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ON DOMAINS OF UNIFORM LOCAL ATTRACTION IN EXTREME VALUE THEORY

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An absolutely continuous distribution F is said to be in the domain of uniform local attraction of the absolutely continuous distribution H if the density of the normalized maximum of an independent sample of size n converges locally uniformly to the density of H as $n \to \infty$.

Under the sole restriction that F is eventually increasing, the domains of uniform local attraction to the three types of extreme value distribution are shown to be characterized by the usual Von Mises' conditions. The equivalent form of conditions used here greatly simplifies and shortens proofs of known results. In particular, L_p convergence and convergence of the k upper sample extremes are investigated and extensions to known results derived.

1. Introduction. A simple condition for a local limit theorem to hold for a distribution in the domain of attraction of the Type III extreme value distribution was given by Pickands (1967). More recently, de Haan and Resnick (1982) have made a more detailed study of local limit theorems in this case, and also for the Type I distribution. It is shown here in Theorem 1 (iii) that the domain of uniform local attraction, in the sense of uniform convergence in compact subsets, of the Type III distribution, is characterized by the slow variation of a certain function. This condition is equivalent to the Von Mises'-type condition given in de Haan and Resnick (1982), as well as Pickand's condition, but is both simpler to state and simpler to work with. Furthermore, our condition leads to a unified approach to the characterization of all three domains of uniform local attraction. In view of the equivalence of our conditions and those given in de Haan and Resnick (1982) certain parts of Theorem 1 follow from results in de Haan and Resnick (1982), but independent proofs are included here for completeness, and since these are relatively short.

The advantage of using our form of conditions becomes apparent when we come to make a more detailed study along the lines of de Haan and Resnick (1982). In Theorem 2 we explore the uniformity of convergence over the entire range of the limit distribution and the characterization given is new. We use our conditions in Theorem 3 to prove L_p density convergence results. For the Type I and III distributions L_p results were obtained by de Haan and Resnick (1982); the proofs given in Section 3 here however are far simpler. Finally the joint density convergence of the k upper sample extremes is given in Theorem 4.

2. Statement of results. We employ the notation $H_{1,\gamma}(x) = \exp(-x^{-\gamma})$, $x > 0$, $H_{2,\gamma}(x) = \exp(-(-x)^{\gamma})$, $x < 0$ and $H_3(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, where γ is a

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positive parameter. Let $X_1, \cdots, X_n$ be independent random variables with common distribution function F, and set $\alpha = \inf\{x:F(x) > 0\}$, $\omega = \sup\{x:F(x) < 1\}$. Let $H_n(x) = P(X_{(n)} \le x) = F^n(x)$. The distribution F is in the domain of attraction of the nondegenerate distribution H if there exist constants c_n and $d_n > 0$ such that $\lim_{n\to\infty} H_n(c_n + d_n x) = H(x)$ at all continuity points of H. The three types $H_{1,\gamma}$, $H_{2,\gamma}$ and H_3 exhaust the possibilities for the limit distribution H, and classical extreme value theory characterizes their domains of attraction, which we denote here by $\mathscr{D}_{1,\gamma}$, $\mathscr{D}_{2,\gamma}$ and $\mathscr{D}_3$ respectively. (See Galambos, 1978, for example.)

Suppose that the distribution F is absolutely continuous with density f. We shall say that F is in the domain of uniform local attraction of H if there exist constants c_n and $d_n > 0$ such that

$$(1) \qquad f_n(x) \to h(x) \quad \text{locally uniformly} \quad \text{as } n \to \infty,$$

where $f_n(x) = nd_n f(c_n + d_n x) F^{n-1}(c_n + d_n x)$ is the density of $(X_{(n)} - c_n)/d_n$ and h is the density of H. Since density convergence implies convergence in distribution, the three extreme value distributions exhaust the possibilities for H, and we denote their corresponding domains of uniform local attraction by $\mathscr{L}_{1,\gamma}$, $\mathscr{L}_{2,\gamma}$, and $\mathscr{L}_3$.

We shall say that F is *eventually increasing* if there exists $x_0 \in (\alpha, \omega)$ such that $F(x_1) < F(x_2)$ whenever $x_0 < x_1 < x_2 < \omega$. In Theorems 1–4 below F is assumed to be an eventually increasing absolutely continuous distribution function. Theorem 1 characterizes the three domains of uniform local attraction in a simple and appealing way. Define for $t > t_0$ the inverse function $a(t) = F^{-1}(1 - t^{-1})$ and $b(t) = tf(a(t))$. The function $a(t)$ is increasing and continuous, and $\lim_{t\to\infty} a(t) = \omega$.

THEOREM 1. (i) *$F \in \mathscr{L}_{1,\gamma}$ iff $\omega = \infty$ and $a(t)b(t) \to \gamma$. In this case one may take $c_n = 0$ and $d_n = a(n)$.*

(ii) *$F \in \mathscr{L}_{2,\gamma}$ iff $\omega < \infty$ and $\{\omega - a(t)\}b(t) \to \gamma$. In this case one may take $c_n = \omega$ and $d_n = \omega - a(n)$.*

(iii) *$F \in \mathscr{L}_3$ iff $b(t)$ is slowly varying. In this case one may take $c_n = a(n)$ and $d_n = \{b(n)\}^{-1}$.*

Pointwise convergence in case (iii) was proved by Pickands (1967) under a condition equivalent to the slow variation of b. Local uniformity in case (iii) was proved by de Haan and Resnick (1982) under the Von Mises'-type condition.

$$(2) \qquad \lim_{x\uparrow\omega} \frac{f(x) \int_x^\omega (-\log F(t))\, dt}{(\log F(x))^2} = 1.$$

(The case $\omega = \infty$ was treated in de Haan and Resnick, 1982 for convenience.) We shall show that (2) is equivalent to the slow variation of b, and so Theorem 1 (iii) states that the Von Mises' condition (2) is necessary and sufficient for local uniform convergence to the Type III distribution. Uniform convergence in case (i) was obtained by Anderson (1971) and characterized by de Haan and Resnick

(1982) under the Von Mises' condition

$$\lim_{x\to\infty} \frac{x\, f(x)}{-\log F(x)} = \gamma,$$

again equivalent to $a(t)b(t) \to \gamma$. The condition given in case (ii) is equivalent to

$$\lim_{x\uparrow\omega} \frac{(\omega - x)f(x)}{-\log F(x)} = \gamma.$$

Note that no assumptions about the boundedness of f are needed in Theorem 1. We can however deduce the *eventual* boundedness of f in cases (i) and (iii), but this is not necessary in case (ii). Finally note that in all cases our conditions imply that f is eventually positive. (Eventual boundedness and eventual positivity of f are defined on (α, ω) in an obvious way.)

THEOREM 2. (i) *If F is in $\mathscr{L}_{1,\gamma}$ for $\gamma > 0$, $\mathscr{L}_{2,\gamma}$ for $\gamma > 1$, or $\mathscr{L}_3$ the convergence in* (1) *is uniform in a neighbourhood of ∞, 0, ∞ respectively.*

(ii) *For F as given in* (i) *the convergence in* (1) *is uniform in $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}$ respectively iff there are constants B, $C > 0$ such that*

$$f(x) \le C\{F(x)\}^{-B} \quad \textit{for all} \quad x \in (\alpha, \omega). \tag{3}$$

REMARK. Theorem 2(ii) characterizes the three domains of uniform local attraction, in the sense of uniformity over the entire range of the limit distribution. Uniform convergence in the case of Type I and III distributions was proved in de Haan and Resnick (1982) under the assumption that f is bounded. The condition (3) permits f to become unbounded as $x \downarrow \alpha$.

THEOREM 3. (i) *If $F \in \mathscr{L}_{1,\gamma}$ then if $p > (1 + \gamma)^{-1}$ and $A > 0$*

$$\lim_{n\to\infty} \int_A^\infty |f_n(x) - h_{1,\gamma}(x)|^p\, dx = 0.$$

(ii) *If $F \in \mathscr{L}_{2,\gamma}$ then for $0 < p < (1 - \gamma)^{-1}$ if $\gamma < 1$, and for all $p > 0$ if $\gamma \ge 1$, and every $A > 0$*

$$\lim_{n\to\infty} \int_{-A}^0 |f_n(x) - h_{2,\gamma}(x)|^p\, dx = 0.$$

(iii) *If $F \in \mathscr{L}_3$ then for all $p > 0$ and every $A > 0$*

$$\lim_{n\to\infty} \int_{-A}^\infty |f_n(x) - h_3(x)|^p\, dx = 0.$$

If additionally $f \in L_p$ then the limit results in (i)–(iii) *remain true on setting $A = -\infty$ in* (i) *and $A = +\infty$ in* (ii) *and* (iii).

REMARK. L_p results in cases (i) and (iii) were proved by de Haan and Resnick (1982). Our proofs given in Section 3 are considerably shorter than theirs.

Finally, Theorem 4 gives the joint density convergence of the k upper sample extremes $X_{(n)}, \cdots, X_{(n-k+1)}$, when F is in one of the domains of uniform local attraction. The results here are local forms of the results given in Weissman (1978), and would be essential for statistical use whenever one wished to approximate the likelihood function based on these extremes. The theorem is most simply formulated in terms of appropriate asymptotically independent variables. Although the normalizing constants in Theorem 1 are used, the results will clearly hold for any other equivalent normalizing sequences.

THEOREM 4. (i) *Suppose* $F \in \mathscr{L}_{1,\gamma}$ *and let* $h_{k,n}(z_1, \cdots, z_{k-1}, x_k)$ *be the joint density of* $Z_j = X_{(n-j+1)}/X_{(n-j)}$, $j = 1, \cdots, k-1$, *and* $X_{(n-k+1)}/a(n)$. *Then as* $n \to \infty$

$$h_{k,n}(z_1, \cdots, z_{k-1}, x_k) \to \{\prod_{j=1}^{k-1} j\gamma z_j^{-j\gamma-1}\}\{\gamma x_k^{-k\gamma-1}\exp(-x_k^{-\gamma})/(k-1)!\}$$

uniformly for $x_k \in (\varepsilon, \infty)$ *and* $z_j \in (1, \infty)$, $j = 1, \cdots, k-1$, *for every* $\varepsilon > 0$. *If additionally condition* (3) *holds then one may take* $\varepsilon = 0$.

(ii) *Suppose* $F \in \mathscr{L}_{2,\gamma}$ *and let* $h_{k,n}(z_1, \cdots, z_{k-1}, x_k)$ *be the joint density of* $Z_j = \{X_{(n-j+1)} - \omega\}/\{X_{(n-j)} - \omega\}$, $j = 1, \cdots, k-1$, *and* $\{X_{(n-k+1)} - \omega\}/\{\omega - a(n)\}$. *Then as* $n \to \infty$

$$h_{k,n}(z_1, \cdots, z_{k-1}, x_k) \to \{\prod_{j=1}^{k-1} j\gamma z_j^{j\gamma-1}\}\{\gamma(-x_k)^{k\gamma-1}\exp(-(-x_k)^{\gamma})/(k-1)!\}$$

uniformly for $x_k \in (-A, -\varepsilon)$ *and* $z_j \in (\varepsilon_j, 1)$, $j = 1, \cdots, k-1$, *for all* $0 < \varepsilon < A$ *and* $\varepsilon_j > 0$. *If* $\gamma > 1$ *one may take* $A = +\infty$, *and if further condition* (3) *holds then one may take* $\varepsilon = \varepsilon_j = 0$, $j = 1, \cdots, k-1$.

(iii) *Suppose* $F \in \mathscr{L}_3$ *and let* $h_{k,n}(z_1, \cdots, z_{k-1}, x_k)$ *be the joint density of* $Z_j = b(n)\{X_{(n-j+1)} - X_{(n-j)}\}$, $j = 1, \cdots, k-1$, *and* $b(n)\{X_{(n-k+1)} - a(n)\}$. *Then as* $n \to \infty$

$$h_{k,n}(z_1, \cdots, z_{k-1}, x_k) \to \{\prod_{j=1}^{k-1} j \exp(-jz_j)\}\{\exp(-kx_k)\exp(-\exp(-x_k))/(k-1)!\}$$

uniformly for $x_k \in (-A, \infty)$ *and* $z_j \in (0, \infty)$, $j = 1, \cdots, k-1$ *for every* A. *If additionally condition* (3) *holds then one may take* $A = +\infty$.

3. Proof of results. We first prove those parts of Theorems 1-3 relating to $\mathscr{L}_3$; these results will be used in the proofs of the remaining parts. For ease of reference we record the following standard results on slowly varying functions.

LEMMA 1. *Suppose* U *is slowly varying and let* $\varepsilon > 0$. *Then there exists* t_0 *such that for all* $t > t_0$

(i) $t^{-\varepsilon} < U(t) < t^{\varepsilon}$;
(ii) $(1-\varepsilon)x^{-\varepsilon} < U(tx)/U(t) < (1+\varepsilon)x^{\varepsilon}$ *for all* $x \geq 1$.

Define $r(t) = t\int_{a(t)}^{\omega} \{1 - F(y)\}\,dy$ and for $r(t) < \infty$ let

$$u_t(x) = -\log t\{1 - F(a(t) + xr(t))\}.$$

Thus for each t, u_t is an eventually increasing continuous function and it follows from Galambos (1978), Theorem 2.1.3, for example, that $F \in \mathscr{D}_3$ iff $r(t) < \infty$ for all $t > 1$ and $u_t(x) \to x$ locally uniformly (l.u.) as $t \to \infty$. Furthermore the normalizing constants may be chosen as $c_n = a(n)$, $d_n = r(n)$. The inverse of u_t is $u_t^{-1}(x) = \{a(te^x) - a(t)\}/r(t)$, and it is readily shown that $u_t(x) \to x$ l.u. iff $u_t^{-1}(x) \to x$ l.u.

The next lemma establishes the equivalence of condition (2) and the slow variation of b. Note that (2) is equivalent to the condition $\lim_{t\to\infty} r(t)b(t) = 1$.

LEMMA 2. *b is slowly varying iff* $\lim_{t\to\infty} r(t)b(t) = 1$.

PROOF. Note that $r(t) = \int_0^1 \{b(t/u)\}^{-1}\,du$ so that, writing $U(t) = \{t^2 b(t)\}^{-1}$, we have

$$r(t)b(t) = \left(\int_t^\infty U(s)\,ds\right) \Big/ tU(t).$$

It follows from Karamata's theorem (de Haan, 1970) that $r(t)b(t) \to 1$ iff U is regularly varying with index -2; that is, iff b is slowly varying as required. □

PROOF OF THEOREM 1 (iii). From Lemma 1(i), b slowly varying implies that $\{b(t)\}^{-1} = ta'(t)$ eventually exists and $r(t) \sim \{b(t)\}^{-1}$ from Lemma 2. Write $A(t) = a(e^t)$ and note that $A'(t) = \{b(e^t)\}^{-1}$ to give $u_t^{-1}(x) = \int_0^x \{r(t)b(te^v)\}^{-1}\,dv \to x$ l.u. Thus $F \in \mathscr{D}_3$ and $F^{n-1}(a(n) + r(n)x) \to H_3(x)$ uniformly in x. It therefore suffices to show that

(4) $$nr(n)f(a(n) + r(n)x) \to e^{-x} \quad \text{l.u.}$$

Write $a(n) + r(n)x = a(n \exp(y_n))$ where $y_n = u_n(x) \to x$ l.u. Then the left-hand side of (4) is $\exp(-y_n)r(n)b(n\exp(y_n)) \to \exp(-x)$ l.u. as required.

Conversely, suppose there exist sequences $c_n, d_n > 0$ such that (1) holds. Then $F \in \mathscr{D}_3$ and so $H_n(a(n) + r(n)x) \to H(x)$ uniformly in x. By Lemma 2.2.3 in Galambos (1978) we have $r(n) \sim d_n$ and $\{a(n) - c_n\}/d_n \to 0$, and it follows from (1) that (4) holds. Let $x \in \mathbb{R}$ and write $n = [t]$, $x_t = x + \log(t/n)$. Then $r(n)b(te^x) = \exp(x_t)n\, r(n)f(a(n) + r(n)y_t)$ where $y_t = u_n^{-1}(x_t) \to x$, since $F \in \mathscr{D}_3$. It follows from (4) that $r(n)b(te^x) \to 1$ for all x, from which the slow variation of b then follows. □

PROOF OF THEOREM 2 ($\mathscr{L}_3$). (i) Suppose $F \in \mathscr{L}_3$; by Lemma 2.2.3 in Galambos (1978) it suffices to show that $f_n(x_n) \to 0$ for every sequence $x_n \to \infty$ with $c_n = a(n)$, $d_n = r(n)$. But

$$f_n(x_n) \le nr(n)f(a(n) + r(n)x_n) = \exp(-y_n)r(n)b(n \exp(y_n))$$

where $y_n = u_n(x_n)$, and it is easily shown that $y_n \to \infty$. Finally, since b is slowly varying and $r(n) \sim \{b(n)\}^{-1}$, it follows from Lemma 1(ii) that for $n > n_0$

(5) $$f_n(x_n) \le c_1\exp(-(1-\varepsilon)y_n) \to 0$$

as required.

(ii) Write $t_n = n\exp(y_n)$, so that

(6) $$f_n(x_n) = \exp(-y_n)r(n)b(t_n)(1 - t_n^{-1})^{n-1}.$$

Since $x_n \to -\infty$ iff $y_n \to -\infty$, we need to show that (6) tends to zero for every sequence $y_n \to -\infty$. Choose n_0 such that $y_n < 0$ for $n > n_0$. Since b is slowly varying and $r(n) \sim \{b(n)\}^{-1}$, it follows from Lemma 1(ii) that there exists t_0 such that whenever $n > n_0$ and $t_n > t_0$ we have $r(n)b(t_n) \leq c_2\exp(-\varepsilon y_n)$. Using $1 - x \leq e^{-x}$ for $x > 0$ in (6) now gives

$$(7) \qquad f_n(x_n) \leq c_2\exp(-(1+\varepsilon)y_n)\exp(-(1-n^{-1})\exp(-y_n) \to 0$$

as $n \to \infty$ through any subsequence for which $t_n > t_0$. Suppose however that $t_n \leq t_0$. Then $a(n) + r(n)x_n \leq a(t_0) < \omega$, and so $F(a(n) + r(n)x_n) \leq \eta$ where $\eta = F(a(t_0)) < 1$. Thus from (3) $f_n(x_n) \leq Cnr(n)\eta^{n-B-1} \to 0$ through any subsequence for which $t_n \leq t_0$, since r is slowly varying.

Conversely suppose that for every B and C we can find a corresponding x such that (3) fails. We can then find a sequence (z_n) such that $f(z_n) > F(z_n)^{-n+1}$. Set $x_n = \{z_n - a(n)\}/r(n)$; then for this sequence we have $f_n(x_n) > nr(n) \to \infty$ since r is slowly varying, from which we deduce that the convergence in (1) cannot be uniform as $h_3(x)$ is bounded. □

Proof of Theorem 3(iii). In view of Theorem 1(iii) it suffices for the first assertion to show that $\limsup_{n\to\infty} \int_A^\infty f_n^p(x)\,dx \to 0$ as $A \to \infty$. Put $s = \exp(u_n(x))$ and note that

$$f_n^p(x)\,dx = s^{-1-p}\{r(n)b(ns)\}^{p-1}\{1-(ns)^{-1}\}^{p(n-1)}\,ds.$$

Let $0 < \varepsilon < p/|p-1|$; since b is slowly varying and $r(n) \sim \{b(n)\}^{-1}$, from Lemma 1(ii) there exists n_1 such that for $n > n_1$ we have $\{r(n)b(ns)\}^{p-1} \leq c_3 s^{\varepsilon|p-1|}$. Therefore, writing $s_n = \exp(u_n(A))$,

$$(8) \quad \int_A^\infty f_n^p(x)\,dx \leq c_3 \int_{s_n}^\infty s^{-(p+1-\varepsilon|p-1|)}\,ds \leq c_4\exp(-(p-\varepsilon|p-1|)u_n(A))$$

and the result follows, as $\lim_{n\to\infty} u_n(A) = A$.

For the second assertion, let $A > 0$, $y = u_n(x)$ and $t_n = ne^y$. Choose n_0 so that $y < 0$ whenever $x < -A$ and $n > n_0$; then from Lemma 1(ii) there exists t_0 such that whenever $n > n_0$ and $t_n > t_0$ we have $\{r(n)b(t_n)\}^{p-1} \leq c_5\exp(-\varepsilon|p-1|y)$. Set $A_n = -u_n^{-1}(\log(t_0/n)) = \{a(n) - a(t_0)\}/r(n)$; then if $-A_n < x < A$ and $n > n_0$ we have

$$f_n^p(x)\,dx \leq c_6\exp(-(p+\varepsilon|p-1|)y)\exp(-p(1-n^{-1})e^{-y})\,dy$$

from which $\lim_{A\to\infty}\limsup_{n\to\infty} \int_{-A_n}^{-A} f_n^p(x)\,dx = 0$ follows.

Finally, we need to show that $\lim_{n\to\infty} \int_{-\infty}^{-A_n} f_n^p(x)\,dx = 0$. But if $x \leq -A_n$ we have $f_n(x) \leq nr(n)f(a(n) + r(n)x)\eta^{n-1}$ where $\eta = F(a(t_0)) < 1$ and so

$$\int_{-\infty}^{-A_n} f_n^p(x)\,dx \leq (n\eta^{n-1})^p r^{p-1}(n) \int_\alpha^{a(t_0)} f^p(u)\,du \to 0$$

as $n \to \infty$, since r is slowly varying. □

We now prove the remaining assertions in Theorems 1-3. Let g be a continuously differentiable increasing real-valued function on $\mathbb{R}$, and let $X_1 = g(X)$,

$F_1(x) = P(X_1 \leq x)$, $f_1(x) = F_1'(x)$. Let $a_1(t) = F_1^{-1}(1 - t^{-1})$, $b_1(t) = tf_1(a_1(t))$; then

$$a_1(t) = g(a(t)), \quad b_1(t) = b(t)/g'(a(t)). \tag{9}$$

PROOF OF THEOREM 1; (i) AND (ii). We deduce (i) from (iii), and then (ii) from (i).

(i) Since $\mathscr{L}_{1,\gamma} \subset \mathscr{D}_{1,\gamma}$ it suffices to assume that $\omega = \infty$ and prove that $F \in \mathscr{L}_{1,\gamma}$ iff $a(t)b(t) \to \gamma$. Let g be a continuously differentiable increasing function on $\mathbb{R}$ with $g(x) = \log x$, $x > 1$, let $X_3 = g(X)$, and a_3, b_3 be the functions a, b for X_3. Then from (9) for $t > t_1$ we have $a_3(t) = \log a(t)$, $b_3(t) = a(t)b(t)$, and so $a(t)b(t) \to \gamma$ is equivalent to $b_3(t) \to \gamma$. But if f_n is the density of $X_{(n)}/a(n)$ and $f_{3,n}$ that of $\gamma(X_{3(n)} - a_3(n))$, these are related for $xa(n) > 1$ by

$$f_n(x) = \gamma x^{-1} f_{3,n}(\gamma \log x). \tag{10}$$

It is now a relatively straightforward matter to deduce from Theorem 1(iii) and (10) that $b_3(t) \to \gamma$ iff $F \in \mathscr{L}_{1,\gamma}$ (using Lemma 2.2.3 in Galambos, 1978, for the necessity).

(ii) Since $\mathscr{L}_{2,\gamma} \subset \mathscr{D}_{2,\gamma}$ it suffices to assume that $\omega < \infty$ and prove that $F \in \mathscr{L}_{2,\gamma}$ iff $\{\omega - a(t)\}b(t) \to \gamma$. Let $g(x) = (\omega - x)^{-1}$, $x < \omega$, and $X_1 = g(X)$. Then, with obvious notation, we have from (9) that $a_1(t) = \{\omega - a(t)\}^{-1}$, $b_1(t) = b(t)\{\omega - a(t)\}^2$, and therefore the given condition is equivalent to $a_1(t)b_1(t) \to \gamma$. But if f_n is the density of $\{X_{(n)} - \omega\}/\{\omega - a(n)\}$ and $f_{1,n}$ that of $X_1(n)/a_1(n)$, we have for $x < 0$

$$f_n(x) = x^{-2} f_{1,n}(-x^{-1}). \tag{11}$$

The result readily follows from this and Theorem 1(i), using Lemma 2.2.3 in Galambos (1978) for the necessity. □

For the proofs of the remaining parts of Theorems 2 and 3 we need the following lemma.

LEMMA 3. *Suppose that $b(t) \to \gamma > 0$ and let $\varepsilon > 0$. Then there exists t_0 such that*

(i) $(1 - \varepsilon)x - \varepsilon \leq u_t^{-1}(x) \leq (1 + \varepsilon)x + \varepsilon$ *for all* $x > 0$ *and* $t > t_0$, *and*

(ii) $(1 + \varepsilon)x - \varepsilon \leq u_t^{-1}(x) \leq (1 - \varepsilon)x + \varepsilon$ *for all* $x < 0$ *and* $te^x > t_0$.

PROOF. Write $U(t) = e^{a(t)}$; then $u_t^{-1}(x) = \{r(t)\}^{-1}\log\{U(te^x)/U(t)\}$. Since $F \in \mathscr{L}_3$ we have $u_t^{-1}(x) \to x$ and $r(t) \to \gamma^{-1}$, and it follows that U is regularly varying with exponent γ^{-1}. Thus from Lemma 1(ii), for all $\delta > 0$ there exists t_0 such that for all $t > t_0$ and $s \geq 1$

$$(1 - \delta)s^{\gamma^{-1}-\delta} \leq U(ts)/U(t) \leq (1 + \delta)s^{\gamma^{-1}+\delta}.$$

The result (i) follows straightforwardly, with an appropriate choice of δ.

Result (ii) follows in a similar way on taking $s < 1$ and applying Lemma 1(ii) to $U(TX)/U(T)$ with $T = ts$ and $X = s^{-1}$. □

PROOF OF THEOREM 2 FOR $\mathscr{L}_{1,\gamma}$, $\mathscr{L}_{2,\gamma}$. Let $F \in \mathscr{L}_{1,\gamma}$. The first assertion is immediate from (10) and the fact that $f_{3,n}(z_n) \to 0$ as $z_n \to \infty$ from Theorem 2 ($\mathscr{L}_3$). For the second assertion, we need to show that $f_n(x_n) \to 0$ when $x_n \to x_0 \leq 0$. Write $z_n = \gamma \log x_n$, $t_n = n\exp(u_{3,n}(z_n))$, with $u_{3,n}$ defined in terms of the functions a_3, r_3 for the random variable X_3. Then it is easy to see that there exists c' such that $a(n)x_n > c'$ implies that $t_n > t_0$. Suppose then that $a(n)x_n > c'$; we show that $\exp(-z_n/\gamma)f_{3,n}(z_n) \to 0$ as $z_n \to -\infty$. But from Lemma 3(ii) we have $z_n \geq (1 + \varepsilon)u_{3,n}(z_n) - \varepsilon$ for $n > n_0$, and the result now follows from (7) for $a(n)x_n > c'$.

Suppose now that $a(n)x_n \leq c'$ and set $\eta = F(c') < 1$. Then from (1) and (3), $f_n(x_n) \leq Cna(n)\eta^{n-B-1} \to 0$ from the regular variation of $a(t) = \exp(a_3(t))$(from the proof of Lemma 3).

On the other hand if (3) fails to hold, choose a sequence (z_n) satisfying $f(z_n) > F(z_n)^{-n+1}$. Setting $x_n = z_n/a(n)$ gives $f_n(x_n) > na(n) \to \infty$, and the convergence in (1) cannot be uniform as $h_{1,\gamma}(x)$ is bounded.

If $F \in \mathscr{L}_{2,\gamma}$, $\gamma > 1$, in order to deduce that $f_n(x_n) \to 0$ for (i) $x_n \to 0$, and (ii) $x_n \to -\infty$, $-x_n \leq \{c'(\omega - a(n))\}^{-1}$, it suffices to show from (10) and (11) that $\exp(z_n/\gamma)f_{3,n}(z_n) \to 0$ for (i) $z_n \to \infty$ and (ii) $z_n \to -\infty$, $t_n > t_0$ respectively. In the latter case the result is immediate from (7). In case (i) we have $z_n \leq (1 + \varepsilon)u_{3,n}(z_n) + \varepsilon$ for $n > n_0$. Since $\gamma > 1$, from (5) we can choose ε sufficiently small to give the desired convergence when $z_n \to \infty$.

Suppose now that $-x_n > \{c'(\omega - a(n))\}^{-1}$ and set $\eta = F(\omega - c'^{-1}) < 1$. Then from (1) and (3), $f_n(x_n) \leq Cn(\omega - a(n))\eta^{n-B-1} \to 0$ from the regular variation of $a_1(t) = (\omega - a(t))^{-1}$.

On the other hand if (3) fails to hold, choose a sequence (z_n) satisfying $f(z_n) > F(z_n)^{-n+1}$. Setting $x_n = (z_n - \omega)/(\omega - a(n))$ gives $f_n(x_n) > n(\omega - a(n)) > n^{1-\gamma^{-1}-\varepsilon} \to \infty$ for $\varepsilon < 1 - \gamma^{-1}$, and the convergence in (1) cannot be uniform as $h_{2,\gamma}(x)$ is bounded for $\gamma > 1$. □

PROOF OF THEOREM 3, (i) AND (ii). (i) In view of Theorem 1(i) it suffices for the first assertion to show that $\lim \sup_{n\to\infty} \int_A^\infty f_n^p(x)\, dx \to 0$ as $A \to \infty$. For $a(n)x > 1$ we have from (10) $f_n^p(x)\, dx = \{\gamma e^{-z/\gamma}\}^{p-1} f_{3,n}^p(z)\, dz$ where $z = \gamma \log x$. For $p > 1$ the result follows immediately from (8). For $p < 1$ Lemma 3(i) gives $\{e^{-z/\gamma}\}^{p-1} \leq \exp[\gamma^{-1}(1 - p)\{(1 + \varepsilon)u_n(z) + \varepsilon\}]$ and it now follows as in (8) that, provided $(\gamma + 1)p > 1$, we can choose ε sufficiently small so that

$$\int_A^\infty \{e^{-z/\gamma}\}^{p-1} f_{3,n}^p(z)\, dz \leq c_7 \exp(-\{p - \gamma^{-1}(1 - p) - (1 + \gamma^{-1})\varepsilon(1 - p)\}u_n(A))$$

and the result follows, as $\lim_{n\to\infty} u_n(A) = A$.

For the second assertion the proof of Theorem 2(i) gives uniform convergence on $a(n)x > c'$, and it suffices to show that $\lim_{n\to\infty} \int_{-\infty}^{A_n} f_n^p(x)\, dx = 0$ where $A_n = c'/a(n)$. But if $x \leq c'/a(n)$ we have $f_n(x) \leq na(n)f(a(n)x)\eta^{n-1}$ where $\eta = F(c')$

< 1 and so

$$\int_{-\infty}^{A_n} f_n^p(x)\,dx \le (n\eta^{n-1})^p a^{p-1}(n) \int_{\alpha}^{c'} f^p(u)\,du \to 0$$

as $n \to \infty$ from the regular variation of $a(n)$.

(ii) The proof is very similar to (i); for $-x \le \omega - a(n)$ we have from (10) and (11) $f_n^p(x)\,dx = \{\gamma e^{z/\gamma}\}^{p-1} f_{3,n}(z)\,dz$ where $z = -\gamma\log(-x)$. Again using Lemma 3(i) one finds that for $p > 1$

$$\int_A^{\infty} \{e^{z/\gamma}\}^{p-1} f_{3,n}^p(z)\,dz \le c_8 \exp(-\{p - \gamma^{-1}(p-1) - (1+\gamma^{-1})\varepsilon(p-1)\}u_n(A))$$

and the result follows, since $\lim_{n\to\infty} u_n(A) = A$ and $p - \gamma^{-1}(p-1) > 0$.

For the final assertion consider the range $-A_n < x < -A < 0$, where $A_n = \{c'(\omega - a(n))\}^{-1}$. The only possible difficulty occurs when $p < 1$, in which case Lemma 3(ii) gives $\{e^{z/\gamma}\}^{p-1} \le \exp[-\gamma^{-1}(1-p)\{(1+\varepsilon)u_n(z) - \varepsilon\}]$, and $\lim_{A\to\infty} \limsup_{n\to\infty} \int_{-A_n}^{-A} f_n^p(x)\,dx = 0$ follows as in the proof of Theorem 3(iii). Finally, $\lim_{n\to\infty} \int_{-\infty}^{-A_n} f_n^p(x)\,dx = 0$ follows in an entirely similar way to (i) above. □

PROOF OF THEOREM 4. If F is in the domain of uniform local attraction of H, and the density convergence is uniform on (a, b), it follows from (1) that $nd_n f(c_n + d_n x) \to h(x)/H(x) = \ell(x)$, say, uniformly on (a, b). But if $f_{k,n}(x_1, \cdots, x_k)$ is the joint density of $(X_{(n-j+1)} - c_n)/d_n$, $j = 1, \cdots, k$, we have

$$f_{k,n}(x_1, \cdots, x_k) = \{\textstyle\prod_{j=1}^k nd_n f(c_n + d_n x_j)\}\{\prod_{j=1}^k (n-j+1)/n\} F^{n-k}(c_n + d_n x_k)$$

for $x_k < \cdots < x_1$, and so

$$f_{k,n}(x_1, \cdots, x_k) \to \{\textstyle\prod_{j=1}^k \ell(x_j)\} H(x_k) \tag{12}$$

uniformly on $a < x_k < \cdots < x_1 < b$.

(i) Here the right-hand side in (12) is $\gamma^k(\prod_{j=1}^k x_j)^{-\gamma-1}\exp(-x_k^{-\gamma})$ and from Theorem 2 the convergence is uniform on $\varepsilon < x_k < \cdots < x_1 < \infty$ for every $\varepsilon > 0$, and if (3) holds, $\varepsilon = 0$. The result follows on transformation to Z_j, $j = 1, \cdots, k-1$.

(ii) The right-hand side in (12) is $\gamma^k\{\prod_{j=1}^k (-x_j)\}^{\gamma-1}\exp(-(-x_k)^\gamma)$ and the convergence is uniform on $-A < x_k < \cdots < x_1 < -\varepsilon$ for every $0 < \varepsilon < A$. If $\gamma > 1$, from Theorem 2, $A = \infty$, and if furthermore (3) holds, $\varepsilon = 0$. The result follows on transformation to Z_j, $j = 1, \cdots, k-1$.

(iii) The right-hand side in (12) is $\exp(-\sum_{j=1}^k x_j)\exp(-e^{-x_k})$ and from Theorem 2 the convergence is uniform on $-A < x_k < \cdots < x_1 < \infty$ for every $A > 0$, and if (3) holds, $A = \infty$. The result follows on transformation to Z_j, $j = 1, \cdots, k-1$. □

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REFERENCES

ANDERSON, C. W. (1971). Ph.D. Thesis, London University.

GALAMBOS, J. (1978). *The Asymptotic Theory of Extreme Order Statistics.* Wiley, New York.

HAAN, L. DE (1970). *On Regular Variation and its Application to the Weak Convergence of Sample Extremes.* Mathematical Centre Tract 32, Amsterdam.

HAAN, L. DE and RESNICK, S. I. (1982). Local limit theorems for sample extremes. *Ann. Probab.* **10** 396–413.

PICKANDS, J. (1967). Sample sequences of maxima. *Ann. Math. Statist.* **38** 1570–1574.

WEISSMAN, I. (1978). Estimation of parameters and large quantiles based on the k largest observations. *J. Amer. Statist. Assoc.* **73** 812–815.

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