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CONJECTURE: IN GENERAL A MIXING TRANSFORMATION IS NOT TWO-FOLD MIXING

BY STEVE ALPERN

London School of Economics

A new topology is introduced on the group of μ-preserving automorphisms of a Lebesgue space (X, Σ, μ) so that $f_k \to f$ if $\mu(f_k^n A \cap B) \to \mu(f^n A \cap B)$ uniformly in n for all A, B in Σ. The subspace of mixing automorphisms is a Baire space in the relative topology. A conjecture (about the extent to which a mixing stationary process is determined by its two-dimension distributions) is stated, which if true implies that the two-fold mixing automorphisms are of first category in the mixing ones. So if the conjecture is true then by Baire's Theorem there is a mixing but not two-fold mixing automorphism.

1. Introduction. Let $\mathscr{G}$ denote the space of all μ-preserving automorphisms of a Lebesgue space (X, Σ, μ). Rokhlin (1949) has defined an automorphism g in $\mathscr{G}$ to be k-fold mixing if

$$\lim_{\inf n_i \to\infty; \inf_{i\neq j}|n_i - n_j|\to\infty} \mu(g^{n_0}C_0 \cap g^{n_1}C_1 \cap \cdots \cap g^{n_k}C_k) = \mu(C_0)\mu(C_1) \cdots \mu(C_k)$$

for any measurable sets $C_0, \cdots, C_k$. Let $\mathscr{M}^k$ denote the set of all k-fold mixing automorphisms, so that $\mathscr{M} = \mathscr{M}^1$ denotes the (strong) mixing automorphisms. It is clear that $\mathscr{M}^{k+1} \subset \mathscr{M}^k$ for all k, but it is not known whether the inclusion is proper for any k. In particular the existence of an automorphism g^* which is mixing but not two-fold mixing is open, although S. Kalikow (1985) has shown that g^* cannot have rank 1 and J. P. Thouvenot has shown that if such an example exists then there is one with zero entropy (Kalikow, 1985). The purpose of this note is to outline a possible approach to this problem based on a Baire Category technique analogous to that used by Rokhlin (1948) to prove that "in general a measure preserving transformation is not mixing".

The key to this approach is the introduction of a new topology on $\mathscr{G}$, called the "two-fold" topology, defined by the sequential convergence of automorphisms f_k to a limit f if $\mu(f_k^n A \cap B)$ converges to $\mu(f^n A \cap b)$ uniformly in n for any measurable sets A, B. This topology was contrived so that the mixing subspace $\mathscr{M}$ is a Baire space in the relative topology. An attempt is made to establish that $\mathscr{M}^2$ is of first category in $\mathscr{M}$, which would imply that $\mathscr{M}^2 \neq \mathscr{M}$. There is a gap in the proof of this proposition which can be bridged by the following conjecture on stochastic processes.

CONJECTURE 1. *Let L and M be positive integers, with M even, and let $\varepsilon > 0$. Let X_n, $n = 0, \pm 1, \cdots$, be a mixing stationary stochastic process taking values in $\{1, 2, \cdots, M\}$ equiprobably. Then there are integers p, $q \geq L$ and another mixing*

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stationary process Y_n taking values equiprobably in $\{1, \cdots, M\}$ satisfying

$$|\,\mathrm{Prob}(Y_0 = i \text{ and } Y_n = j) - \mathrm{Prob}(X_0 = i \text{ and } X_n = j)\,| < \varepsilon \quad \text{for all} \quad i, j \text{ and } n,$$

and

$$|\,\mathrm{Prob}(Y_0 \le M/2 \text{ and } Y_p \le M/2 \text{ and } Y_{p+q} \le M/2) - 1/8\,| > 1/100.$$

2. The two-fold topology. In the introduction the two-fold topology was described by the sequential convergence of f_k to f if for all mesurable sets A and B,

$$(*) \qquad \mu(f_k^n A \cap B) \to \mu(f^n A \cap B) \quad \text{uniformly in } n.$$

We have two observations on this description. The first is that for any measurable set A we have $\frac{1}{2}\mu(f_k A \,\Delta\, fA) = \mu(A) - \mu(f_k A \cap fA)$ which is seen to go to 0 for f_k, f satisfying $(*)$ by fixing $n = 1$ and setting $B = fA$. Hence the two-fold topology is finer than the coarse (sometimes called the weak) topology defined by the sequential convergence condition $\mu(f_k A \,\Delta\, fA) \to 0$ for all A. Next, observe that if $(*)$ holds whenever A and B are dyadic intervals of equal rank (we are taking our space X to be the "standard" Lebesgue space, the unit interval) then it holds for all measurable sets A, B. In light of this observation we may explicitly define the two-fold topology by giving as a basis for the open sets the family of all sets of the form $\mathscr{N}(f; m, \varepsilon) = \{g\colon |\,\mu(g^n A \cap B) - \mu(f^n A \cap B)\,| < \varepsilon$, for all $n \ge 1$ and rank m intervals $A, B\}$.

The two-fold topology can also be given by a metric d_2 defined as follows. Let (A_i, B_i) be an enumeration of all ordered pairs of dyadic intervals of equal rank. For any f in $\mathscr{G}$, $i = 1, 2, \cdots$, and $n = 0, 1, 2, \cdots$ define $f(i, n) = \mu(f^n A_i \cap B_i)$ and let $f(i)$ be the vector in ℓ_∞ (Banach space of bounded sequences with the sup norm $\|\;\|$) whose nth coordinate is $f(i, n)$. Then for any f, g in $\mathscr{G}$ we define $d_2(f, g) = \sum_{i=1}^{\infty} 2^{-i} \| f(i) - g(i) \|$. Observe that the powers of any mixing automorphism form a d_2-Cauchy sequence which is not d_2-convergent, so that d_2 is not complete. However we can show,

LEMMA 1. *The metric space $(\mathscr{G}, d_2)$ is topologically complete (hence Baire).*

PROOF. Let d_1 denote a complete metric on $\mathscr{G}$ inducing the coarse topology (e.g., $d_1(f, g) = \sum_{i=1}^{\infty} 2^{-i}(\mu(fE_i \,\Delta\, gE_i) + \mu(f^{-1}E_i \,\Delta\, g^{-1}E_i))$ for a countable dense family of sets E_i). Since we have already demonstrated that the two-fold topology is finer than the coarse topology, it follows that the sum metric $d^* = d_1 + d_2$ is topologically equivalent to d_2, and hence also induces the two-fold topology. We will show that d^* is complete.

Suppose the sequence f_k is Cauchy with respect to d^*. Since $d_1 \le d^*$ it follows that the f_k are also Cauchy with respect to d_1. But d_1 is complete so there is an automorphism f in $\mathscr{G}$ with $d_1(f_k, f) \to 0$. We claim that also $d_2(f_k, f) \to 0$. To establish this fix any i and observe that $d_2(f_k, f_{k'}) \ge 2^{-i} \| f_k(i) - f_{k'}(i) \|$. Consequently the sequence $f_k(i)$, $k = 1, 2, \cdots$, is a Cauchy sequence in ℓ_∞ with the sup norm $\| y \| = \sup_n | y(n) |$. The completeness of ℓ_∞ guarantees the existence of an

$x = x(n)$ in ℓ_∞ such that $\| f_k(i) = x \| \to 0$. Since $\mathscr{G}$ is a topological group with respect to the weak topology (Halmos, 1956) the weak convergence of f_k to f implies, for each fixed n, the weak convergence of f_k^n to f^n. Hence for each n, $\mu(f_k^n A_i \cap B_i) \to \mu(f^n A_i \cap B_i)$ so that $x(n) = f(i, n)$, or $x = f(i)$. Thus for each i, $\| f_k(i) - f(i) \| \to 0$ and hence $d_2(f_k, f) \to 0$. This implies that $d_1(f_k, f) \to 0$ and therefore $d^*(f_k, f) \to 0$. So we have shown that d^* is complete and consequently that the two-fold topology is topologically complete and hence Baire.

We conclude this section with two easy propositions designed mainly to give the reader some examples of convergence in the two-fold topology. These propositions will not be used in the rest of the paper.

PROPOSITION 1. *$\mathscr{G}$ with the two-fold topology is not a topological group.*

PROOF. Let $D_{m,j}$ denote the jth dyadic interval of rank m, $[(j-1)/2^m, j/2^m)$. For each positive integer k define an automorphism f_k in $\mathscr{G}$ which linearly maps $D_{k,j}$ onto $D_{k,j'}$, where $j' = j + 1$ if j is odd and $j' = j - 1$ if j is even. Similarly define a sequence of automorphisms g_k mapping $D_{k,j}$ linearly onto D_{k,j^*} where $j^* = j - 1$ if j is odd and $j^* = j + 1$ if j is even, with arithmetic mod 2^k. In cycle notation as permutations of the index j for fixed k, f_k is given by $[(1, 2)(3, 4) \cdots (2^k - 1, 2^k)]$ and g_k by $[(2, 3)(4, 5) \cdots (2^k, 1)]$. It is not difficult to check that $f_k \to e$, $g_k \to e$, but $f_k g_k \not\to e$ where convergence is in the two-fold topology and e denotes the identity automorphism.

Since the two-fold topology is very strong (fine) we offer the following result (without proof) which we will not use, just to show that it is far from being discrete.

PROPOSITION 2. *For each f in $\mathscr{G}$ the conjugacy class $c(f) = \{g^{-1}fg : g \in \mathscr{G}\}$ is arcwise connected in the two-fold topology.*

3. Mixing and two-fold mixing. In this section we analyze the structure of the mixing and two-fold mixing subspaces of $\mathscr{G}$ with respect to the two-fold topology.

LEMMA 2. *$\mathscr{M}$ is closed in $\mathscr{G}$ with respect to the two-fold topology.*

PROOF. This follows easily from the definitions (of mixing and two-fold topology).

We now analyze the topological type of the two-fold mixing automorphisms $\mathscr{M}^2$ with respect to the two-fold topology. If f is two-fold mixing then taking $A = B = E$, where E denotes the left half of the unit interval ($E = [0, 1/2)$), in the definition of two-fold mixing, yields $\lim_{p,q\to\infty} \mu(E \cap f^p E \cap f^{p+q} E) = 1/8$. Therefore for some integer j, $p, q \geq j$ implies that $| \mu(E \cap f^p E \cap f^{p+q}) - 1/8 | \leq 1/100$. It follows that $\mathscr{M}^2 \subset \bigcup_{j=1} \mathscr{S}_j$, where $\mathscr{S}_j = \bigcap_p \bigcap_q \{f : | \mu(E \cap f^p E \cap f^{p+q} E) - 1/8 | \} \leq 1/100$. The following easy lemmas show that each $\mathscr{S}_j$ is nowhere dense.

LEMMA 3. *For each j, $\mathscr{S}_j$ is closed in the two-fold topology.*

LEMMA 4. *Assume that Conjecture 1 is true. Then for each j, $\mathscr{S}_j$ is dense in $\mathscr{M}$ with respect to the two-fold topology.*

THEOREM 1. *Assume Conjecture 1 is true. Then the two-fold mixing automorphisms form a first category subset of the Baire space of mixing automorphisms, with respect to the two-fold topology. In particular, there is a mixing automorphism which is not two-fold mixing.*

PROOF. (All topological terms refer to the two-fold topology.) Since $\mathscr{M}$ is a closed (Lemma 2) subspace of the topologically complete (Lemma 1) $\mathscr{G}$, $\mathscr{M}$ is Baire. By Lemmas 3 and 4 (where Conjecture 1 is needed) each set $\mathscr{S}_j$ is nowhere dense in $\mathscr{M}$. It follows that $\bigcup_j \mathscr{S}_j$ is a first category subset of $\mathscr{M}$, and hence the same is true of its subset $\mathscr{M}^2$. The Baire Category Theorem consequently implies that $\mathscr{M} - \mathscr{M}^2$ is non-empty.

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LONDON SCHOOL OF ECONOMICS
HOUGHTON STREET
LONDON WC2A 2AE
ENGLAND