assume that $X_{n,n} = G(U_{1,n})$ where $U_{1,n} = \min_{1 \le i \le n} U_i$, and where $U_1, U_2, \cdots$ is an i.i.d. sequence of random variables uniformly distributed on (0, 1). By the characterization of the almost sure upper and lower classes of $U_{1,n}$ due to Barndorff-Nielsen (1961) (see Devroye, 1982b, pages 237–238), we have for $\varepsilon > 0$, almost surely,

$$\liminf_{n\to\infty} n(\mathrm{Log}\, n) U_{1,n} = 0, \quad \lim_{n\to\infty} n(\mathrm{Log}\, n)^{1+\varepsilon} U_{1,n} = +\infty,$$

and

$$\limsup_{n\to\infty} n(\mathrm{Log}_2 n)^{-1} U_{1,n} = 1.$$

It follows that a necessary condition for strong stability of $X_{n,n}$ is that

$$\lim\inf_{u\downarrow 0}\{G(u) - G(u\mathrm{Log}(1/u))\} = 0.$$

while a sufficient condition for strong stability of $X_{n,n}$ is that

$$\lim_{u\downarrow 0}\{G(u) - G(u\mathrm{Log}(1/u))\} = 0.$$

If we assume that (H5) is true (i.e. $\{1 - F(x)\}/f(x)$ is ultimately nonincreasing as $x \uparrow \infty$), then both conditions are equivalent, since then $G(u) - G(u\mathrm{Log}(1/u))$ is nonincreasing as $u \downarrow 0$. This last result follows from the inequality

$$\frac{d}{du}\left\{G(u) - G\left(u\mathrm{Log}\left(\frac{1}{u}\right)\right)\right\} = -\frac{1}{u}\left\{\frac{u}{f(G(u))} - \frac{u\mathrm{Log}(1/u) - u}{f(G(u\mathrm{Log}(1/u)))}\right\}$$

$$\leq \frac{1}{u}\left\{\Psi\left(u\mathrm{Log}\left(\frac{1}{u}\right)\right) - \Psi(u)\right\} \leq 0,$$

where $\Psi(u) = u/f(G(u))$, and $\Psi(1 - F(x)) = \{1 - F(x)\}/f(x)$.

Therefore we obtain the lemma:

LEMMA 3. *Under* (H5), $X_{n,n}$ *is strongly stable iff*

$$\lim_{u\downarrow 0}\{G(u) - G(u\mathrm{Log}(1/u))\} = 0.$$

REMARKS.

1. If $X_{n,n}$ is strongly stable, then $X_{n,n} - G(1/n) \to 0$ a.s. as $n \to \infty$.
2. A typical example of distribution such that $X_{n,n}$ is strongly stable is given by the normal distribution. A counterexample is given by the exponential distribution.

Next, we consider an i.i.d. sample of uniformly distributed random variables on (0, 1) and the corresponding order statistics

$$0 = U_{0,n} < U_{1,n} < \cdots < U_{n,n} < 1 = U_{n+1,n}.$$

Let us assume, without loss of generality, that $X_{i,n} = G(U_{n-i+1,n})$, $1 \leq i \leq n$. We have then

$$M_k^{(n)} \leq M_1^{(n)} = \max_{1\leq i\leq n-1}\{G(U_{i,n}) - G(U_{i+1,n})\}.$$

Since there exists almost surely an n_0 such that, for $n \geq n_0$,

$$U_{i+1,n} - U_{i,n} \leq \frac{2\mathrm{Log}\, n}{n}, \quad 0 \leq i \leq n + 1,$$

it follows that there exists almost surely an n_1 such that for $n \geq n_1$:

$$M_k^{(n)} \leq M_1^{(n)} \leq \sup_{U_{1,n}\leq u\leq 1}\left\{G(u) - G\left(u + \frac{2\mathrm{Log}\, n}{n}\right)\right\} \leq A_1 + A_2 + A_3,$$

where, for $0 < c < 1$,

$$A_1 = \sup_{c \le u < 1}\left\{G(u) - G\left(u + \frac{2\mathrm{Log}\, n}{n}\right)\right\},$$

$$A_2 = \sup_{n^{-1}\mathrm{Log} n \le u \le c}\{G(u) - G(3u)\},$$

$$A_3 = \sup_{n^{-1}\mathrm{Log}^{-2} n \le u \le n^{-1}\mathrm{Log} n}\{G(u) - G(u\mathrm{Log}^4(1/u))\}.$$

Note that

$$u + \frac{2\mathrm{Log}\, n}{n} \le \frac{3\mathrm{Log}\, n}{n} \le u\mathrm{Log}^4\left(\frac{1}{u}\right)$$

for $n^{-1}\mathrm{Log}^{-2}n \le u \le n^{-1}\mathrm{Log}\, n$ and large n.

Here, we have used the characterization of the upper class of the uniform spacings given in Section 1, and the bounds (see Barndorff-Nielsen, 1961), satisfied almost surely as $n \to \infty$:

$$n^{-1}\mathrm{Log}^{-2}n < U_{1,n}.$$

If we assume now that (H5) is satisfied and that $X_{n,n}$ is strongly stable, then by Lemma 3, it follows that for a fixed $c \in (0, 1)$, $A_1 \to 0$ and $A_3 \to 0$ as $n \to \infty$. Since we can also choose c such that A_2 is arbitrarily small, we have proved that $M_k^{(n)} \to 0$ almost surely as $n \to \infty$.

Conversely, if we assume that $M_k^{(n)} \to 0$ almost surely as $n \to \infty$, then we must have

$$\min_{1 \le i \le k}\{X_{i+1,n} - X_{i,n}\} \to 0 \text{ a.s.}, \quad n \to \infty.$$

It is well known (Malmquist, 1950, see David, 1981, page 21) that for $i = 1, \cdots, k$, the ratios

$$\xi_{i,n} = \left\{\frac{U_{i,n}}{U_{i+1,n}}\right\}^i, \quad 1 \le i \le n,$$

are independent and uniformly distributed random variables on (0, 1).

LEMMA 4. *Let $k > 1$ be fixed. Then, almost surely for infinitely many n, we have*

$$\min_{1 \le \ell \le k}\xi_{\ell,n} < 1/\mathrm{Log}\, n.$$

PROOF. We define a sequence of random stopping times by

$$m_0 = k,\ m_j = \min\{n > m_j;\ U_{k,n} < U_{1,m_j}\}, \quad j = 1, 2, \cdots.$$

Let

$$\eta_j = \min_{1 \le i \le k}\xi_{i,m_j}.$$

It is easily seen that $\eta_1, \eta_2, \cdots$ is an i.i.d. sequence of random variables with distribution given by

$$P(\eta_i > u) = (1 - u)^k, \quad 0 < u < 1, \quad i = 1, 2, \cdots.$$

By Borel-Cantelli, it follows that $\eta_i < 1/2ki$ infinitely often with probability one.

Next, we remark that if we define the sequence of record times by

$$n_1 = 1, \; n_j = \min\{n > n_j; \; U_{1,n} < U_{1,n_j}\}, \quad j = 1, 2, \cdots,$$

we have

$$m_j \le n_{kj}, \quad j = 1, 2, \cdots.$$

It is well known (Renyi, 1962, Deheuvels, 1983d) that, almost surely as $j \to \infty$,

$$n_j = \exp(j + O(\{j \operatorname{Log}_2 j\}^{1/2})).$$

It follows evidently that, almost surely as $i \to \infty$,

$$\operatorname{Log} m_i \le ki + O(\{i \operatorname{Log}_2 i\}^{1/2}).$$

This in turn implies that, infinitely often with probability one,

$$\eta_i = \min_{1\le \ell \le k} \xi_{\ell, m_i} \le \frac{1}{2ki} < \frac{1}{\operatorname{Log} m_i}$$

for large i, a.s. This completes the proof of Lemma 4.

Proof of Theorem 5 (*continued*). If we assume that

$$\min_{1\le i\le k}\{X_{i+1,n} - X_{i,n}\} = \min_{1\le i\le k}\{G(U_{i,n}\xi_{i,n}^{-1/i}) - G(U_{i,n})\} \to 0,$$

we must have by Lemma 4

$$\liminf_{u\downarrow 0}\{G(u(\operatorname{Log}(1/u))^{1/k}) - G(u)\} = 0,$$

which implies that the condition of Lemma 3 is satisfied and completes the proof of Theorem 5.

Note here that (by the change of variables $u = v(\operatorname{Log}(1/v))^{r-s}$), if

$$\lim_{u\downarrow 0}\{G(u(\operatorname{Log}(1/u))^s) - G(u)\} = 0, \quad s \ne 0,$$

then we have equivalently

$$\lim_{u\downarrow 0}\{G(u(\operatorname{Log}(1/u))^r) - G(u)\} = 0 \quad \text{for all } r.$$

4. Acknowledgments. The author thanks the referee for some valuable suggestions.

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7 AVENUE DU CHATEAU
92340 BOURG-LA-REINE
FRANCE