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SAMPLE PATH PROPERTIES OF SELF-SIMILAR PROCESSES WITH STATIONARY INCREMENTS[1]

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A real-valued process $X = (X(t))_{t\in\mathbb{R}}$ is self-similar with exponent H (H-ss), if $X(a\cdot) =_d a^H X$ for all $a > 0$. Sample path properties of H-ss processes with stationary increments are investigated. The main result is that the sample paths have nowhere bounded variation if $0 < H \le 1$, unless $X(t) \equiv tX(1)$ and $H = 1$, and apart from this can have locally bounded variation only for $H > 1$, in which case they are singular. However, nowhere bounded variation may occur also for $H > 1$. Examples exhibiting this combination of properties are constructed, as well as many others. Most are obtained by subordination of random measures to point processes in $\mathbb{R}^2$ that are Poincaré, i.e., invariant in distribution for the transformations $(t, x) \mapsto (at + b, ax)$ of $\mathbb{R}^2$. In a final section it is shown that the self-similarity and stationary increment properties are preserved under composition of independent processes: $X_1 \circ X_2 = (X_1(X_2(t)))_{t\in\mathbb{R}}$. Some interesting examples are obtained this way.

0. Introduction. In the present paper stochastic processes are random functions $X = (X(t))_{t\in T} = (X(t, \omega))_{t\in T}$ on $T = \mathbb{R}$ or $T = \mathbb{R}_+ := [0, \infty)$ with values in $\overline{\mathbb{R}} := [-\infty, \infty]$. Two stochastic processes X and Y are said to be equal in distribution, notation $X =_d Y$, if they have the same finite-dimensional distributions, i.e., $(X(t))_{t\in I} =_d (Y(t))_{t\in I}$ for all finite $I \subset T$. We say that X is self-similar with exponent $H \in \mathbb{R}$ (H-ss), if $X(t) \in \mathbb{R}$ with probability 1 (wp1) for each fixed $t \in T$ and

$$X(a\cdot) =_d a^H X \quad \text{for all real} \quad a > 0. \tag{0.1}$$

Lamperti (1962) has shown that a stochastically continuous process X is ss (i.e., H-ss for some H) iff there is a process Y and a positive function d on $\mathbb{R}_+$ such that

$$\left.\begin{matrix} Y(s\cdot)/d(s) \to_d X \\ d(s) \to \infty \end{matrix}\right\} \quad \text{as} \quad s \to \infty \quad \text{through } \mathbb{R}_+. \tag{0.2}$$

Here $\to_d$ denotes convergence of the finite-dimensional distributions. Actually, Lamperti introduced the term "semi-stable" instead of "self-similar", which we

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also abbreviate to "ss". The present name is due to Mandelbrot (1977). We say that X has *stationary increments*, or is a stationary increment (si) process, if

$$X(\cdot + b) - X(b) =_d X - X(0) \quad \text{for all} \quad b \in T. \tag{0.3}$$

It is easy to see that X in (0.2) is si, if Y is discretely si, i.e., satisfies (0.3) only for $b \in T \cap \mathbb{Z}$. So ss si processes arise as limits in (0.2) with si or discretely si Y. Futhermore, all nontrivial measurable ss si processes have $H > 0$ in (0.1) (cf. Section 1), so all such processes arise in (0.2), as (0.1) yields (0.2) with $Y = X$, $d(s) = s^H$ and $=_d$ instead of $\to_d$. This explains the importance of this class of processes.

Classical examples of ss si processes are obtained by the additional assumption that the increments are independent, i.e., $(X(t + b) - X(b))_{t\in\mathbb{R}_+}$ and $(X(t))_{t\in T\cap(-\infty,b]}$ are independent for all $b \in T$. These assumptions characterize the *strictly stable* processes. Recall (cf. Taylor, 1973, or Fristedt, 1974) that stable processes are characterized as processes X with stationary independent nondegenerate increments such that there exist reals $c(a)$ and $d(a)$ with $d(a) > 0$ and

$$X(a\cdot) =_d d(a)X + c(a) \tag{0.4}$$

for all real $a > 0$. It then follows that $d(a) = a^H$ for some $H \in [½, \infty)$ and that all these H indeed occur. In particular, X is Brownian motion if $H = ½$. In our definition (0.1) of self-similarity we do not allow translations, which amounts to the restriction $c(a) = 0$ in (0.4). The adverb "strictly" in "strictly stable" refers to this restriction. In the theory of stable processes one usually refers to $\alpha := 1/H \in (0, 2]$ as the (characteristic) exponent of the process. This terminology is motivated by the formula $\mathbb{E}e^{i\lambda X(t)} = \exp(-ct\,|\,\lambda\,|^{\alpha})$ for symmetric stable processes $(X =_d -X)$.

The ss Markov process was the first class of ss processes to be characterized completely (cf. Lamperti, 1972, Kiu, 1975). The second such class is the class of ss stationary extremal processes (O'Brien, Torfs and Vervaat, 1984+). Although the second paper will appear later, its underlying research (1980) preceded that of the present paper and provided the author with the necessary experience and motivation. There is already an extensive literature on ss si processes. For surveys the reader is referred to Major (1981) and Taqqu (1982). Recent papers are Maejima (1983), Surgailis (1981) and Taqqu and Wolpert (1983). However, the character of these papers is somewhat different from those concerned with the other classes of ss processes. So far, all papers dealt with rather special subclasses of the ss si processes (e.g., Gaussian processes) or described specific methods for generating such processes (e.g. by stochastic integrals). Although it is nowhere stated explicitly, the reader can feel the hope that by pursuing this "natural history" of ss si processes finally so many examples and constructions will be found, that all ss si processes are covered. However, in O'Brien and Vervaat (1985) all ss si jump processes are studied, and so many processes arise already there, that there is no hope for an exhaustive list of examples and constructions. This changed the author's points of view and led him to the investigation of general properties of ss si processes. The present paper deals with sample path properties, and a companion paper by O'Brien and Vervaat (1983) with properties of marginal distributions. They seem to be the first papers in this direction.

The main result of the present paper is in Section 3, with as principal conclusion that, apart from trivial possibilities for $H = 1$, an H-ss si process can have sample paths of locally bounded variation only if $H > 1$, in which case the sample paths are singular. Sections 1 and 2 contain preparations of a general character. Section 1 deals with the impossibility of $H \leq 0$, and with regularity assumptions like separability for X. Generalities about the variation, absolute continuity or singularity of sample paths of ss si processes are dealt with in Section 2. Section 4 starts with a review of the main results of O'Brien and Vervaat (1985) about ss si jump processes. Such processes can be characterized most conveniently by their saltus processes, point processes Π in $\mathbb{R}^2$, whose distributions are invariant for the transformations $(t, x) \mapsto (at + b, ax)$ (a, b real, $a > 0$) of $\mathbb{R}^2$. Section 4 then introduces the most important production rule for new ss si processes in this paper, subordination of measures to saltus processes Π as above. A similar subordination was studied by Surgailis (1981) for Poisson Π. General properties of subordinated processes are studied at the end of Section 4.

Section 5 presents examples of ss si processes, all constructed by the principles of Section 4. Some examples serve to demonstrate that possibilities which are not excluded in the theorems of Section 3 actually do occur, other like fractional processes (cf. Maejima, 1983, and Taqqu and Wolpert, 1983) are reviewed briefly because of their intrinsic interest and occurrence in the literature.

In the concluding Section 6 it is shown that the composition $X_1 \circ X_2 = (X_1(X_2(t)))_{t \in T}$ of two independent ss si processes X_1 and X_2 is again ss si. By this principle, examples of H-ss si processes with $H > 1$ and nowhere bounded variation can be constructed, that supplement those of Section 4.

In contrast to much of the previous literature on ss si processes, the domain T and the range $\overline{\mathbb{R}}$ of the processes are one-dimensional. It is not too hard to generalize several results to higher dimensions, but the wish to keep things readable and a proper amount of laziness discouraged the author from doing so.

1. Regularity assumptions. Henceforth the domain T of $X = (X(t))_{t \in T}$ is all of $\mathbb{R}$ unless stated otherwise. The most important regularity assumption on X is measurability. Recall that X is measurable if $(t, \omega) \mapsto X(t, \omega)$ is jointly measurable. The next theorem contains a useful necessary condition for measurability, which can be seen as a partial converse to Theorem 2.6 of Doob (1953).

THEOREM 1.1. *If the stochastic process X is measurable, then there is a subset T_0 of T such that $T \setminus T_0$ is a Lebesgue null set and for each $t \in T_0$ there is a decreasing sequence $t_n \downarrow t$ in T_0 with $X(t_n) \to_p X(t)$ (convergence in probability).*

PROOF. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be the underlying probability space of X, call two $\mathbb{R}$-valued rv's on $(\Omega, \mathscr{F}, \mathbb{P})$ equivalent if they are equal wp1, and let L_0 be the collection of equivalence classes of such rv's, endowed with the topology of convergence in probability. From Theorem IV.30 in Dellacherie & Meyer (1978) it follows that the mapping ϕ: $T_0 \ni t \mapsto X(t) \in L_0$ is Borel measurable, and by Lusin's theorem (page 211, Bauer, 1981) that for each $\varepsilon > 0$ there is a Borel set $U_\varepsilon \subset T$ with Lebesgue measure $<\varepsilon$, such that the restriction of ϕ to $T \backslash U_\varepsilon$ is continuous. Let V_ε be the union of U_ε with those points of $T \backslash U_\varepsilon$ which are right-isolated in $T \backslash U_\varepsilon$ (a countable set). Then the conclusion of the theorem holds with $T \backslash T_0 = \bigcap_{n=1}^{\infty} V_{1/n}$. □

Although H-self-similarity with $H < 0$ is a relevant possibility for other classes of processes like extremal processes (cf. O'Brien, Torfs and Vervaat, 1984+), it is not for si processes, as the next theorem shows.

LEMMA 1.2. *If X is H-ss with $H \neq 0$, then $X(0) = 0$ wp1.*

PROOF. By (0.1) we have $X(0) =_d a^H X(0) \to 0$ wp1 as $a \downarrow 0$ if $H > 0$, $a \to \infty$ if $H < 0$.

THEOREM 1.3. *Let X be H-ss si.*

(a) *If $H < 0$, then $X(t) = 0$ wp1 for each real t, so $X \equiv 0$ wp1 in case X is separable.*
(b) *If $H = 0$ and X is measurable, then $X(t) = X(0)$ wp1 for each real t, so $X \equiv X(0)$ in case X is separable.*

REMARK. There are nontrivial nonmeasurable 0-ss si processes. For example, $(X(t))_{t \in \mathbb{R}}$ iid and nondegenerate.

PROOF OF THE THEOREM. (a) From (0.1) we have

$$X(t) =_d |t|^H X(\operatorname{sgn} t) \to_d 0 \quad \text{as} \quad |t| \to \infty.$$

Hence $X(t) = X(t) - X(0) =_d X(b+t) - X(b) \to_d 0 - 0$ as $b \to \infty$ by Lemma 1.2 and (0.3). So $X(t) = 0$ wp1 for each t separately.

(b) Also $Y := X - X(0)$ is 0-ss si and measurable. Moreover, $Y(0) = 0$. From $Y(a\cdot) =_d Y$ it follows that $Y(a) =_d Y(1)$ for $a > 0$, so $Y(t+h) - Y(t) =_d Y(1)$ for $h > 0$. By Theorem 1.1 we must have $Y(1) = 0$ wp1. Hence $Y(t) = 0$ wp1, so $X(t) = X(0)$ wp1 for each t separately. □

From now on we will restrict our attention to H-ss si processes with $H > 0$. Moreover, we want to exclude the trivial event $[X \equiv 0]$. Therefore we require that this event has zero probability. If $H > 0$, then $X(0) = 0$ wp1 by Lemma 1.2 and X is continuous in probability (or stochastically continuous):

$$X(t+h) - X(t) =_d X(h) =_d |h|^H X(\operatorname{sgn} h) \to_d 0 \quad \text{as} \quad h \to 0. \tag{1.1}$$

By Doob (1953, Theorem II.2.6) or Neveu (1965, page 91) there is a measurable separable version of X, for which each countable dense subset S of T is a separant. Let us explain briefly the terminology. A version of X is a process Y such that $Y(t) = X(t)$ wp1 for each t separately. In particular $Y =_d X$. The process Y is separable with separant S (S countable and dense in T), if wp1 the graph of Y is contained in the closure in $T \times \overline{\mathbb{R}}$ of the graph of $(Y(t))_{t \in S}$. From now on we will assume that X is measurable and separable with each countable dense subset S of T as separant.

Formula (1.1) and the implied continuity in probability might suggest that X has smooth sample paths. This suggestion is wrong. There are H-ss si processes X, the so-called fractional stable processes (cf. 5.4 or Maejima, 1983), for which X is wp1 nowhere bounded (the author even conjectures that the graph of X is wp1 dense in $T \times \overline{\mathbb{R}}$). In this case separability is a meaningless restriction (but measurability is not).

Nowhere bounded processes occur more often in the literature. For instance, stationary Gaussian processes are known to have either continuous or nowhere bounded sample paths (Belyaev, 1960, 1961).

Very often though, X does allow a version whose sample paths have only jump discontinuities, i.e., wp1 X has left and right limits at every $t \in T$. In particular this is the case if X has locally bounded variation. Making X right-continuous does not change the finite-dimensional distributions outside the set of fixed discontinuities

$$\Delta := \{t \in \mathbb{R}: \mathbb{P}[\lim_{u \uparrow t} X(u) \neq \lim_{u \downarrow t} X(u)] > 0\}.$$

By Billingsley (1968, page 124) Δ is countable (i.e., finite or countably infinite), and by (0.3) Δ is translation invariant. Hence Δ is empty. So X allows a version in $D(T)$, the set of $\overline{\mathbb{R}}$-valued right-continuous functions on T that have everywhere left limits (except at inf T if lying in T). Since the distribution of a $D(T)$-valued rv is determined by its finite-dimensional distributions (Billingsley, 1968, page 123), formulae (0.1) and (0.3) now hold with X interpreted as $D(T)$-valued rv.

We now summarize the standard hypotheses that will be assumed throughout the paper.

STANDARD HYPOTHESES 1.4. *The process* $(X(t))_{t \in T}$ *is* $\overline{\mathbb{R}}$*-valued and* $X(t) \in \mathbb{R}$ *wp*1 *for each* $t \in T$. *We have* $T = \mathbb{R}$, *unless it is stated that* $T = \mathbb{R}_+$. *The process* X *is* H*-self-similar* (*i.e.* (0.1) *holds*) *and has stationary increments* (*i.e.* (0.3) *holds*). *Moreover*, $H > 0$, X *is separable and measurable, each countable dense subset* S *of* T *is a separant for* X, *and* $\mathbb{P}[X \equiv 0] = 0$. *Whenever* X *allows a version in* $D(T)$, *we take* X *equal to this version.*

2. Basic properties.

Random measures. Let $\mathscr{R}$ be the ring of finite unions of intervals $(a, b] \cap T$ $(a, b \in \mathbb{R})$. We say that μ is a formal measure on $\mathscr{R}$, if μ is a finitely additive function on $\mathscr{R}$ with values in $\mathbb{R}^* := \overline{\mathbb{R}} \cup \{*\}$. Here $*$ is to be interpreted as "undefined", and the measure-theoretic addition conventions in $\overline{\mathbb{R}}$ are extended to $\mathbb{R}^*$ by

$$\infty - \infty = -\infty + \infty = *, \quad * + x = x + * = * \quad \text{for all} \quad x \in \mathbb{R}^*.$$

If f is an $\overline{\mathbb{R}}$-valued function on $\mathbb{R}$, then it generates a formal measure μ by

$$\mu(a, b] = f(b) - f(a). \tag{2.1a}$$

If $f(0)$ is finite, we can recover $f - f(0)$ from μ by

$$f(t) - f(0) = \mu(I_t)\operatorname{sgn} t, \tag{2.1b}$$

where henceforth

$$I_t := \begin{cases} (0, t] & \text{if} \quad t > 0, \\ (t, 0] & \text{if} \quad t < 0, \\ \varnothing & \text{if} \quad t = 0. \end{cases} \tag{2.2}$$

Consequently, with each stochastic process X such that $X(0) = 0$ wp1 we can associate a random formal measure $N = N_X$ by

$$\begin{cases} N(a, b] := X(b) - X(a) \\ X(t) \quad\;\; = N(I_t)\operatorname{sgn} t. \end{cases} \tag{2.3}$$

Here N is considered to be an $\mathbb{R}^*$-valued random function on $\mathscr{R}$.

Obviously, X is H-ss (i.e., (0.1) holds) iff N is H-ss as a random function on $\mathscr{R}$:

$$N(a\cdot) =_d a^H N \quad \text{for all real} \quad a > 0. \tag{2.4}$$

Moreover, X is si (i.e., (0.3) holds) iff N is *stationary* as a random function on $\mathscr{R}$:

$$N(\cdot + b) =_d N \quad \text{for all} \quad b \in T. \tag{2.5}$$

The advantage of considering random measures is that the si property (0.3) translates into the more natural stationarity property (2.5). In particular, the generalization to $T = \mathbb{R}_d$ or $\mathbb{R}_+^d$ is awkward for (0.3), but straightforward for (2.5). In the present paper we will shift freely our considerations hence and forth between X and N, connected by (2.3), depending on the situation.

Even if f in (2.1a) is $\mathbb{R}$-valued, the measure theory behind μ can be virtually empty, for instance if f has nowhere bounded variation (nbv), i.e., no bounded variation in any interval. There is a relevant measure theory for the case that f in (2.1a) is $\mathbb{R}$-valued and has locally bounded variation (lbv), which corresponds to μ being locally bounded, signed and finitely additive. If, in addition, f is right-continuous then μ is a *Radon measure*, i.e., locally bounded, signed and σ-additive. To this case all of traditional measure theory applies.

We will see below that for H-ss si X there is the following dichotomy. Either X has lbv, in which case X has a version in $D(T)$ and the corresponding random measure N is Radon, or X has nbv and the random measure N is purely formal.

If N is a random Radon measure, then N allows a Hahn decomposition $N = N^+ - N^-$ with N^+, N^- positive Radon measures, and a Jordan decomposition

$$N = N_{\mathrm{ac}} + N_{\mathrm{cs}} + N_{\mathrm{d}} \;(= N_{\mathrm{ac}} + N_{\mathrm{s}} = N_{\mathrm{c}} + N_{\mathrm{d}}),$$

where N_{ac} is absolutely continuous with respect to Lebesgue measure, N_{cs} is singular with respect to Lebesgue measure and diffuse or "continuous", and N_{d}

discrete or atomic, i.e., concentrated on a countable set. Inspecting the constructions ($J \in \mathscr{R}$)

$$N^+(J) = \sup\{\textstyle\sum_k (N(J_k))^+ : J_k \in \mathscr{R}, \text{ disjoint, } \bigcup_k J_k \subset J\},$$

$$N_{\mathrm{ac}}(J) = \int_J X'(t)\, dt,$$

where X' exists a.e. by Lebesgue's differentiation theorem,

$$N_d := \textstyle\sum_{t \in T} (X(t) - X(t-))\iota_t,$$

where ι_t is the degenerate probability measure concentrated at t, we see that N^+, N^-, N_{ac}, N_{cs} and N_{d} all are H-ss and/or stationary, if N is.

Intensive affine properties. Let f be an $\overline{\mathbb{R}}$-valued function on T. We consider properties $\mathscr{P}$ that f may or may not have on bounded intervals $I \subset T$. We call property $\mathscr{P}$ *intensive*, if f having property $\mathscr{P}$ on I implies $\mathscr{P}$ for f on all subintervals $J \subset I$. We call $\mathscr{P}$ *affine*, if f having property $\mathscr{P}$ on I implies $\mathscr{P}$ for $af + b$ on I and for $f(a \cdot + b)$ on $(I - b)/a$ (a, b real, $a > 0$).

2.1. *Examples of intensive affine properties.*
f is bounded on I;
f is finite-valued on I;
f has bounded variation on I;
f is continuous on I;
f is singular on I;
f is nowhere bounded on I, i.e., unbounded on all subintervals of I;
f has nbv in I, i.e., f does not have bounded variation in any subinterval of I.

THEOREM 2.2. *If X is a separable measurable H-ss si process and $\mathscr{P}$ is an intensive affine property such that [X has property $\mathscr{P}$ on I] is an event for all bounded intervals $I \subset T$, then wp1 either X has property $\mathscr{P}$ on all bounded intervals $I \subset T$ or on none of them.*

REMARK. The present paper contains many statements of the type "wp1 either A or B", as in the conclusion of the theorem. They mean that almost surely one and only one of the events A and B occur. They do not exclude that $0 < \mathbb{P}(A) < 1$.

PROOF OF THEOREM 2.2. For $b \in T$ we have by (0.3), strengthened beyond finite-dimensional distributions by separability and measurability,

$$p := \mathbb{P}[X \text{ has } \mathscr{P} \text{ on } I] = \mathbb{P}[X(b + \cdot) - X(b) \text{ has } \mathscr{P} \text{ on } I]$$
$$= \mathbb{P}[X(b + \cdot) \text{ has } \mathscr{P} \text{ on } I] = \mathbb{P}[X \text{ has } \mathscr{P} \text{ on } I - b].$$

Let b be an interior point of I. Then 0 is an interior point of $I - b =: I_0$ (to be replaced by $I_0 \cap \mathbb{R}_+$ in case $T = \mathbb{R}_+$). We now have for all $a > 0$ by (0.1)

$$p = \mathbb{P}[X \text{ has } \mathscr{P} \text{ on } I_0] = \mathbb{P}[X(a^{-1}\cdot) \text{ has } \mathscr{P} \text{ on } aI_0]$$
$$= \mathbb{P}[a^{-H}X \text{ has } \mathscr{P} \text{ on } aI_0] = \mathbb{P}[X \text{ has } \mathscr{P} \text{ on } aI_0].$$

Since $\mathscr{P}$ is an intensive property, the last event is decreasing in a, so the last probability decreases to (consequently, being constant, equals)

$$\mathbb{P}(\cap_{a>0} [X \text{ has } \mathscr{P} \text{ on } aI_0]) = \mathbb{P}[X \text{ has } \mathscr{P} \text{ everywhere}].$$

It follows that $[X$ has $\mathscr{P}$ on $I]$ and the smaller event $[X$ has $\mathscr{P}$ everywhere$]$ differ only by a set of probability 0, and so do $[X$ has $\mathscr{P}$ on some $I]$ and $[X$ has $\mathscr{P}$ everywhere$]$.

COROLLARY 2.3. *If X is separable, measurable, H-ss and si, then wp1:*

(a) *X is constant (so $X \equiv 0$) or nowhere constant;*
(b) *X is finite-valued, or infinite-valued on a dense subset of T;*
(c) *X is locally bounded or nowhere bounded;*
(d) *X has lbv or nbv.*

At this point it is useful to collect some characterizations of monotonicity of the sample paths of X.

THEOREM 2.4. *Let X be H-ss, si, separable and measurable. Then*

(a) $[X(1) = 0] = [X \equiv 0]$ *modulo null events,*
(b) $X(1) \le 0$ *wp1 iff X is nonincreasing wp1.*

REMARK. Another result on monotone sample paths is Theorem 3.5(c).

PROOF. (a) By Lemma 3 of O'Brien and Vervaat (1983) both events have equal probability. Moreover, $[X \equiv 0] \subset [X(1) = 0]$.

(b) If X increases somewhere with positive probability, then there are $s, t \in \mathbb{R}$ with $s < t$ and $\mathbb{P}[X(s) < X(t)] > 0$. By (0.1) and (0.3) it follows that

$$\mathbb{P}[X(t) - X(s) > 0] = \mathbb{P}[X(t-s) > 0] = \mathbb{P}[(t-s)^H X(1) > 0] = \mathbb{P}[X(1) > 0],$$

so $X(1) > 0$ with positive probability. This proves one implication in (b). The other is trivial.

2.5. *σ-finite measures.* So far we have not considered σ-finite σ-additive signed measures μ. Whenever such a measure arises from finite-valued f by (2.1a), it is necessarily Radon. However, measures like $\mu(dt) = |t|^{-1}\,dt$ are σ-finite, but not Radon. So it seems natural to consider also σ-finite σ-additive random measures N (not originating from an X as in (2.3)). Now Theorem 2.2 can easily be rephrased and proved for H-ss stationary random measures in this generality. Since $|N| := N^+ + N^-$ being bounded on I is an intensive affine property, an

H-ss stationary random measure is wp1 either locally bounded or everywhere unbounded. The latter possibility contradicts σ-finiteness. So all σ-finite H-ss stationary random measures are in fact Radon.

2.6. *Extreme distributions and ergodicity.* Let X be a process such that $X(0) = 0$ wp1, with corresponding random measure N as coupled by (2.3). Recall that X is si iff N is stationary iff the distributions of X and N are invariant under the (equivalent) transformations

$$\left.\begin{aligned} X &\to X(b + \cdot) - X(b) \\ N &\to N(b + \cdot) \end{aligned}\right\} \quad \text{for} \quad b \in T. \tag{2.6}$$

Similarly X is H-ss iff N is H-ss iff the distributions of X and N are invariant under the (equivalent) transformations

$$\left.\begin{aligned} X &\to a^{-H}X(a\cdot) \\ N &\to a^{-H}N(a\cdot) \end{aligned}\right\} \quad \text{for real} \quad a > 0. \tag{2.7}$$

In case $T = \mathbb{R}$, all compositions of transformations (2.6) and (2.7) form a group isomorphic to the group of positive affine transformations $t \mapsto at + b$ ($a, b \in R$, $a > 0$) of $\mathbb{R}$ with composition of functions as product. In case $T = \mathbb{R}_+$, the set of compositions is isomorphic to the subsemigroup consisting of $t \mapsto at + b$ with a, $b \in \mathbb{R}$, $a > 0$, $b \geq 0$.

Let (2.7, H) denote (2.7) with a given fixed H and let * denote (2.6), (2.7, H) or (2.6) and (2.7, H). Let $\mathscr{I}_*$ be the σ-field of events which are invariant up to null sets under *, and let us call X or N * stationary, if their probability distributions are invariant under *. So X is si iff it is (2.6) stationary, and H-ss iff it is (2.7, H) stationary. We call X (or N) * ergodic if X is * stationary and $P[X \in A] = 0$ or 1 for all $A \in \mathscr{I}_*$.

We will not pursue an analysis of ergodicity in the examples of the present paper. In general, they are either very obviously not (2.6), (2.7, H) ergodic, or they are, but a proof may be complicated. Analysis of ergodicity is more prominent in O'Brien and Vervaat (1985).

3. Variation of the sample paths. Let X be a strictly stable process with exponent $\alpha = 1/H \in [\frac{1}{2}, \infty)$, i.e., X is H-ss with stationary *independent* increments. The following is known in this case. If $\frac{1}{2} \leq H \leq 1$ $(1 \leq \alpha \leq 2)$, then wp1 the sample paths have nowhere bounded variation (nbv). If $H > 1$ $(0 < \alpha < 1)$, then wp1 the sample paths have locally bounded variation (lbv) and vary only by jumps. These results are reobtained (for $\alpha \neq 2$) in the context of H-ss si jump processes in O'Brien and Vervaat (1985). The following theorems generalize these and other well-known properties of strictly stable processes to ss si processes. They will be proved at the end of this section. Each theorem is followed by a discussion of its consequences and occasionally an indication of examples.

AUXILIARY THEOREM 3.1. *Let Hypothesis* 1.4 *hold for* X *and suppose that*

$\mathbb{E}X(1)$ exists (i.e., at least one of $\mathbb{E}X^+(1)$, $\mathbb{E}X^-(1)$ is finite).

(a) *If $H < 1$, then $\mathbb{E}X(1) = 0$.*
(b) *If $H = 1$, then $X(t) \equiv tX(1)$ wp1.*
(c) *If $H > 1$, then either wp1 $X(1) \geq 0$, $\mathbb{E}X(1) = \infty$ and X is strictly increasing, or the same holds wp1 for $-X$.*

COROLLARY 3.2. *If $H > 1$, then the support of the distribution of $X(1)$ extends to ∞ or is contained in $(-\infty, 0]$. If $H = 1$, then the support of the distribution of $X(1)$ is unbounded above unless $X(t) \equiv tX(1)$ wp1. A more detailed result for $H > 1$ is Theorem 5 of O'Brien and Vervaat (1983). For $H = 1$ the last section of the same paper is supplemented by our observation. The author does not know whether the support of $X(1)$ can be bounded above in case $H < 1$.*

THEOREM 3.3. *Let Hypothesis 1.4 hold for X and set $A := [X$ has lbv$]$. If $\mathbb{P}(A) > 0$, then either $H > 1$ and $A \subset [dX(t)/dt = 0$ for almost all $t]$ modulo null events, or $H = 1$ and $A = [X(t) \equiv tX(1)]$ modulo null events.*

3.4. *Remarks and examples.* Note that X has nbv wp1 in case $H \leq 1$, unless $H = 1$ and $X(t) \equiv tX(1)$ with positive probability. We will see that even in case $H > 1$ the paths of X may have nbv. We briefly discuss the examples.

CASE (A). $H < 1$. The most obvious examples are Brownian motion and other strictly stable processes with $\alpha = 1/H > 1$. Other examples are abundant in the literature up to 1979. Sample paths can be continuous (cf. Yeh, 1973, for fractional Brownian motion), vary only by jumps (symmetric stable processes with $\alpha = 1/H \in (1, 2)$ or more generally conditionally convergent jump processes in O'Brien and Vervaat 1985, with $0 < H < 1$) or be nowhere bounded (cf. Maejima, 1982, 1983, and Taqqu and Wolpert, 1983, for fractional stable processes and our Section 5.4 for other fractional processes).

CASE (B). $H > 1$. Strictly stable processes with $\alpha = 1/H \in (0, 1)$ have lbv and vary only by jumps. The class of all processes with this type of variation is studied in O'Brien and Vervaat (1984) (cf. also Section 4). Up to 1979 no other examples than stable processes were known. The first new example was found by Kesten and Spitzer (1979), with lbv and continuous singular sample paths. After that, other examples were found by Surgailis (1981), Taqqu and Wolpert (1983) and O'Brien and Vervaat (1985). Based on the last paper, the present paper provides a new class of examples of X with lbv and continuous sample paths (by subordination to jump processes, cf. 5.1). A similar class could have been produced within the subordination set-up of Surgailis (1981). The present paper provides the first known examples of X with nbv (cf. 5.3, 6.5 and 6.6).

CASE (C). $H = 1$. The lbv case is trivial, so we discuss only the nbv case. The only strictly stable processes with $\alpha = 1/H = 1$ are the Cauchy processes, possibly with drift. Examples with continuous sample paths are given by Kesten and Spitzer (1979). The examples in O'Brien and Vervaat (1985) vary only by jumps. No attempt is made in the present paper to construct examples with continuous paths by subordination to jump processes, but probably it can be done.

THEOREM 3.5. *Let Hypothesis* 1.4 *hold for* X *and set for* $t \in T$. $\beta \in \mathbb{R}$

$$D_\beta(t) := \limsup_{h\downarrow 0} h^{-\beta}(X(t+h) - X(t)),$$

$$M_\beta := \sup_{0<t\le 1} t^{-\beta} X(t),$$

$$\phi_\beta(x) := \begin{cases} \infty \cdot x & \text{if } \beta > 0 \\ x & \text{if } \beta = 0 \\ x1_{[|x|=\infty]} & \text{if } \beta < 0 \end{cases} \quad \text{for} \quad x \in \bar{\mathbb{R}}.$$

Then the following holds.

(a) $D_\beta(t) = D_\beta(0)$ *wp*1 *for each t separately, and the random set of t for which* $D_\beta(t) \neq D_\beta(0)$ *has wp*1 *Lebesgue measure zero.*
(b) $D_\beta(0) = \phi_{\beta-H}(M_\beta)$ *wp*1. *In particular,* $D_\beta(0) \in \{-\infty, 0, \infty\}$ *wp*1 *if* $\beta \neq H$.
(c) $[M_\beta \le 0] = [X$ *is strictly decreasing*$]$ *modulo null events.*
(d) *Let* $\mathscr{I}$ *be the* (2.7, H) *invariant* σ*-field. Then*

$$D_H(0) = \sup\{x: \mathbb{P}^{\mathscr{I}}[X(1) > x] > 0\} \quad wp1.$$

In particular, if X *is* (2.7, H) *ergodic, then wp*1 $D_H(0)$ *equals the right end of the support of the distribution of* $X(1)$.

3.6. *Corollaries and remarks*

CASE $\beta = H$. *If* $H \ge 1$ *and* $\mathbb{P}[X(t) \equiv tX(1)] = 0$, *then* $D_H(0) = \infty$ *wp*1 *by* (d) *and Corollary* 3.2, *unless* X *is strictly decreasing wp*1. *In the latter case* $D_H(0)$ *can take any value in* $(-\infty, 0]$, *by Theorem* 5 *of O'Brien and Vervaat* (1983). *If* $H < 1$, *then* $D_H(0) > 0$ *wp*1 *since* X *cannot be monotone by Theorems* 3.1 *or* 3.3. *The author does not know whether* $D_H(0) \in (0, \infty)$ *can occur with positive probability.*

CASE $\beta > H$. *If* $H \le 1$ *and* $\mathbb{P}[X(t) \equiv tX(1)] = 0$, *then* X *cannot be monotone, so* $D_\beta(0) = \infty$ *wp*1 *by* (b) *and* (c). *If* $H > 1$, *then* $D_\beta(0) = \infty$ *unless* X *is strictly decreasing in which case* $D_\beta(0) = 0$ *or* $-\infty$. *If* $D_H(0) < 0$, *then certainly* $D_\beta(0) = -\infty$. *It is unclear what can happen if* $D_H(0) = 0$. *The author does not know examples with* $D_\beta(0) = 0$.

CASE $\beta < H$. *If* X *is strictly decreasing, then* $D_\beta(0) = 0$ *wp*1, *since* $D_H(0) \in (-\infty, 0]$ *wp*1. *If* X *has lbv and* $H > 1$, *then* $D_1(0) = 0$ *so* $D_\beta(0) = 0$ *for* $\beta \le 1$, *by Theorem* 3.3. *If* X *is nowhere bounded (examples are only known for* $H < 1$, *cf. Section* 5.4), *then* $D_\beta(0) = \infty$. *If* X *is locally bounded but not decreasing, then* $D_\beta(0) = 0$ *or* ∞, *but the author does not know whether* $D_\beta(0) = \infty$ *actually can occur with positive probability.*

Turning to the proofs, we start with an extension of the Birkhoff Ergodic Theorem, which may be well known, but for which the author could not find a proof in the literature.

THEOREM 3.7. *Let $(\xi_n)_n$ be a stationary sequence of rv's, $\mathscr{I}$ its invariant σ-field, and let ϕ be a measurable function on $\mathbb{R}$ such that $\mathbb{E}\phi(\xi_1)$ exists (possibly infinite-valued). Then*

$$(1/n) \textstyle\sum_{k=1}^{n} \phi(\xi_k) \to \mathbb{E}^{\mathscr{I}} \phi(\xi_1) \quad wp1. \tag{3.1}$$

PROOF. The theorem is well-known for the case $\mathbb{E} \mid \phi(\xi_1) \mid < \infty$, and we will reduce the complementary case to this. So suppose $\mathbb{E}\phi^{-}(\xi_1) < \infty$. Then $\mathbb{E} \mid \phi(\xi_1) \wedge c \mid < \infty$ for $c > 0$, and applying Birkhoff's Ergodic Theorem for all these c we find

$$\liminf_{n\to\infty}(1/n) \textstyle\sum_{k=1}^{n} \phi(\xi_k) \geq \sup_c \lim_{n\to\infty}(1/n) \sum_{k=1}^{n} (\phi(\xi_k) \wedge c)$$
$$= \lim_{c\to\infty}\mathbb{E}^{\mathscr{I}}(\phi(\xi_1) \wedge c) = \mathbb{E}^{\mathscr{I}}\phi(\xi_1) \quad \text{wp1}.$$

This already proves (3.1) on $[\mathbb{E}^{\mathscr{I}}\phi(\xi_1) = \infty]$. Set for $c > 0$

$$A(c) := [\mathbb{E}^{\mathscr{I}}\phi(\xi_1) \leq c] = [\mathbb{E}^{\mathscr{I}}\phi(\xi_n) \leq c],$$
$$\eta_{n,c} := \phi(\xi_n)1_{A(c)}.$$

Then $(\eta_{n,c})_{n=1}^{\infty}$ is stationary, $\mathbb{E}\eta_{1,c}^{-} \leq \mathbb{E}\phi^{-}(\xi_1) < \infty$ and

$$\mathbb{E}\eta_{1,c} = \mathbb{E}\mathbb{E}^{\mathscr{I}}\phi(\xi_1)1_{A(c)} = \mathbb{E}1_{A(c)}\mathbb{E}^{\mathscr{I}}\phi(\xi_1) \leq c,$$

so $\mathbb{E} \mid \eta_{1,c} \mid < \infty$. Hence

$$1_{A(c)} \cdot (1/n) \textstyle\sum_{k=1}^{n} \phi(\xi_k) = (1/n) \sum_{k=1}^{n} \eta_{k,c} \to \mathbb{E}^{\mathscr{I}}\eta_{1,c}$$
$$= \mathbb{E}^{\mathscr{I}}\phi(\xi_1)1_{A(c)} = 1_{A(c)}\mathbb{E}^{\mathscr{I}}\phi(\xi_1) \quad \text{wp1}.$$

Considering the outmost sides for all $c > 0$ we see that (3.1) also holds on $\bigcup_c A(c) = [\mathbb{E}^{\mathscr{I}}\phi(\xi_1) < \infty]$. □

PROOF OF THEOREM 3.1. Let $\mathscr{I}$ be the invariant σ-field of the stationary sequence $(X(n) - X(n-1))_{n=1}^{\infty}$ completed with the events of probability zero. By Theorem 3.7 (which also holds with this completed $\mathscr{I}$) we have

$$(1/n)X(n) = (1/n) \textstyle\sum_{k=1}^{n} (X(k) - X(k-1)) \to \mathbb{E}^{\mathscr{I}}X(1) \quad \text{wp1}$$

(recall that $X(0) = 0$ wp1 by Lemma 1.2). By self-similarity

$$\frac{1}{n} X(n) =_d n^{H-1}X(1) \to \begin{cases} 0 & \text{if } H < 1, \\ X(1) & \text{if } H = 1, \\ \infty \cdot X(1) & \text{if } H > 1. \end{cases}$$

It follows that $\mathbb{E}^{\mathscr{I}}X(1) =_d \lim_{n\to\infty} h^{H-1}X(1)$.

(a) If $H < 1$, then $\mathbb{E}^{\mathscr{I}}X(1) = 0$ wp1, so $\mathbb{E}X(1) = \mathbb{E}\mathbb{E}^{\mathscr{I}}X(1) = 0$.
(b) If $H = 1$, then $\mathbb{E}^{\mathscr{I}}X(1) =_d X(1)$, so by Smit (1983) $X(1) = \mathbb{E}^{\mathscr{I}}X(1)$ wp1

(here the completion of $\mathscr{I}$ is needed). Hence $X(1)$ is $\mathscr{I}$ measurable, i.e., $X(1) = X(n) - X(n-1)$ wp1 for $n = 1, 2, \cdots$, so $X(n) = nX(1)$ wp1 for these n. By self-similarity $X(t) = tX(1)$ wp1 for all positive rational t, hence for all positive real t simultaneously wp1 by separability. Since X has si, this identity extends to negative t.

(c) If $H > 1$, then $\mathbb{E}^{\mathscr{I}}X(1) =_d \infty \cdot X(1)$. Since $\mathbb{E}\mathbb{E}^{\mathscr{I}}X(1) = \mathbb{E}X(1)$ exists, $\mathbb{E}^{\mathscr{I}}X(1)$ cannot assume either value $\pm\infty$ with positive probability, so $X(1) \le 0$ wp1 or $X(1) \ge 0$ wp1. In either case X is strictly monotone by Theorem 2.4(b), Corollary 2.3(a) and Hypothesis 1.4. Moreover, $\mathbb{E}X(1) = 0$ corresponds to $X \equiv 0$ wp1, again excluded by Hypothesis 1.4. □

PROOF OF THEOREM 3.3. The event A is invariant under transformations (2.6) and (2.7), so the conditional process X given A is H-ss si as well. We therefore may assume that $\mathbb{P}(A) = 1$ for the time being, by considering the conditioned process.

If X has lbv, then X can be written as a difference of two nondecreasing H-ss si processes X_1 and X_2: $X = X_1 - X_2$, corresponding to the Hahn decomposition of the related Radon measure N (cf. Section 2). Since $X_j(1) \ge 0$ for $j = 1, 2$, $\mathbb{E}X_j(1)$ exists and Theorem 3.1 applies to either X_j. If $H < 1$, then by Theorem 3.1(a) $\mathbb{E}X_j(1) = 0$, so $X_j(1) = 0$ wp1, so $X(1) = 0$ wp1. By Theorem 2.4(a) $X \equiv 0$ wp1, which is excluded in Hypothesis 1.4. It follows that $H \ge 1$. If $H = 1$, then $X_j(t) \equiv tX_j(1)$ wp1 by Theorem 3.1(b), so $X(t) \equiv tX(1)$ wp1.

Suppose $H > 1$ and X has lbv. Then by Lebesgue's differentiation theorem $dX(t)/dt$ exists (with finite values) and equals $D_1(t)$ (see Theorem 3.5) almost everywhere. By Theorem 3.5(b) (whose proof is independent of the present theorem) it follows that $dX(t)/dt = 0$ almost everywhere.

Dropping the assumption $\mathbb{P}(A) = 1$, i.e., returning to the unconditioned process X, the proof is already complete for $H > 1$, and is completed for $H = 1$ by observing that the process $tX(1)$ has lbv. □

PROOF OF THEOREM 3.5. (a) Since X is measurable and separable with each countable dense subset of T as separant, the process D_β is well-defined and measurable, and its finite-dimensional distributions depend uniquely on the finite-dimensional distribution of X. From (0.1) and (0.3) it then follows that D_β is $(H - \beta)$-ss (except that $D_\beta(t)$ may assume infinite values with positive probability), and stationary, i.e.,

$$D_\beta(a\cdot + b) =_d a^{H-\beta}D_\beta \quad \text{for} \quad a, b \in T, \quad a > 0. \tag{3.2}$$

The process D_β need not be separable, in fact it is far from separable in many cases. From (3.2) at 0 with $b = 0$, $a \downarrow 0$ or $a \to \infty$ we see $D_\beta(0) \in \{-\infty, 0, \infty\}$ wp1 in case $H \ne \beta$. From (3.2) with $a = 1$ at 0 we see that $D_\beta(t) =_d D_\beta(0)$. Hence $a^{H-\beta}D(t) = D(t)$ wp1. So the case $H \ne \beta$ in (3.2) reduces to the case $H = \beta$ with the additional condition that $D_\beta(t) \in \{-\infty, 0, \infty\}$ wp1. From (3.2) we see that $(D_\beta(b), D_\beta(b + a)) =_d (D_\beta(0), D_\beta(1))$ in $\overline{\mathbb{R}}^2$. Letting $a \downarrow 0$, we see by Theorem 1.1 that D_β can be measurable only if $D_\beta(0) = D_\beta(1)$ wp1. Hence $D_\beta(t) = D_\beta(0)$ wp1 for each t separately. By arguments similar to those on page 113 of Lamperti (1966) it follows that wp1 $D_\beta(t) = D_\beta(0)$ for almost every t.

(b) Interpreting $D_\beta(0)$ more explicitly, we have

$$D_\beta(0) = \inf_{h>0}\sup_{0<t\le h} t^{-\beta}X(t) = \lim_{h\downarrow 0}\sup_{0<t\le h} t^{-\beta}X(t), \tag{3.3}$$

in particular,

$$D_\beta(0) \le M_\beta. \tag{3.4}$$

The limit on the right-hand side of (3.3) refers to convergence wp1, hence also to convergence in distribution. By self-similarity

$$\sup_{0<t\le h} t^{-\beta}X(t) =_d \sup_{0<t\le h} t^{-\beta}h^H X(t/h) = \sup_{0<u\le 1} u^{-\beta}h^{H-\beta}X(u) = h^{H-\beta}M_\beta$$

so

$$D_\beta(0) =_d \lim_{h\downarrow 0} h^{H-\beta}M_\beta = \phi_{H-\beta}(M_\beta). \tag{3.5}$$

Applying $\phi_{H-\beta}$ to (3.4) and recalling that $D_\beta(0) \in (-\infty, 0, \infty)$ wp1 for $\beta \ne H$ (cf. proof of (a)), we find

$$\phi_{H-\beta}(D_\beta(0)) = D_\beta(0) \le \phi_{H-\beta}(M_\beta) \quad \text{wp1}.$$

This combined with (3.5) gives $D_\beta(0) = \phi_{H-\beta}(M_\beta)$ wp1, as in general $\xi =_d \eta$ and $\xi \le \eta$ wp1 imply $\xi = \eta$ wp1.

(c) Set

$$M_\beta(t) := \sup_{0<h\le 1} h^{-\beta}(X(t+h) - X(t)),$$

and note that modulo null events

$$[X \text{ nonincreasing}] = \cap_{t\in S}\, [M_\beta(t) \le 0],$$

if S is a separant for X. By the obvious analogue of (a) for $M_\beta(t)$ we have $M_\beta(t) = M_\beta(0) = M_\beta$ wp1 for all $t \in S$ simultaneously. Now (c) follows by Corollary 2.3(a) and Hypothesis 1.4.

(d) From (a) and the definition of $D_H(0)$ it follows that D_H is almost invariant under the transformations (2.7, H) applied to X, so D_H is $\mathscr{I}$ measurable. Furthermore,

$$\mathbb{P}^{\mathscr{I}}[X(1) > x] = \mathbb{P}^{\mathscr{I}}[t^{-H}X(t) > x] \quad \text{wp1}$$

for each t separately. Consequently, we have for countable dense sets S in $[0, 1]$

$$\begin{aligned}
[\mathbb{P}^{\mathscr{I}}[X(1) > x] = 0] &= \cap_{t\in S}\, [\mathbb{P}^{\mathscr{I}}[t^{-H}X(t) > x] = 0] \\
&= [\mathbb{P}^{\mathscr{I}}[\sup_{t\in S} t^{-H}X(t) > x] = 0] = [\mathbb{P}^{\mathscr{I}}[M_H > x] = 0] \\
&= [\mathbb{P}^{\mathscr{I}}[D_H(0) > x] = 0] = [D_H(0) \le x]
\end{aligned}$$

modulo null events. In the last identity we used that $D_H(0)$ is $\mathscr{I}$ measurable, the second last is based on (b). Part (d) now follows by the identity between the outmost sides.

4. Subordination to point processes.

Stationary ss discrete random measures. In O'Brien and Vervaat (1985) H-ss si jump processes X of lbv are studied and characterized. Here we quote the results we need, mostly rephrased for the corresponding random ss stationary discrete Radon measure N, associated with X by (2.3). Throughout this section we assume that $T = \mathbb{R}$.

If X is a jump process of lbv, i.e., if its associated random measure N is Radon and discrete, then its saltus process is the point process

$$\Pi := \{(t, N\{t\}): t \in \mathbb{R}, N\{t\} \neq 0\},$$

a random subset of

$$E := \mathbb{R} \times (\mathbb{R}\backslash\{0\}).$$

If $X \not\equiv 0$ wp1, assumed in Standard Hypotheses 1.4, then $\Pi \neq \varnothing$ wp1, and $\{t: (t, x) \in \Pi\}$ is dense in $\mathbb{R}$ wp1 by Corollary 2.3(a). However, Π is locally finite in $\mathbb{R} \times \overline{\mathbb{R}}\backslash\{0\}$. By abuse of notation Π is identified with the point process (= random integer-valued measure) counting its points. So two ways to write $N(A)$ for Borel sets $A \subset \mathbb{R}$ are

$$N(A) = \textstyle\sum_{(t,x)\in\Pi, t\in A} x = \displaystyle\int_{\mathbb{R}\backslash\{0\}} x\Pi(A, dx).$$

In its second interpretation, Π is a random Radon measure on the Borel field of $\mathbb{R} \times (\overline{\mathbb{R}}\backslash\{0\})$. Define

$$x^{\uparrow\alpha} := |x|^{\alpha}\operatorname{sgn} x \quad \text{for real} \quad \alpha \quad \text{and} \quad x \neq 0,$$

and set

$$\Pi^{\alpha} := \{(t, x^{\uparrow\alpha}): (t, x) \in \Pi\}$$

for $\alpha \in \mathbb{R}$. Then each random stationary H-ss discrete Radon measure N_H can be represented as

$$N_H = \int_{\mathbb{R}\backslash\{0\}} x\Pi^H(\cdot, dx) = \int_{\mathbb{R}\backslash\{0\}} x^{\uparrow H}\Pi(\cdot, dx), \tag{4.1}$$

where Π is a point process in E, locally finite in $\mathbb{R} \times (\overline{\mathbb{R}}\backslash\{0\})$, whose distribution is invariant under the transformations

$$(t, x) \mapsto (at + b, ax) \quad (a, b \in \mathbb{R}, a > 0) \tag{4.2}$$

of E. We express this invariance by saying that Π is (4.2) stationary or *Poincaré*.

Let $\mathbb{E}\Pi$ be the intensity of Π, defined by $(\mathbb{E}\Pi)(B) := \mathbb{E}(\Pi(B))$ for Borel sets $B \subset E$. If Π is Poincaré, then $\mathbb{E}\Pi$ is as well, so

$$\mathbb{E}\Pi(dt, dx) = c(\operatorname{sgn} x)\, dt\, dx/x^2 \quad \text{for} \quad (t, x) \in E, \tag{4.3}$$

where $c(\pm 1) =: c_{\pm} \in [0, \infty]$ and $c_+ + c_- > 0$, since $\Pi \neq \varnothing$ wp1. We call $\mathbb{E}\Pi$ finite if $c_+ + c_- < \infty$, otherwise infinite.

Let Π be Poincaré and let H be a fixed real. Consider $N_H(I)$ in (4.1) for bounded intervals I. By Theorem 2.2 it follows that wp1 the right-hand sides of (4.1) either converge absolutely for all such I, or for none of them. Let $\mathscr{H}_a$, depending on the sample point ω, be the random set of H (necessarily an interval extending to ∞ or empty), for which there is absolute convergence in (4.1). Then by Theorem 2.1 of O'Brien and Vervaat (1985) we have $\mathscr{H}_a \subset (1, \infty)$ wp1, and $\mathscr{H}_a = (1, \infty)$ wp1 if $\mathbb{E}\Pi$ is finite. In the present paper (Section 5.2) we will construct examples with infinite $\mathbb{E}\Pi$ and $\mathscr{H}_a = (H_a, \infty)$, where $1 < H_a < \infty$.

Conditional convergence in (4.1) is studied extensively in Section 4 of O'Brien and Vervaat (1985). In the present paper it will occur only incidentally. The order of summation for conditional convergence in (4.1) is specified by

$$N_H := \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} x\Pi^H(\cdot, dx). \tag{4.4}$$

We define $\mathscr{H}_c$ to be the random set of all H such that (4.4) converges at all intervals $(a, b]$ for almost all $(a, b) \in \mathbb{R}^2$, $a < b$. It is not hard to see that $H \in \mathscr{H}_c$ wp1 iff (4.4) converges wp1 at one fixed interval. However, the author does not know whether this implies that (4.4) converges wp1 at all intervals simultaneously.

Both $\mathscr{H}_a$ and $\mathscr{H}_c$ are (4.2) invariant, so essentially nonrandom, if N_H is (2.6), (2.7, H) ergodic (cf. Section 2.6), which is the same as Π being (4.2) ergodic.

If Π is Poisson with intensity (4.3) then X_H associated with N_H is strictly stable with exponent $\alpha = 1/H$. Several other examples of Poincaré Π are presented in Sections 3, 5 and 6 of O'Brien and Vervaat (1985).

Subordination. Let Π be a Poincaré point process in E, and let μ be a formal measure on the ring $\mathscr{R}$ of finite unions of finite intervals $(a, b]$. Let N_H for $H > 0$ be the random measure

$$N_H := \textstyle\sum_{(t,x)\in\Pi} x^{\uparrow H}\mu\left(\frac{\cdot - t}{|x|}\right) = \int\int_E x^{\uparrow H}\mu\left(\frac{\cdot - t}{|x|}\right)\Pi(dt, dx), \tag{4.5}$$

supposed to be convergent in some sense. Then N_H is obviously stationary, and moreover H-ss:

$$\begin{aligned} N_H(a\cdot) &= \textstyle\sum_{(t,x)\in\Pi} x^{\uparrow H}\mu((a\cdot - t)/|x|) \\ &= \textstyle\sum_{(t,x)\in\Pi} x^{\uparrow H}\mu((\cdot - ta^{-1})/|x|a^{-1}) \\ &= \textstyle\sum_{(at',ax')\in\Pi} (ax')^{\uparrow H}\mu((\cdot - t')/|x'|) \\ &=_d a^H \textstyle\sum_{(t,x)\in\Pi} x^{\uparrow H}\mu((\cdot - t)/|x|) = a^H N_H \quad \text{for} \quad a > 0. \end{aligned} \tag{4.6}$$

Consequently,

$$X_H(t) := N_H(I_t)\operatorname{sgn} t \tag{4.7}$$

defines an H-ss si process. We call N_H the random measure subordinated to Π

by μ, and Π the subordinator in (4.5). By setting $\mu = \iota_0$ (probability measure concentrated at 0) in (4.5) we reobtain (4.1). The random measure expression (4.5) is simpler and intuitively more appealing than the corresponding expression for X_H. With

$$F(t) := \mu(I_t)\operatorname{sgn} t \tag{4.8}$$

it becomes

$$\begin{aligned} X_H(t) &= \textstyle\sum_{(u,x)\in\Pi} x^{\uparrow H}\mu((I_t - \mu)/|x|)\operatorname{sgn} t \\ &= \textstyle\sum_{(u,x)\in\Pi} x^{\uparrow H}(F((t-u)/|x|) - F(-u/|x|)). \end{aligned} \tag{4.9}$$

Formulae (4.5) and (4.9) allow a generalization in which also the measure μ is random. To this end, let $((t_n, x_n))_{n=1}^{\infty}$ be an enumeration of the points of Π which is measurable, i.e., such that (t_n, x_n) is an $\mathbb{R}^2$-valued rv for each n (cf. Kallenberg, 1976, Lemma 2.3). Let $(\mu_n)_{n=1}^{\infty}$ be a collection of iid random formal measures on $\mathscr{R}$, independent of Π. If π is a random permutation of $\mathbb{N}$ and π is independent of $(\mu_n)_{n=1}^{\infty}$, in particular if π depends functionally on Π, then $(\mu_{\pi n})_{n=1}^{\infty} =_d (\mu_n)_{n=1}^{\infty}$ and is independent of Π. The generalization of (4.5) and (4.9) now reads

$$N_H = \textstyle\sum_{n=1}^{\infty} x_n^{\uparrow H}\mu_n\left(\dfrac{\cdot - t_n}{|x_n|}\right) = \displaystyle\int\int_E x^{\uparrow H}\mu_{n(t,x)}\left(\frac{\cdot - t}{|x|}\right)\Pi(dt, dx). \tag{4.10}$$

Now an analogue of (4.6) and the stationarity of N_H remain true.

A special case of (4.9) occurs in Surgailis (1981), where Π is replaced by $\Pi - \mathbb{E}\Pi$ and Π is Poisson. In this case (4.9) can be written down only with integrals whose integrators have uncountable support, and a random μ version like (4.10) then seems hard to define.

Expressions analogous to (4.5) and (4.9) in the context of extremal processes were discovered independently and exploited thoroughly in the research leading to O'Brien, Torfs and Vervaat (1984+). In fact this was the main hint for the present author to investigate (4.5) in its present context. It turns out that fractional stable processes (cf. Section 5.4) as studied by Taqqu and Wolpert (1983) and Maejima (1983) are a special case of (4.5) with Π replaced by $\Pi - \mathbb{E}\Pi$, Π Poisson.

Several examples will be presented and discussed in Section 5. The remainder of the present section is devoted to the following question. If the subordinator Π is concentrated on $E_+ := \mathbb{R} \times (0, \infty)$ and μ is nonnegative, when will N_H in (4.5) be Radon, i.e., finite on bounded intervals? We will find sufficient conditions. The following are our assumptions.

HYPOTHESES 4.1. (a) Π *is Poincaré and concentrated on* E_+ *with finite intensity* $\mathbb{E}\Pi(dt, dx) = c_+ dt\, dx/x^2$ *and measurable enumeration* $((t_n, x_n))_{n=1}^{\infty}$ *of its points.*

(b) $(\mu_n)_{n=1}^{\infty}$ *is a sequence of iid nonnegative Radon measures, independent of* Π.

Recall that N_H in (4.10) is stationary and H-ss, whenever convergent, under

Hypotheses 4.1. For the next lemmas we decompose N_H in (4.10) into $N_H = N_H^{(1)} + N_H^{(2)}$, where

$$N_H^{(1)} := \textstyle\sum_{n:x_n\le 1} x_n^H \mu_n((\cdot - t_n)/x_n) \tag{4.11}$$

is the sum of the contributions by the small jumps $x_n \le 1$, and $N_H^{(2)} := N_H - N_H^{(1)}$ the sum of the contributions by the large jumps $x_n > 1$.

LEMMA 4.2. *If Hypotheses* 4.1 *are satisfied and* $\mathbb{E}\mu_n(\mathbb{R}) < \infty$, *then* $N_H^{(1)}(I) < \infty$ *wp*1 *for each bounded interval* I *and each* $H > 1$.

PROOF. Obviously, $N_H^{(1)}$ is stationary and increasing in I, so we may assume $I = I_a = (0, a]$ for some $a > 0$. Set $m := \mathbb{E}\mu_n(\mathbb{R})$ ($m > 0$, otherwise $N_H \equiv 0$ wp1) and $\mu := m^{-1}\mathbb{E}\mu_n$. Then μ is a (nonrandom) probability measure on the Borel field of $\mathbb{R}$. Since (μ_n) and Π are independent, we have

$$\begin{aligned}\mathbb{E}N_H^{(1)}(I_a) &= \mathbb{E}\int_{\mathbb{R}}\int_{(0,1]} x^H \mu_{n(t,x)}\left(\frac{I_a - t}{x}\right)\pi(dt, dx)\\ &= \int_{\mathbb{R}}\int_{(0,1]} x^H m\mu\left(\frac{I_a - t}{x}\right) c_+\, dt\, dx/x^2\\ &= mc_+\int_0^1 x^{H-1}\left(\int_{\mathbb{R}} \mu(I_{a/x} + u)\, du\right) dx.\end{aligned}$$

Since μ is a probability measure, we have

$$\int_{\mathbb{R}} \mu(I_{a/x} + u)\, du = \frac{a}{x}.$$

Hence

$$\mathbb{E}N_H^{(1)}(I_a) = mc_+a\int_0^1 x^{H-2}\, dx = \frac{mc_+a}{H-1} < \infty$$

if $H > 1$, so $N_H^{(1)}(I_a) < \infty$ wp1.

LEMMA 4.3. *Let Hypotheses* 4.1 *be satisfied. Let* $\mu_n(\mathbb{R}) < \infty$ *wp*1 *and let* S *be wp*1 *a common support of* μ_n: $\mu_n(\mathbb{R}\backslash S) = 0$ *wp*1. *Set*

$$S^y := \textstyle\bigcup_{|b|<y} (S + b) \quad \textit{for} \quad y > 0 \tag{4.12}$$

and suppose that

$$\int_0^1 \lambda(S^y)\, dy/y < \infty, \tag{4.13}$$

where λ *denotes Lebesgue measure. Then* $N_H^{(2)}(I) < \infty$ *wp*1 *for each finite interval* I *and each* $H > 0$.

PROOF. Obviously, $N_H^{(2)}$ is stationary and increasing in I, so we may assume $I = (-a, a)$ for some $a > 0$. For this I we have

$$N_H^{(2)}(I) = \sum_{n:x_n>1} x_x^H \mu_n((I - t_n)/x_n)$$

$$\leq \sum \{x^H \mu_{n(t,x)}(\mathbb{R}): (t, x) \in \Pi,\ x > 1,\ -t/x \in S^{a/x}\} \quad \text{wp1}.$$

The latter series is finite, if it has finitely many terms, i.e., if

$$\Pi\{(t, x) \in E_+: x > 1, -t/x \in S^{a/x}\} < \infty. \tag{4.14}$$

The left-hand side has expectation

$$c_+ \int_1^\infty \lambda(xS^{a/x})\, dx/x^2 = c_+ \int_0^a \lambda(S^y)\, dy/y,$$

which is finite iff (4.13) holds. So (4.14) holds wp1, and the lemma follows.

REMARK 4.4. Since $S^y = (\operatorname{clos} S)^y$, we may assume S to be closed in Lemma 4.3. For closed S, Condition (4.13) is equivalent to

$$\begin{cases} S \quad \text{is bounded}, \\ \lambda(S) = 0, \\ \sum_{n=1}^\infty \ell_n \,|\log \ell_n| < \infty, \end{cases} \tag{4.15}$$

where $(\ell_n)_{n=1}^\infty$ is an enumeration of the lengths of the disjoint open intervals whose union is $[\inf S, \sup S]\backslash S$. This follows from the lemma on page 326 of Carleson (1952) (reference due to John Hawkes).

THEOREM 4.5. *Let Hypotheses* 4.1 *be satisfied. Let* $\mathbb{E}\mu_n(\mathbb{R}) < \infty$, *and let* S *be wp1 a common support of* μ_n: $\mu_n(\mathbb{R}\backslash S) = 0$ *wp1. Suppose, moreover, that* S *is closed and satisfies* (4.15). *Then* N_H *in* (4.10) *has domain of absolute convergence* $\mathscr{A}_a = (1, \infty)$ *wp1.*

PROOF. From Lemmas 4.2 and 4.3, Remark 4.4 and the independence of $\mathscr{A}_a$ of I in $N_H(I)$ as noted after formula (4.3) it follows that $(1, \infty) \subset \mathscr{A}_a$ wp1. Observe that N_H is nonnegative, so X_H is nondecreasing. By Theorem 3.3 it follows that $(1, \infty) \supset \mathscr{A}_a$ wp1, unless N_1 converges and $N_1 = N_1(I_1)\lambda$. If μ_n had a singular part with positive probability, then N_1 would have some, so μ_n is absolutely continuous wp1 if N_1 converges. Consequently, N_H is absolutely continuous wp1 for $H > 1$, which contradicts Theorem 3.3 and excludes the possibility $N_1 = N_1(I_1)\lambda$.

5. Examples. In the present section we discuss examples of random H-ss stationary measures N_H subordinated to Poincaré point processes Π as in (4.5) or (4.10). In particular we will show that several combinations of H, lbv or nbv, continuity or discreteness, not excluded in Theorem 3.3, indeed do occur. Recall that $E = \mathbb{R} \times (\mathbb{R}\backslash\{0\})$ and $E_+ = \mathbb{R} \times (0, \infty)$.

5.1. *A process of lbv with continuous singular sample paths.* Let Π be a Poincaré point process with finite $\mathbb{E}\Pi$ in E_+. Let C be the Cantor set:

$$C := \{x \in [0, 1]: x \text{ has a triadic expansion without digits } 1\}.$$

Then C satisfies (4.15) referred to in Theorem 4.5, so N_H in (4.10) or (4.5) converges wp1 for $H > 1$ if $\mu_n = \mu$ is a fixed nonrandom probability measure concentrated on C. Let μ be the uniform distribution on C, i.e., the probability distribution of $\sum_{n=1}^{\infty} \xi_n 3^{-n}$ with $(\xi_n)_{n=1}^{\infty}$ iid such that $\xi_n = 0$ or 2 with equal probability. Then μ is diffuse (nonatomic) and nonnegative, and so is N_H for $H > 1$ (Π is concentrated on E_+). Hence the corresponding H-ss si X_H is continuous and nondecreasing, even strictly increasing by Corollary 2.3(a). By Theorem 3.3 X_H must be singular. For other examples with the same properties, see 6.5 and 6.6.

5.2. *Absolute convergence boundary* > 1. Recall the definition of domain of absolute convergence $\mathscr{A}_a$ for Poincaré Π in the first part of Section 4. If $\mathbb{E}\Pi$ is finite, then $\mathscr{A}_a = (1, \infty)$ wp1, whereas $\mathscr{A}_a \subset (1, \infty)$ wp1 if $\mathbb{E}\Pi$ is infinite (O'Brien and Vervaat, 1985, Theorem 2.1). Here we present examples with strict inclusion in the latter case.

We construct a process subordinated to the Poisson process Π in E_+ with intensity $\mathbb{E}\Pi(dt, dx) = dt\, dx/x^2$ and measurable enumeration $((t_n, x_n))_{n=1}^{\infty}$. The subordinated measures (μ_n) are nonnegative and finite wp1, and concentrated on a countable subset of the Cantor set C. By Lemma 4.3 and Remark 4.4 the sum $N_H^{(2)}$ of the contributions for $x > 1$ to N_H in (4.10) is finite wp1 for $H > 1$ in this case. By Lemma 4.2 the sum $N_H^{(1)}$ of contributions for $x \le 1$ to N_H would be finite, if $\mathbb{E}\mu_n(\mathbb{R}) < \infty$, but this condition will be violated in our example. Let $(\nu_n)_{n=1}^{\infty}$ be a sequence of iid nonnegative rv's, independent of Π, with Laplace–Stieltjes transform $\mathbb{E}e^{-\tau\nu_n} = \exp(-\tau^{\alpha})$ for $\tau \ge 0$, so ν_n is one-sided stable with exponent α. Let $(c_n)_{n=1}^{\infty}$ be an enumeration of the endpoints of the disjoint open intervals whose union is $[0, 1]\backslash C$, and set

$$\mu_n := \textstyle\sum_{m=0}^{[\nu_n]} \iota_{c_n}.$$

Then μ_n is discrete, and so is N_H in (4.10) whenever convergent. Since $N_H^{(2)}$ converges wp1 for all $H > 1$, N_H in (4.10) converges wp1 for such an H iff $N_H^{(1)}$ in (4.11) does. For $H > 1$ and $a > 1$ we have

$$\int_{(0,a-1]} \int_{(0,1]} x^H \nu_{n(t,x)} \Pi(dt, dx)$$
$$\le N_H^{(1)}\,(0, a] \le \int_{(-1,a)} \int_{(0,1]} x^H (\nu_{n(t,x)} + 1) \Pi(dt, dx).$$

Since the last expression with $\nu_{n(t,x)}$ omitted is finite wp1, it follows that $N_H^{(1)}(0, a]$ is finite wp1 iff

$$Z := \int_I \int_{(0,1]} x^H \nu_{n(t,x)} \Pi(dt, dx) < \infty \quad \text{wp1}$$

for finite intervals I. As Laplace–Stieltjes transforms can be considered for

$[0, \infty]$-valued rv's, we have for all $H > 0$ and $\tau > 0$, whether Z is finite or not,

$$\begin{aligned}\mathbb{E}e^{-\tau Z} &= \exp - \int_I \int_{(0,1]} (1 - \mathbb{E}\exp(-\tau x^H \nu_1))\, dt\, dx/x^2 \\ &= \exp - \lambda(I) \int_{(0,1]} x^{-2}(1 - \exp(-\tau^\alpha x^{H\alpha}))\, dx \\ &= \exp - \tau^{1/H}\lambda(I) \int_{(0,\tau^{1/H}]} y^{-2}(1 - \exp(-y^{H\alpha}))\, dy.\end{aligned}$$

The last expression is positive and converges to 1 as $\tau \downarrow 0$ iff $H\alpha > 1$, so $Z < \infty$ wp1 in this case only. If $H\alpha \le 1$, then $\mathbb{E}\exp(-\tau Z) = 0$, so $Z = \infty$ wp1. We conclude that N_H converges wp1, iff $H \in \mathscr{A}_a = (1/\alpha, \infty)$.

Variants. The convergence of N_H can be investigated by direct inspection of the sample paths, without Laplace-Stieltjes tranforms or appeal to Lemma 4.3, if one takes as subordinator Π not a Poisson process, but the left points of neighboring pairs in the triadic lattice process $\Pi_{3,c}$, introduced in Section 3 of O'Brien and Vervaat (1985). The result is the same: $\mathscr{A}_a = (1/\alpha, \infty)$ wp1. If one replaces ν_n by nonnegative integer-valued rv's ν_n such that

$$P[\nu_n \ge m] \sim (m(\log m)^\beta)^{-\alpha} \quad \text{as} \quad m \to \infty$$

with $0 < \alpha < 1 < \beta$, then $\mathscr{A}_a = [1/\alpha, \infty)$ wp1. If one takes

$$P[\nu_n \ge m] \sim 1/\log m \quad \text{as} \quad m \to \infty,$$

then $\mathscr{A}_a = \varnothing$ wp1. If

$$P[\nu_n \ge m] \sim 1/m \quad \text{as} \quad m \to \infty,$$

then $\mathscr{A}_a = (1, \infty)$ wp1, whereas the saltus process of N_H still has infinite intensity. We omit the details.

5.3. *Processes of nbv with $H > 1$.* Let Π be a Poincaré point process in E_+ with measurable enumeration $((t_n, x_n))_{n=1}^\infty$. Then its symmetrization is the Poincaré point process in E defined by

$$\Pi_s := \{(t_n, \varepsilon_n x_n): n \in \mathbb{N}\},$$

where $(\varepsilon_n)_{n=1}^\infty$ is a sequence of iid rv's, independent of Π and such that $\varepsilon_n = \pm 1$ with equal probability. Obviously, $\mathscr{A}_a$ is the same for Π and Π_s. Let $\mathscr{A}_c$ be the domain of conditional convergence of Π_s. Then $\mathscr{A}_c = \tfrac{1}{2}\mathscr{A}_a$ by Theorem 5.1 of O'Brien and Vervaat (1985). Consequently, N_H in (4.1) with Π_s instead of Π converges only conditionally wp1 for $H \in \mathscr{A}_s \backslash \mathscr{A}_a$, hence has nbv wp1.

For given $H > 1$ we can find $\alpha \in (0, 1)$ such that $1/(2\alpha) < H < 1/\alpha$, and a Poincaré point process Π on E_+ with $\mathscr{A}_a = (1/\alpha, \infty)$ (cf. 5.2). Hence its symmetrization Π_s produces a convergent N_H in (4.1) for the given $H > 1$, with nbv. Obviously, the present N_H is not continuous. For an example with continuous sample paths, see 6.6.

5.4. *Fractional processes.* This time we consider (4.9) rather than (4.5), since μ in (4.5) need not be Radon in the present example. Set

$$(5.1) \qquad \begin{cases} F(t) := \gamma^{-1}(|t|^{\gamma} - 1) & \text{for } \; t \neq 0, \; 0 \; \text{ for } \; t = 0 \quad (\gamma \neq 0), \\ F_0(t) := \log |t| & \text{for } \; t \neq 0, \; 0 \; \text{ for } \; t = 0. \end{cases}$$

Formula (4.9) now becomes

$$(5.2) \quad X_{H,\gamma}(t) = \begin{cases} \iint_E x^{\uparrow(H-\gamma)}\gamma^{-1}(|t-u|^{\gamma} - |u|^{\gamma})\Pi(du, dx) & \text{for } \; \gamma \neq 0, \\ \iint_E x^{\uparrow H}(\log|t-u| - \log|u|)\Pi(du, dx) & \text{for } \; \gamma = 0. \end{cases}$$

Again $X_{H,\gamma}$ is H-ss and si, if it converges in some sense. We call $X_{H,\gamma}$ the fractional process generated by Π. These processes have been studied in the literature for Π Poisson, or rather Π Poisson with $\Pi - \mathbb{E}\Pi$ instead of Π in (5.2), and have been called fractional stable processes. Our present choice for $\gamma = 0$ seems to have been overlooked so far, but all quoted results hold for $\gamma = 0$ as well. Maejima (1983a) proves that (5.2) converges in probability (easily to be strengthened to "wp1") in the sense of

$$(5.3) \qquad \lim_{n\to\infty}\lim_{\varepsilon\downarrow 0} \int_{1/n<|u|,|u-t|<n} \int_{|x|\le\varepsilon}$$

in case $0 < H < 1$, $H - \gamma > \frac{1}{2}$. Taqqu and Wolpert (1983), who consider the one-sided fractional stable processes with $F_\gamma(t) = 0$ for $t > 0$, $F_0(t) = 1$ for $t < 0$ in (5.1), obtain convergence in probability in (5.2) for the same H and γ, as an application of a newly developed stochastic integral.

For us it is important that $X_{H,\gamma}$ in (5.2) can converge for negative γ. In this case X_H is wp1 nowhere bounded, as already observed in Maejima (1983b), since $\Pi(dt, \mathbb{R}\backslash\{0\})$ has a dense support in $\mathbb{R}$. The author conjectures that the graph of X is wp1 dense in $\mathbb{R}^2$.

If $H \ge 1$, then X_H does not converge in the sense of (5.3). So we have here a domain of convergence bounded away from $+\infty$, which contrasts the results for subordination of positive measures at the end of Section 4.

If Π is replaced by a general Poincaré point process Π on E with finite intensity, then an explorative analysis suggests that $X_{H,\gamma}$ in (5.2) converges wp1 in the sense of (5.3) if

$$0 < H < 1, \quad H - \gamma \in \mathscr{A}_c,$$

where $\mathscr{A}_c$ is the domain of conditional convergence of Π.

In the special case that Π is symmetric and Poisson, it is easy to see that the marginal distributions of $X_H(t)$ in (5.2) are symmetric and stable with exponent $1/(H - \gamma)$. Nevertheless, X_H is H-ss, and not $(H - \gamma)$-ss. Therefore fractional stable processes are interesting examples when one studies the properties of marginal distribution functions of ss si processes, as is done in O'Brien and Vervaat (1983).

5.5. *Polynomial processes.* Analogues of the polynomial processes in Surgailis

(1981) for centered Poisson Π are obtained for general Π by considering

$$N_H^{(n)}(I) := \left(\int\int_E\right)^n (x_1x_2 \cdots x_n)^{\uparrow H/n} \int_I \left(\mu\left(\frac{ds - t_1}{|x_1|}\right) \cdots \mu\left(\frac{ds - t_n}{|x_n|}\right)\right)^{\uparrow 1/n} \tag{5.4}$$
$$\cdot\ \Pi(dt_1, dx_1) \cdots \Pi(dt_n, dx_n)$$

for bounded intervals $I \subset \mathbb{R}_+$ whenever convergent in some sense. Here the measures μ are σ-additive σ-finite and signed, and

$$(\Pi_{k=1}^n \mu_k(ds))^{\uparrow 1/n} := \left(\Pi_{k=1}^n \frac{d\mu_k}{d\nu}(s)\right)^{\uparrow 1/n} \nu(ds)$$

for any measure ν dominating $\mu_1, \mu_2, \cdots, \mu_n$. The left-hand side is well-defined, since the right-hand side is the same for all such ν. It is easy to see that $N_H^{(n)}$ is H-ss and stationary. The proof is the same as for $n = 1$ after formula (4.5). For $n > 1$ we cannot interpret (5.4) for formal measures μ, as we did for $n = 1$ in (4.5). In the particular case that μ is absolutely continuous with respect to Lebesgue measure with density f, the inner integral in (5.4) takes the form

$$|x_1x_2 \cdots x_n|^{-1} \int_I \left(f\left(\frac{s - t_1}{|x_1|}\right) \cdots f\left(\frac{s - t_n}{|x_n|}\right)\right)^{\uparrow 1/n} ds$$

(actually, Surgailis, 1981, considers this particular case, rather than (5.4)). More particularly, let $N_{H,\gamma}^{(n)}$ for $\gamma \in \mathbb{R}$ be defined by (5.4) for μ with density

$$f(s) = s^{\uparrow(\gamma-1)}.$$

Then $N_{H,\gamma}^{(1)} = N_{H,\gamma}$ as in 5.4, so $N_{H,\gamma}^{(n)}$ is the polynomial analogue of the fractional process and the (two-sided) analogue of the Hermite processes (with Brownian motion instead of Π and the one-sided $f(s) = -|s|^{\gamma-1}$ for $s < 0$, 0 for $s > 0$) in Taqqu (1979).

In the particular case that Π is Poisson, the polynomial processes (5.4) with absolutely continuous μ play a role in series representations for $L^2(\Omega, \sigma(\Pi))$, where $\sigma(\Pi)$ is the σ-field generated by Π (cf. Surgailis, 1981). A similar theory does not seem to be available for general Poincaré Π.

6. Composition of processes. In the present section we investigate the preservation of the ss si properties for composition of processes

$$X_1 \circ X_2 := (X_1(X_2(t)))_{t\in\mathbb{R}} := (X_1(X_2(t, \omega), \omega))_{t\in\mathbb{R}}. \tag{6.1}$$

Note that $T = \mathbb{R}$. We do not assume the regularity conditions of Hypotheses 1.4, but from the right-hand side of (6.1) it is obvious that X_1 must be assumed to be measurable (i.e., $(t, \omega) \mapsto X_1(t, \omega)$ is measurable) in order to guarantee that $X_1 \circ X_2$ is a stochastic process (i.e., $\omega \mapsto X_1 \circ X_2(t, \omega)$ is measurable for each t).

The most general condition that guarantees $X_1 \circ X_2$ to be ss si is that X_1 is conditionally ss si given X_2 and vice versa. The rather delicate interpretation of this condition and proofs of results in this generality can be found in Vervaat

(1982), the first version of the present article. Here we restrict ourselves to the much more tractable and most important subcase that X_1 and X_2 are ss si and independent. The results for this subcase are more or less implicit in pages 115–118 of Major (1981), and a particular case is treated by Lou (1983).

THEOREM 6.1. *If X_1 and X_2 are independent processes, X_1 is measurable and X_j is H_j-ss for $j = 1, 2$, then $X_1 \circ X_2$ is H_1H_2-ss.*

PROOF. In shorthand the proof reads

$$X_1 \circ X_2(a\cdot) =_d X_1(a^{H_2}X) =_d (a^{H_2})^{H_1}X_1 \circ X_2 \quad \text{for} \quad a > 0.$$

More extensively, we have for finite-dimensional measurable subsets B of $\overline{\mathbb{R}}^T$

$$\begin{aligned}\mathbb{P}[X_1 \circ X_2(a\cdot) \in B] &= \mathbb{E}\mathbb{P}^{X_1}[X_1 \circ X_2(a\cdot) \in B] = \mathbb{E}\mathbb{P}^{X_1}[X_1(a^{H_2}X_2) \in B] \\ &= \mathbb{E}\mathbb{P}^{X_2}[X_1(a^{H_2}X_2) \in B] = \mathbb{E}\mathbb{P}^{X_2}[(a^{H_2})^{H_1}X_1 \circ X_2 \in B] \\ &= \mathbb{P}[a^{H_1H_2}X_1 \circ X_2 \in B].\end{aligned}$$

THEOREM 6.2. *If X_1 and X_2 are independent si processes, X_1 is measurable and $X_2(0) = 0$ wp1, then $X_1 \circ X_2$ is si.*

PROOF. We give the proof only in shorthand:

$$\begin{aligned}X_1 \circ X_2(b + \cdot) - X_1 \circ X_2(b) \qquad & \\ &= X_1(X_2(b) + X_2(b + \cdot) - X_2(b)) - X_1(X_2(b)) \\ &=_d X_1(X_2(b + \cdot) - X_2(b)) - X_1(0) =_d X_1 \circ X_2 - X_1(0) \\ &=_d X_1 \circ X_2 - X_1 \circ X_2(0).\end{aligned}$$

COROLLARY 6.3. *If X_1 and X_2 are independent and satisfy Hypotheses 1.4 with ss-exponents H_1 and H_2, then $X_1 \circ X_2$ is H_1H_2-ss si.*

We now present three examples, based on Corollary 6.3.

6.4. *Strictly stable processes.* The particular case of Corollary 6.3 with independent increments and X_2 nondecreasing is well-known (cf. Fristedt, 1974, Ex. 7.2). If X_1 is a strictly stable process with exponent $\alpha_1 = 1/H_1 \in (0, 2]$ and X_2 is a nondecreasing stable process with exponent $\alpha_2 \in (0, 1)$, then $X_1 \circ X_2$ is strictly stable with exponent $\alpha_1\alpha_2$. Note that for $X_1 \circ X_2$ to have independent increments it is essential that X_2 is nondecreasing. For instance, if X_1 and X_2 are strictly stable with exponents $\alpha_1, \alpha_2 \in (1, 2]$ such that $\alpha_1\alpha_2 > 2$, then $X_1 \circ X_2$ cannot be stable, whereas $X_1 \circ X_2$ is $1/\alpha_1\alpha_2$-ss and si by Corollary 6.3.

6.5. *An H-ss si process with $H > 1$ and nbv.* (due to Terry R. McConnell, oral communication). Let X and B be independent, where X is an increasing stable process with exponent $\alpha_1 = 1/4 = 1/H_1$ and B is standard Brownian motion (so $\alpha_2 = 2 = 1/H_2$). Then $X \circ B$ is 2-ss si by Corollary 6.3. The process X increases by jumps only. Let $\tau := \inf\{t: X(t) - X(t-) \geq 1\}$ be the location of its first jump ≥ 1, and let $\sigma := \inf\{s: B(s) = \tau\}$ be the hitting time of τ by B. Then σ is a stopping time for B with its filtering σ-fields enriched by the independent σ-field of X. Hence $B(\sigma + \cdot) - B(\sigma) =_d B$. In particular, B assumes wp1 both values greater and less than $B(\sigma)$ in any right neighborhood of $B(\sigma)$. As $X(B(\sigma)) - X(B(\sigma) -) \geq 1$, it follows that $X \circ B$ makes infinitely many jumps of size at least 1 in any right neighborhood of σ, so does not have bounded variation in any such neighborhood. As σ with these properties can be found wp1, X has nbv wp1 by Corollary 2.3(d).

6.6 *An H-ss si process as in* 6.5, *but with continuous sample paths.* Let X and B be independent, where B is standard Brownian motion and X a 4-ss si process as in Example 5.1 with $H = 4$, but with μ replaced by $\nu := \mu\phi^{-1}$, where ϕ is an increasing differentiable function on [0, 1] with $\phi(0) = 0$ and bounded derivative. Then X has continuous sample paths by the arguments at the end of Example 5.1, provided that the defining sequence in (4.5) or (4.9) converges. It does by Theorem 4.5, since with the Cantor set C also $\phi(C)$, the support of ν, satisfies (4.15). To see this, note that if $((a_n, b_n))_{n=1}^{\infty}$ is the sequence of gaps in C, then $(\phi(a_n), \phi(b_n))_{n=1}^{\infty}$ is the sequence of gaps in $\phi(C)$, whereas $\phi(b_n) - \phi(a_n) \leq (b_n - a_n)\sup_t \phi'(t)$. We conclude that $X \circ B$ is a 2-ss si process with continuous sample paths.

Let $\tau := \inf\{t: (t, x) \in \Pi, x \geq 1\}$ and let $\sigma := \inf\{s: B(s) = \tau\}$ be the hitting time of τ by B. As in 6.5 it follows that

$$B_0 := B(\sigma + \cdot) - B(\sigma) =_d B.$$

At $B(\sigma) = \tau$ the process X starts increasing by a new contribution $x^4\nu((\cdot - t)/x)$ (with $x \geq 1$), so

$$X(\tau + h) - X(\tau) \geq x^4\nu(0, h/x] = x^4\mu(0, \phi^{-1}(h/x)]$$

for $h > 0$. From $\mu(0, 3^{-n}] = 2^{-n}$ for $n = 0, 1, 2, \cdots$ we obtain

$$\mu(0, t] \geq 1 \wedge \tfrac{1}{2}t^{\beta} \quad \text{with} \quad \beta := \log 2/\log 3 \in (0, 1).$$

Consequently,

$$X(\tau + h) - X(\tau) \geq x^4(1 \wedge \tfrac{1}{2}(\phi^{-1}(h/x))^{\beta}) \quad \text{for} \quad h > 0.$$

Set

$$\xi_n := \begin{cases} xB_0(2^{-n}) & \text{if} \quad B_0(2^{-n}) > 0 \quad \text{and} \quad B_0(s) = 0 \quad \text{for some} \quad s \in (2^{-n}, 2^{n+1}], \\ 0 & \text{else.} \end{cases}$$

Then the total variation of $X \circ B$ right of $\tau = B(\sigma)$ is greater than some tail sum of the series

$$2 \sum_{n=1}^{\infty} (X(\tau + \xi_n) - X(\tau)) \geq x^4 \sum_{n=1}^{\infty} (2 \wedge (\phi^{-1}(\xi_n))^{\beta}). \tag{6.2}$$

Since $B_0 =_d B$ is ½-ss and has a trivial tail σ-field at 0 (by the Blumenthal–Getoor 0-1-law, cf. Freedman, 1971, page 106), the sequence of processes

$((2^{n/2}B_0(s2^{-n}))_{s\in[0,1]})_{n=1}^{\infty}$ is stationary and ergodic, and so is the sequence $(\eta_n)_{n=1}^{\infty} := (2^{n/2}\xi_n)_{n=1}^{\infty}$, obtained by applying one fixed functional. Choosing finally $\phi(t) := e^{-1/t}$, so $\phi^{-1}(t) = 1/\log t^{-1}$ for $0 \le t \le e^{-1}$, we find that the two sides of (6.2) are minorized by

$$x^4 \sum_{n=1}^{\infty} (2 \wedge (\tfrac{1}{2}n \log 2 - \log \eta_n)^{-\beta}). \tag{6.3}$$

Since $\mathbb{P}[\xi_1 > 0] = \mathbb{P}[\eta_1 > 0] > 0$, there is a real c such that $\mathbb{P}[\log \eta_1 > c] > 0$. By ergodicity we have for each natural n_0

$$N_n := (1/n) \sum_{k=n_0+1}^{n} 1_{[\log \eta_n > c]} \to \mathbb{P}[\log \eta_1 > c] \quad \text{wp1}. \tag{6.4}$$

Take n_0 such that $4^{-1}n_0 \log 2 > c$. Then (6.3) is minorized by

$$x^4 \sum_{n=n_0+1}^{n} (2 \wedge (4^{-1}n \log 2)^{-\beta}) 1_{[\log \eta_n > c]} \ge x^4 \sum_{n=n_0+1}^{n+nN_n} (2 \wedge (4^{-1}n \log 2)^{-\beta}),$$

which diverges to ∞ wp1, as N_n converges in (6.4) to a positive constant wp1. We conclude that $X \circ B$ has wp1 no bounded variation in any right neighborhood of σ, so has nbv wp1 by Corollary 2.3(d).

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