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A LIMIT THEOREM FOR N_{0n}/n IN FIRST-PASSAGE PERCOLATION

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Let U be the distribution function of the nonnegative passage time of an individual bond of the square lattice, and let θ_{0n} denote one of the first passage times a_{0n}, b_{0n}. We define

$$N_{0n} = \min\{|r|: r \text{ is a route of } \theta_{0n}\},$$

where $|r|$ is the number of bonds in r. It is proved that if $U(0) > 1/2$ then

$$\lim_{n\to\infty} \frac{N_{0n}^{a}}{n} = \lim_{n\to\infty} \frac{N_{0n}^{b}}{n} = \lambda \quad \text{a.s.} \quad \text{and in } L^1,$$

where λ is a constant which only depends on $U(0)$.

1. Introduction. First-passage percolation was first studied by Hammersley and Welsh [2]. We generally follow the notation of [5]. The lattice L of intergral points in the plane is viewed as a graph with points $v = (v', v'')$ of $\mathbb{Z} \times \mathbb{Z}$, and bonds the line segments between adjacent points. Here points v and w are called adjacent if $v' = v'$, $|v'' - w''| = 1$, or if $|v' - w'| = 1$, $v'' = w''$. A *path* of L from v to w is a sequence $(v_0, e_1, v_1, \cdots, e_n, v_n)$ with v_i, v_{i+1} adjacent points in L and such that $v_0 = v$, $v_n = w$, and with e_i the bond connecting v_{i-1} and v_i. The path is called *selfavoiding* if $v_i \neq v_j$ for $i \neq j$. The path $(v_0, e_1, v_1, \cdots, e_n, v_n)$ is called a selfavoiding circuit if $v_i \neq v_j$ for $i \neq j$ with the exception $v_0 = v_n$.

To each bond e is assigned a random variable $X(e)$, called the *passage time* of e. It is assumed that all $X(e)$, $e \in L$, are independent identically distributed with distribution U, satisfying $U(0\text{-}) = 0$ (which means that the $X(e)$ are nonnegative random variables). The passage time of the path $r = (v_0, e_1, \cdots, e_n, v_n)$ is defined as $t(r) = \sum_{i=1}^{n} X(e_i)$ and the length $|r|$ of the path r is defined as the number of the bonds of r.

As in [5] we set for $m \leq n$

$$a_{mn} = \inf\{t(r): r \text{ is a selfavoiding path from } (m, 0) \text{ to } (n, 0)\},$$
$$b_{mn} = \inf\{t(r): r \text{ is a selfavoiding path from } (m, 0) \text{ to the line } x = n\}.$$

A selfavoiding path r from $(m, 0)$ to $(n, 0)$ with $t(r) = a_{mn}$ is called a *route* from a_{mn}, a selfavoiding path r from $(m, 0)$ to $x = n$ with $t(r) = b_{mn}$ is called a *route* for b_{mn}. It is proved in [5], Theorem 4.10 that routes exist with probability one for a_{0n} and for b_{0n}. Finally we define

$$N_{mn}^{\theta} = \min\{|r|: r \text{ is a route for } \theta_{mn}\},$$

where $\theta = a$ or b. Our result is as follows.

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THEOREM. *If $U(0) > 1/2$, then*

$$\lim_{n\to\infty} \frac{N^a_{0n}}{n} = \lim_{n\to\infty} \frac{N^b_{0n}}{n} = \lambda \quad \text{a.s.} \quad \text{and in } L^1, \tag{1.1}$$

where λ is a constant which depends only on $U(0)$.

REMARKS. (i) By a rather involved argument it is possible to show that also

$$\lim_{n\to\infty} \frac{N^t_{0n}}{n} = \lim_{n\to\infty} \frac{N^s_{0n}}{n} = \lambda \quad \text{a.s.} \quad \text{and in } L^1,$$

where s_{0n} and t_{0n} are the cylinder passage times defined in [5], Definition 4.2 and N^θ_{0n} is as above even for $\theta = s$ or t.

(ii) It is not hard to prove that $\lambda(\cdot)$ is a nonincreasing function of $U(0)$ on $U(0) > 1/2$. Moreover, simple estimates on the number of self avoiding paths from $(0, 0)$ to $(n, 0)$ of length $\le n(1 + \delta)$, show that for sufficiently small $\delta > 0$

$$P\{\exists \text{ self avoiding path from } (0,0) \text{ to } (n,0) \text{ with } t(r) \le \delta n \text{ and } |r| \le n(1+\delta)\} \to 0$$

as $n \to \infty$. From this it follows that

$$\lambda > 1 \quad \text{for} \quad U(0) > 1/2.$$

To prove our theorem we need the following known facts from percolation theory (see [5]). A *Bernoulli percolation* is a family of independent identically distributed random variables $\{X(e), e \in L\}$ as above, with U a Bernoulli distribution

$$U(x) = \begin{cases} 0 & x < 0 \\ p & 0 \le x < 1 \\ 1 & 1 \le x. \end{cases}$$

This means $p = P(X(e) = 0)$, $1 - p = P(X(e) = 1)$. It is customary to call e *open* (*closed*) if $X(e) = 0$ (respectively $X(e) = 1$). A path r is open (closed) if each bond of r is open (closed). We define an open (closed) *cluster* to be a maximal connected subgraph which has all its bonds open (closed). It is known (cf [3], Section 9, [5], Theorems 3.2 and 3.14) that

(1.2) If $p > 1/2$, then, with probability one, there is exactly one infinite open cluster, and no infinite closed cluster. Even more, the upper and lower open half plane each contain exactly one infinite open cluster.

(1.3) If $p > 1/2$, then, for any point ν there exist infinitely many disjoint open circuits around ν with probability 1.

A collection of random variables $\{X_{mn}, m \le n, m, n \in \mathbb{N}\}$ is called a *subadditive process* if it satisfies (1.4)–(1.6) below.

$$\text{If } m < p < n \quad \text{then} \quad X_{mn} \le X_{mp} + X_{pn}. \tag{1.4}$$

(1.5) The process $\{X_{m+1,n+1}\}$ has the same joint distributions as the process $\{X_{m,n}\}$.

(1.6) $EX_{0n}^{+} < \infty$ for all $n \in \mathbb{N}$ and $\inf_n(Ex_{0n}/n) \geq A$ for some finite constant A.

2. Proof of the theorem. The proof is broken up into several lemmas. We consider a bond e with $X(e) = 0$ to be open, and with $X(e) > 0$ to be closed. The open edges can then be viewed as the open edges of a Bernoulli percolation. The unique infinite open cluster will be denoted by Λ. The upper (lower) open half plane will be denoted by $L_+(L_-)$ and its closure by $\bar{L}_+(\bar{L}_-)$. We now choose for each site $(p, 0)$ a special open circuit D_p surrounding $(p, 0)$. Since $U(0) > ½$ there exists a minimal $q \geq 1$ such that the square $S = S(p, q) := [p - q, p + q] \times [q, q]$ contains a point of an infinite open cluster in each of L_+ and L_-. D_p is the minimal open circuit containing S in its interior. It follows from (1.3) and (1.4) that D_p exists with probability one. D_p is unique.

The reason for introducing the D_p is that they allow a fairly explicit description of routes for a_{0n} and b_{0n}, as given in the first lemma.

LEMMA 1(a). *Assume that D_m and D_n lie in each other's exterior. Then a path r from $(m, 0)$ to $(n, 0)$ is a route for a_{mn} if and only if it consists of three pieces r_1, r_2, r_3 of the following nature. r_1 connects $(m, 0)$ to D_m inside D_m (except for its endpoint on D_m) and has minimal passage time among all such paths. r_3 connects $(n, 0)$ to D_n inside D_n (except for its endpoint on D_n) and has minimal passage time among all such paths. Finally r_2 is contained in Λ and connects D_m with D_n. In this case*

$$a_{mn} = t(r) = t(r_1) + t(r_2).$$

(b) *Assume that D_0 lies in the half plane $\{x \leq n\}$. Then a path r from* (0, 0) *to the line $\{x = n\}$ is a route for b_{0n} if and only if r consists of two pieces r_1 and r_2, with r_1 as in part* (a) *and r_2 contained in Λ and connecting D_0 to $\{x = n\}$. In this case*

$$b_{mn} = t(r) = t(r_1).$$

PROOF. We only prove part (a); part (b) is quite similar. Consider any path r from $(m, 0)$ to $(n, 0)$. r must intersect D_m and D_n, since $(m, 0)$ lies in the exterior of D_n and $(n, 0)$ in the exterior of D_m. Denote the first intersection of r with D_m by v_1 and the last intersection with D_n by v_2. Any points on D_m and D_n can be connected by a path of zero passage time on Λ, since D_n and D_m belong to Λ, by construction. Thus, for r to be a route for a_{mn}, the piece of r between v_1 and v_2 must have zero passage time. Moreover, the piece from $(m, 0)$ to v_1 must have minimal passage time among all paths from $(m, 0)$ to D_m which lie inside D_m except for their endpoint on D_m. A similar statement holds for the piece of r from $(n, 0)$ to v_2. Part (a) easily follows from these observations. □

LEMMA 2. *If* $U(0) > 1/2$, *then for any* $m > 1$

$$E\,|\,D_n\,|^m < \infty \tag{2.1}$$

and

$$\lim_{n\to\infty} (\,|\,D_n\,|^m/n) = 0 \quad a.s. \tag{2.2}$$

PROOF. Essentially the same argument as in Lemmas 2.3 and 2.4 of [1] show that $E\,|\,D_0\,|^m < \infty$. (2.2) now follows from the Borel-Cantelli lemma, since the $|\,D_n\,|$ are identically distributed.

LEMMA 3. *If* $U(0) > 1/2$, *then there exist a constant* λ, *which depends on* $U(0)$ *only, such that*

$$\lim_{n\to\infty} (1/n)\, N^a_{0n} = \lambda \quad a.s. \quad \text{and in } L^1. \tag{2.3}$$

PROOF. First we show that $\{X_{m,n}\} := \{N^a_{mn} + 2\,|\,D_n\,|^2 + 2\,|\,D_n\,|\}$, $m \le n$, is a subadditive process, i.e., $\{X_{m,n}\}$ satisfies (1.4)–(1.6). (1.5) is trivial since the $X(e)$, $e \in L$, are i.i.d. (1.6) holds with $A = 0$ by virtue of (8.32) in [5] and (2.1). To prove (1.4) let $r^a_{mn} (m \le n)$ be a route for a_{mn} with $|\,r^a_{mn}\,| = N^a_{mn}$ and take $m < p < n$. Let $\bar{D}_p$ denote the union of D_p and its interior. For any fixed sample point, four cases are possible.

(i) Both $(m, 0)$ and $(n, 0)$ are contained in $\bar{D}_p$.
(ii) Neither $(m, 0)$ nor $(n, 0)$ are contained in $\bar{D}_p$.
(iii) $(m, 0)$ is contained in $\bar{D}_p$, but $(n, 0)$ is not.
(iv) $(n, 0)$ is contained in $\bar{D}_p$, but $(m, 0)$ is not.

CASE (i). If $r^a_{mn} \subset \bar{D}_p$, then

$$N^a_{mn} = |\,r^a_{mn}\,| \le \text{number of edges in } \bar{D}_p \le 2\,|\,D_p\,|^2.$$

(1.4) follows trivially in this case.

If r^a_{mn} is not contained in $\bar{D}_p$ then consider the following path r which is contained in $\bar{D}_p$. First r goes along r^a_{mn} till the first intersection of r^a_{mn} and D_p, then along D_p to the last intersection of r^a_{mn} and D_p, finally along r^a_{mn} to $(n, 0)$. (See Figure 1.) r is a route for a_{mn} by the proof of Lemma 1. Hence

$$N^a_{mn} \le |\,r\,| \le 2\,|\,D_p\,|^2,$$

and (1.4) again follows.

CASE (ii). We denote the first intersection of r^a_{mp} and D_p by v_1 and the last intersection of r^a_{pn} and D_p by v_2. Also, we denote the first (last) intersection of r^a_{mn} and Λ by $v_3(v_4)$. See Figure 2. r^a_{mn} must intersect Λ, so that v_3 and v_4 exist, because $(m, 0)$ and $(n, 0)$ are separated by a curve in $\Lambda \cup S(p, q)$, where $S(p, q)$ is the square used in the definition of D_p. Indeed Λ contains a path π_+ in L_+ connecting a point of $S(p, q)$ to ∞ and a path π_- in L_- connecting a point of $S(p, q)$ to ∞; π_+ and π_- can be connected by an arc of $S(p, q)$ which only inter-

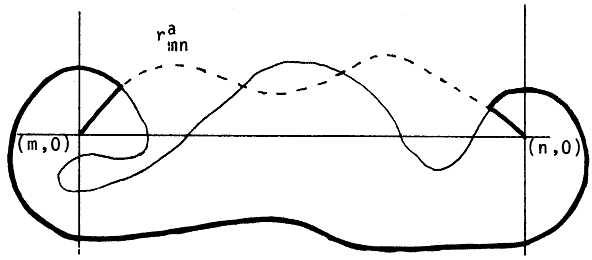

FIG. 1. *r^a_{mn} is dashed; D_p is the solid circuit. r is the boldly drawn path.*

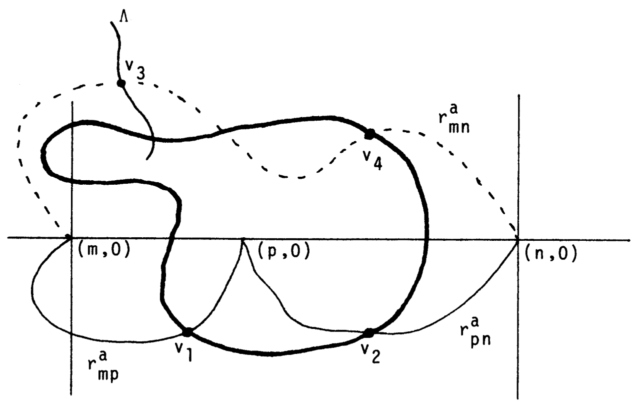

FIG. 2. *r^a_{mn} is the dashed path. D_p is drawn boldly. The solid paths from (m, 0) to (p, 0) and from (p, 0) to (n, 0) are r^a_{mp} and r^a_{pn}, respectively.*

sects the x-axis between $(m, 0)$ and $(n, 0)$. Thus r^a_{mn} must intersect $\pi_+ \cup \pi_- \cup S(p, q)$. But $S(p, q)$ lies in the interior of D_p, hence if r^a_{mn} intersects $S(p, q)$ it intersects D_p, which is part of Λ.

Now let r be the following path from $(m, 0)$ to $(n, 0)$. First r goes along r^a_{mp} from $(m, 0)$ to v_1, then from v_1 to v_2 along an arc of D_p, and finally from v_2 to $(n, 0)$ along r^a_{pn}. Clearly

$$|r| \leq |r^a_{mp}| + |r^a_{pn}| + |D_p| = N^a_{mp} + N^a_{pn} + |D_p| \leq X_{mp} + N^a_{pn}.$$

To complete the proof of (1.4) we now show that r is a route for a_{mn}. Let s_1 and s_2 be paths on Λ which connect v_3 with v_1, and v_4 with v_2, respectively. Then we obtain a path from $(m, 0)$ to $(p, 0)$ by going along r from $(m, 0)$ to v_3, then along s_1 and then along r^a_{mp} from v_1 to $(p, 0)$. The passage time of this path is at least $a_{mp} = t(r^a_{mp})$. Therefore

$$\begin{aligned} &t(\text{piece of } r^a_{mp} \text{ from } (m, 0) \text{ to } v_1) \\ &\quad \leq t(\text{piece of } r^a_{mn} \text{ from } (m, 0) \text{ to } v_3). \end{aligned} \tag{2.4}$$

For the same reason

$$(2.5)\qquad \begin{aligned} &t(\text{piece of } r^a_{pn} \text{ from } v_2 \text{ to } (n, 0)) \\ &\qquad \le t(\text{piece of } r^a_{mn} \text{ from } v_4 \text{ to } (n, 0)). \end{aligned}$$

If we add (2.4) and (2.5) and take into account that any arc of D_p has zero passage time we obtain

$$t(r) \le t(r^a_{mn}) = a_{mn}.$$

This proves that r is a route for a_{mn} and establishes (1.4) in Case (ii).

Cases (iii) and (iv) can be handled by combining cases (i) and (ii), so that $\{X_{mn}\}$ is indeed subadditive. It now follows from Kingman's subadditive ergodic theorem (see [4], Theorem 17) and Lemma 2 that

$$(2.6)\qquad \lim_{n\to\infty}(1/n)N^a_{0n} = \lambda \quad \text{a.s.} \quad \text{and in } L^1$$

for some λ. The limit λ is given by formula 1.3.6 in [4]. But the σ-field of events generated by the $\{X_{mn}\}$ and invariant under the shift in the x-direction is contained in the σ-field of events generated by all the bonds and invariant under the shift in the x-direction. The latter σ-field is trivial (cf. [3], Lemma 3.1 or [5], Lemma 8.11) so that λ is a constant w.p.1.

To show that λ depends on $U(0)$ only, observe that by Lemma 1, as soon as D_0 and D_n lie in each others exterior (hence for all large n) any route r for a_{0n} consists of some bonds inside $D_0 \cup D_n$ plus a piece r^n_2 in Λ. Since $n^{-1} | D_0 \cup D_n | \to 0$, λ depends on min r^n_2, $n = 1, 2, \cdots$ only, where the min is over all paths r^n_2 in Λ connecting D_0 and D_n. Therefore λ depends on Λ only, and hence on $U(0)$ only. □

LEMMA 4. *If $U(0) > 1/2$, then for the λ of* (2.3)

$$\lim_{n\to\infty}(1/n)N^b_{0n} = \lambda \quad \textit{a.s.} \quad \textit{and in } L_1.$$

PROOF. Set

$$A_{mn}(u) = \inf\{t(r){:}r \text{ is a path from } (m, \lfloor mu \rfloor) \text{ to } (n, \lfloor nu \rfloor)\}$$

and

$$N_{mn}(u) = \inf\{| r |{:}r \text{ is a route for } A_{mn}(u)\},$$

where $\lfloor x \rfloor$ denotes the largest integer $\le x$. Note that $a_{mn} = A_{mn}(0)$. The proof of Lemma 3 with some trivial modifications shows that for each u there exists a constant $\lambda_1(u)$ such that

$$(2.7)\qquad \lim_{n\to\infty}(1/n)N_{0n}(u) = \lambda_1(u) \quad \textit{a.s.}$$

Now let $D = D(n, u)$ be a circuit surrounding $(n, \lfloor n \rfloor)$ such that D is part of Λ and such that any two of D_0, D and D_{2n} lie in the exterior of the third. Let r_1 and r_2 be paths of minimal passage time connecting (0, 0) to $(n, \lfloor nu \rfloor)$ and $(n, \lfloor nu \rfloor)$ to $(0, 2n)$, respectively. Then by the arguments of Lemma 1 we can construct a route for $a_{0,2n}$ by going along r_1 from (0, 0) to its first intersection with D, then along D to the first intersection of r_2 with D and finally along D to $(0, 2n)$. It follows that

$$N^a_{0,2n} \le N_{0n}(u) + N((n, \lfloor nu \rfloor), (2n, 0)) + |D(n, u)|, \tag{2.8}$$

where

$$N((n, \lfloor nu \rfloor), (2n, 0)) = \inf\{|r| : r \text{ a path of minimal passage time from } (n, \lfloor nu \rfloor) \text{ to } (2n, 0)\}.$$

$N((n, \lfloor nu \rfloor), (2n, 0))$ has the same distribution as $N_{0n}(u)$ and therefore

$$(1/n)N((n, \lfloor nu \rfloor), (2n, 0)) \to \lambda_1(u) \text{ in probability.} \tag{2.9}$$

(2.7)–(2.9) show that

$$\lambda = \lim(1/2n)N^a_{0,2n} \le \lambda_1(u). \tag{2.10}$$

Now let r_n be a route for b_{0n} with $|r_n| = N^b_{0n}$. Define

$$h(r_n) = \max\{|m| : (k, m) \text{ is a site on } r_n \text{ for some } k\}.$$

From Theorem 8.10 in [5] it follows that for some finite constant K

$$\limsup(1/n)N^b_{0n} \le K \quad \text{a.s.}$$

In fact this holds with $K = \lambda = \lim n^{-1}N^a_{0n}$. To see this, observe that if D_0 and D_n lie in each others exterior, then in the notation of Lemma 1 any route for b_{0n} must have passage time at least $t(r_1)$. This time is achieved by the path which follows a route r for a_{0n} till its first intersection with D_n and then follows an arc of D_n till the line $x = n$. Thus for large n

$$N^b_{0n} \le N^a_{0n} + |D_n| \tag{2.11}$$

and

$$\limsup_{n\to\infty}(1/n)N^b_{0n} \le \lim_{n\to\infty}(1/n)N^a_{0n} = \lambda \quad \text{w.p.1.} \tag{2.12}$$

As a consequence of (2.12) we have

$$\limsup_{n\to\infty}(1/n)h(r_n) \le \lambda \quad \text{a.s.} \tag{2.13}$$

Fix a small positive $\varepsilon > 0$. Let (n, v) be the endpoint of r_n on the line $\{x = n\}$ and let $j = j_n$ be the unique integer for which

$$j\varepsilon n \le v < (j + 1)\varepsilon n.$$

By virtue of (2.13) we may assume $|j| \le 2\lambda\varepsilon^{-1}$ for large n. Now let $\Delta(x, y)$ be the smallest circuit in Λ with (x, y) in its interior. Then, as in [1], Lemma 2.6 and Corollary 2.7, we can connect $\Delta(n, v)$ to $\Delta(n, j\varepsilon n)$ by a path $s = s_n$ which consists of pieces of

$$\{\Delta(n, y) : j\varepsilon n \le y \le v\}.$$

Moreover, Lemma 2 shows that for any $m \ge 1$ there exist a constant C such that

$$P\{\text{diameter of } \Delta(x, y) \ge d\} \le P\{|D_0| \ge d\} \le C(d + 1)^{-m}.$$

It follows from this that with probability 1

$$\Delta(n, y) \subset [n - n^{1/2}, n + n^{1/2}] \times [y - n^{1/2}, y + n^{1/2}] \tag{2.14}$$

for all $|y| \le 2\lambda n$ and all sufficiently large n. Thus we may also assume that for large n, D_0 lies in the exterior of each $\Delta(n, y)$ with $|y| \le 2\lambda n$. (It is actually possible to show that $P\{|D_0| \ge d\} \to 0$ exponentially in d, so that $\Delta(n, y) \subset [n - C_1 \log n, n + C_1 \log n] \times [y - C_1 \log n, y + C_1 \log n]$ for all $|y| \le 2\lambda n$ and sufficiently large n. We do not need this sharper result here.)

Finally, if this happens, then we can construct a route for $A_{0n}(j\varepsilon)$ from pieces of r_n, s_n and a path in the interior of $\Delta((n, \lfloor j\varepsilon n \rfloor))$ (compare proof of Lemma 1 and last paragraph of proof of Lemma 3). Consequently,

$$N_{0n}(j\varepsilon) \le |r_n| + |s_n| + |\text{interior of } \Delta(n, \lfloor j\varepsilon n \rfloor)|$$

$$\le N_{0n}^b + \max_{|j| \le 2\lambda\varepsilon^{-1}} \textstyle\sum_{j\varepsilon n \le y < (j+1)\varepsilon n} |\tilde{\Delta}(n, y)|,$$

where

$$|\tilde{\Delta}(n, y)| = \text{number of edges in } \Delta(n, y) \cup \text{interior of } \Delta(n, y).$$

It follows that w.p.1

$$\begin{aligned} \liminf(1/n)N_{0n}^b &\ge \liminf \min_{|j| \le 2\lambda\varepsilon^{-1}}(1/n)N_{0n}(j\varepsilon) \\ &\quad - \limsup(1/n)\max_{|j| \le 2\lambda\varepsilon^{-1}} \textstyle\sum_{j\varepsilon n \le y < (j+1)\varepsilon n} |\tilde{\Delta}(n, y)|, \end{aligned} \tag{2.15}$$

where the lim inf and lim sup in (2.15) are taken along any fixed subsequence of the intergers. The first term in the right hand side of (2.15) equals λ for any choice of such a subsequence, by virtue of (2.7) and (2.10). We next study the second term in the right hand side of (2.15). We claim that if we take the lim sup along any subsequence $\{n_k\}$ of the form

$$n_k = \lfloor (1 + \delta)^k \rfloor,$$

with $\delta > 0$ fixed, then

$$\lim_{\varepsilon \downarrow 0} \limsup_{k \to \infty} (1/n_k)\max_{|j| \le 2\lambda\varepsilon^{-1}} \textstyle\sum_{j\varepsilon n_k \le y < (j+1)\varepsilon n_k} |\tilde{\Delta}(n_k, y)| = 0 \quad \text{w.p.1.} \tag{2.16}$$

(2.16) follows by a simple Borel-Cantelli argument. Indeed, by virtue of (2.14) we may replace $|\tilde{\Delta}(n, y)|$ by

$$\Gamma(n, y) := |\tilde{\Delta}(n, y)| \, I\{\Delta(n, y) \subset [n - n^{1/2}, n + n^{1/2}] \times [y - n^{1/2}, y + n^{1/2}]\}$$

($I\{E\}$ is the indicator function of E.) For $|y_1 - y_2| > 2n^{1/2}$, $\Gamma(n, y_1)$ and $\Gamma(n, y_2)$ are independent. Since $\Delta(n, 0)$ is contained in $\bar{D}_0$, $\Gamma(n, 0)$ has finite moments of all orders (see Lemma 2). Consequently

$$\sigma^2\{\textstyle\sum_{j\varepsilon n_k \le y < (j+1)\varepsilon n_k} \Gamma(n_k, y)\} \le 5\varepsilon(n_k)^{3/2}\sigma^2(\Gamma(n_k, 0)) = O(n_k^{3/2})$$

and

$$E\{\textstyle\sum_{j\varepsilon n_k \le y < (j+1)\varepsilon n_k} \Gamma(n_k, y)\} \le (\varepsilon n_k + 1)E|D_0|.$$

Thus

$$P\{\max_{|j|\le 2\lambda\varepsilon^{-1}} \textstyle\sum_{j\varepsilon n_k \le y < (j+1)\varepsilon n_k} \Gamma(n_k, y) \ge 2\varepsilon n_k E\,|\,D_0\,|\,\}$$

$$\le 5\lambda\varepsilon^{-1} \frac{O(n_k^{3/2})}{(\varepsilon n_k E\,|\,D_0\,|\,)^2} = O(n_k^{-1/2}) = O((1+\delta)^{-k/2}). \tag{2.17}$$

The Borel-Cantelli lemma now gives (2.16).

It is now easy to complete the proof of the lemma. If $n_k \le n < n_{k+1}$ and $D_0 \subset \{x \le n_k\}$, then by Lemma 1(b) any route for b_{on} contains a route for b_{0n_k} so that

$$N_{0n_k}^b \le N_{0,n}^b.$$

Thus, from (2.15) and (2.16)

$$\liminf_{n\to\infty} \frac{1}{n} N_{0n}^b \ge \liminf_{k\to\infty} \frac{n_k}{n_{k+1}} \frac{1}{n_k} N_{0n_k}^b \ge \frac{1}{1+\delta}\lambda \quad \text{w.p.1.}$$

Since $\delta > 0$ is arbitrary

$$\liminf (1/n) N_{0n}^b \ge \lambda \quad \text{w.p.1.}$$

In view of (2.12) this completes the proof of the a.s. convergence. The L^1 convergence follows now from (2.11) and Lemmas 2, 3. □

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