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LIMITING MULTIVARIATE DISTRIBUTIONS OF INTERMEDIATE ORDER STATISTICS[1]

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Let $Z_m^{(n)}$ represent the mth largest order statistic in a random sample of size n from a distribution F. If $m = m(n)$ is an intermediate sequence such that $m \to \infty$ and $m/n \to 0$ as $n \to \infty$, the intermediate order statistics of the form $Z_{[mt_1]}^{(n)}, \cdots, Z_{[mt_k]}^{(n)}$, for $0 < t_1 < \cdots < t_k$, can be used jointly for making statistical inferences about the upper tail of F. We find the asymptotic joint distribution of order statistics of this form, for various types of underlying distributions F, by determining the limit (weak convergence) of a stochastic process of the form $(Z_{[mt]}^{(n)} - \beta_{mt}^{(n)})/\alpha_{mt}^{(n)}$, $t > 0$, for appropriate normalizing functions $\alpha_{mt}^{(n)} > 0$, $\beta_{mt}^{(n)}$.

1. Introduction. Let $X_1, \cdots, X_n$ be mutually independent random variables with common distribution function F and let $Z_k^{(n)}$ represent the kth largest order statistic. The possible asymptotic distributions of the maximum $Z_1^{(n)}$ have been extensively investigated. Gnedenko (1943) gives necessary and sufficient conditions on F under which there exist sequences $a_n > 0$ and b_n such that the normalized maximum $(Z_1^{(n)} - b_n)/a_n$ has a nondegenerate limiting distribution Λ_c in the following sense:

$$P[(Z_1^{(n)} - b_n)/a_n \le x] = F^n(a_n x + b_n) \to \Lambda_c(x) \quad \text{as} \quad n \to \infty \tag{1.1}$$

for all x at which $\Lambda_c(x)$ is continuous. When such a limiting distribution exists, F is said to be in the domain of attraction of Λ_c, $F \in \Delta_{\text{ext}}(\Lambda_c)$, and Λ_c can always be written in the following parametric form (von Mises, 1936)

$$\Lambda_c(x) = \exp(-g_c(x)) \tag{1.2}$$

where

$$g_c(x) = \exp\left(-\int_0^x [(1 + cu)_+]^{-1}\, du\right) \tag{1.3}$$

for some $c \in R$ and for any x such that $0 < \Lambda_c(x) < 1$. As a distribution function, $\Lambda_c(x)$ is therefore continuous whenever $x \in R$ and the convergence in (1.1) is therefore uniform for all $x \in R$. The function $g_c(x)$ is itself a tail function whenever $x \ge 0$. When $c > 0$, $g_c(x)$ is the tail function of a Pareto distribution, $g_c(x) = (1 + cx)^{-1/c}$, so that $-1/c$ is the exponent of regular variation for the tail function $1 - F(x)$. When $c \le 0$, $g_c(x)$ is a natural generalization of this Pareto

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tail function ($g_0(x) = e^{-x}$). The upper tail of a distribution and its relationship to the limiting extremal distribution has been studied by Resnick (1971) and Pickands (1975). Of course, analogous results also hold for the normalized minimum $Z_n^{(n)}$.

Various extensions of the basic results in extreme value theory are possible when the underlying random variables are not mutually independent. Important work in this area includes Berman (1964), Loynes (1965), and Leadbetter (1974). The asymptotic multivariate distribution of the k largest order statistics was found by Dwass (1966) and Weissman (1975). More recent work in this particular area has been done by Weissman (1978) and Serfozo (1982). Galambos (1978) is an important general reference on the probability theory underlying the study of extreme values.

A related area of investigation concerns the limiting distribution of the intermediate order statistic $Z_m^{(n)}$ where the sequence of integers $m = m(n)$ is such that $1 \le m \le n$ for all n, and

$$(1.4) \qquad m \to \infty, \quad m/n \to 0 \quad \text{as} \quad n \to \infty.$$

In this case, $m(n)$ is called an intermediate sequence. Chibisov (1964) and Wu (1966) have both shown that normal and lognormal distributions are possible limiting distributions. Smirnov (1949), Cheng (1965) and Mejzler (1978) also present conditions under which these limiting distributions apply for intermediate order statistics. Balkema and de Haan (1978a, b) provide a comprehensive study of the limiting univariate distributions of various types of order statistics and they consider the intermediate order statistics in this more general framework. Watts, Rootzen, and Leadbetter (1982) show that the same limiting univariate distributions also apply when the original random variables $X_1, \cdots, X_n$ satisfy a general dependence restriction.

Intermediate order statistics can be used to estimate probabilities of future extreme observations and to estimate tail quantiles of the underlying distribution that are extreme relative to the available sample size. Pickands (1975) has shown that intermediate order statistics of the form $Z_{[mt_1]}^{(n)}, \cdots, Z_{[mt_k]}^{(n)}$ for $0 < t_1 < \cdots < t_k$, where $m = m(n)$ is an intermediate sequence ($[x]$ denotes the integer part of x), can be used in constructing consistent estimators of the shape parameter c of the limiting extremal distribution Λ_c and in finding consistent estimators for conditional tail functions of the form:

$$(1.5) \qquad h_m^{(n)}(x) = P[X > Z_m^{(n)} + x \mid X > Z_m^{(n)}],\ x > 0.$$

De Haan and Resnick (1980), Teugels (1981) and Mason (1982) have also found estimators that are based, in part, on intermediate order statistics. In this context, the joint asymptotic distribution of intermediate order statistics that are based on the same intermediate sequence is of some interest. One can of course study such asymptotic joint distributions by considering a stochastic process of the form

$$(1.6) \qquad v_m^{(n)}(t) \equiv (Z_{[mt]}^{(n)} - \beta_{mt}^{(n)})/\alpha_{mt}^{(n)}, \quad t > 0$$

and determining when a limiting (weak convergence) stochastic process exists for appropriate normalizing sequences $\alpha_{mt} > 0$ and β_{mt}. In general, we will write $v_m^{(n)}(t) \to_D v(t)$, $t > 0$, for t fixed with respect to n, whenever all finite dimensional distributions of the process $v_m^n(t)$ converge to those of $v(t)$:

$$(v_m^{(n)}(t_1), \cdots, v_m^{(n)}(t_k)) \to_d (v(t_1), \cdots, v(t_k))$$

for any values $0 < t_1 < \cdots < t_k < \infty$.

In addition to considering the more general process of (1.6), we will show that whenever $F \in \Delta_{\text{ext}}(\Lambda_c)$, one can construct an intermediate sequence $m = m(n)$ and corresponding normalizing sequences $\alpha_m^{(n)} > 0$ (which does not need to be indexed by t) and $\beta_{mt}^{(n)}$ such that

$$w_m^{(n)}(t) \equiv (Z_{[mt]}^{(n)} - \beta_{mt}^{(n)})/\alpha_m^{(n)} \tag{1.7}$$

converges weakly to a Gaussian stochastic process as $n \to \infty$. When F satisfies an additional constraint, such a limiting stochastic process exists for all intermediate sequences $m = m(n)$. Define the tail quantile $z_p = \inf\{u: F(u) \geq 1 - p\}$, $0 < p < 1$. The existence of a limiting process for $w_m^{(n)}(t)$ makes it possible to estimate functions of ratios of differences between extreme quantiles of the form $z_{mt/n}$, $t > 0$, or the limits of such functions as $n \to \infty$, by using the same functions of the corresponding intermediate order statistics. This approach is possible if $w_m^{(n)}(t)$ has a limiting process because the locationally adjusted order statistics $Z_{[mt]}^{(n)} - \beta_{mt}^{(n)}$, $t > 0$, must then be of the same stochastic order for all $t > 0$. For example, Pickands (1975) has shown that consistent estimators of this form exist for $h_m^{(n)}(x)$ of (1.5) and for the shape parameter c whenever F is continuous and $F \in \Delta_{\text{ext}}(\Lambda_c)$. A knowledge of the limiting stochastic process for $w_m^{(n)}(t)$ also makes it possible to evaluate such estimators in terms of their asymptotic distributions.

2. The intermediate domains of attraction.

Let $m(n)$ be a nondecreasing intermediate sequence. Wu (1966) proved that whenever there exist sequences $\alpha_m^{(n)}$ and $\beta_m^{(n)}$ such that $P[Z_m^{(n)} \leq \alpha_m^{(n)} x + \beta_m^{(n)}]$ has a nondegenerate limit, the limiting distribution must be of the form $\Phi(h(x))$, where Φ is the standard normal distribution and where $h(x)$ is of the following form (up to an affine transformation of x):

$$h(x) = -\log(g_\lambda(x)), \quad \inf\{y: g_\lambda(y) < \infty\} < x < g_\lambda^{-1}(0) \tag{2.1}$$

for $\lambda \in R$ and for $g_\lambda(x)$ as defined in (1.3). When $\lambda = 0$, $\Phi \bullet h$ is a normal distribution, so that with the appropriate choice of normalizing sequences, the limiting distribution is simply Φ.

DEFINITION 2.1. We will say that F is in the intermediate domain of attraction of $\Phi \bullet h$ for an intermediate sequence $m(n)$, $F \in \Delta_{\text{int}}(\Phi \bullet h, m(n))$, if there exist sequences $\alpha_m^{(n)} > 0$ and $\beta_m^{(n)}$ such that

$$P[Z_m^{(n)} \leq \alpha_m^{(n)} x + \beta_m^{(n)}] \to \Phi(h(x))$$

as $n \to \infty$, for all continuity points of $\Phi \bullet h$.

A necessary and sufficient condition for $F \in \Delta_{\text{int}}(\Phi\bullet h,\ m(n))$ is that there exist sequences $\alpha_m^{(n)} > 0$ and $\beta_m^{(n)}$ such that

$$1 - F(\alpha_m^{(n)}x + \beta_m^{(n)}) = mn^{-1}[1 - m^{-1/2}(h(x) + o(1))] \tag{2.2}$$

as $n \to \infty$ whenever x is a continuity point of $\Phi\bullet h$. This condition is due to Smirnov (1949). Also see Watts, Rootzen and Leadbetter (1982, page 654). The next theorem gives a condition for the general multivariate result. Given any normalizing sequence κ_n, the corresponding normalizing function κ_x will be defined as

$$\begin{aligned} \kappa_x &= \kappa_1, \quad 0 \le x < 1 \\ &= \kappa_{[x]}, \quad x \ge 1. \end{aligned} \tag{2.3}$$

THEOREM 2.1. *Let $m = m(n)$ be an intermediate sequence and define $v_m^{(n)}(t)$ as in* (1.6). *Then $F \in \Delta_{\text{int}}(\Phi\bullet h,\ m(n)t)$ for all $t > 0$ (t fixed with respect to n) if and only if there exist normalizing functions $\alpha_{mt}^{(n)} > 0$ and $\beta_{mt}^{(n)}$ such that*

$$v_m^{(n)}(t) \to_D h^{-1}(t^{-1/2}u(t)) \quad \text{as} \quad n \to \infty, \tag{2.4}$$

where $u(t)$ is standard Brownian motion and h is defined in (2.1).

Note that $F \in \Delta_{\text{int}}(\Phi\bullet h,\ m(n))$ does not imply $F \in \Delta_{\text{int}}(\Phi\bullet h,\ m(n)t)$ for any $t \neq 1$. For example, let $F(x) = 1 - (\log\log x)^{-1}$, $x \ge e^e$. It follows from (2.2) that $F \in \Delta_{\text{int}}(\Phi\bullet(-\log g_1),\ m(n))$ whenever

$$m(n) = 2n(\log n)^{-1}[1 + 3\,(\log\log n - \log 2)(\log n)^{-1}(1 + o(1))] \quad \text{as} \quad n \to \infty,$$

but F is not in any intermediate domain of attraction for sequences $m(n)t$, $t \neq 1$. In this case the normalizing sequences are such that

$$\alpha_m^{(n)} = \beta_m^{(n)}(1 + o(1)) = \exp[\exp(n/m)][1 + o(1)] \quad \text{as} \quad n \to \infty.$$

The proof of Theorem 2.1 will require the following lemma. This result is a special case of a general Bahadur representation for order statistics due to Watts (1977, Theorem 4.3.1).

LEMMA 2.1. *Let $U_k^{(n)}$ represent the kth largest uniform* (0, 1) *order statistic from a random sample of size n. Define*

$$u_m^{(n)}(t) \equiv (mt)^{-1/2}n[U_{[mt]}^{(n)} - (1 - mtn^{-1})], \quad t > 0 \tag{2.5}$$

where $m = m(n)$ is any intermediate sequence. Then $u_m^{(n)}(t) \to_D t^{-1/2}u(t)$, $t > 0$ (t fixed with respect to n), where $u(t)$ is standard Brownian motion.

PROOF. Define

$$Y_j \equiv (n - j + 1)[\log U_{j-1}^{(n)} - \log U_j^{(n)}], \quad 1 \le j \le [mt]$$

with $U_0^{(n)} \equiv 1$. It is well-known that the Y_j are independent exponential random variables with mean 1. See, for example, David (1970, page 19). Then for $t > 0$,

$$\begin{aligned}(1 - [mt]n^{-1})^{-1}U_{[mt]}^{(n)} &= \prod_{j=1}^{[mt]}[(1 - (j-1)n^{-1})U_j^{(n)}][(1 - jn^{-1})U_{j-1}^{(n)}]^{-1}\\ &= \exp(-\sum_{j=1}^{[mt]}[(n - j + 1)^{-1}Y_j + \log(1 - (n - j + 1)^{-1})])\\ &= \exp(-[\sum_{j=1}^{[mt]} (n - j + 1)^{-1}(Y_j - 1)] + O(mn^{-2}))\\ &= 1 - n^{-1}\sum_{j=1}^{[mt]}(Y_j - 1) + O_p(m^{3/2}n^{-2})\end{aligned}$$

as $n \to \infty$. Consequently, as $n \to \infty$

$$u_m^{(n)}(t) = -[mt]^{-1/2} \sum_{j=1}^{[mt]} (Y_j - 1) + O_p(m^{-1/2} + mn^{-1}), \quad t > 0.$$

Therefore, the process $u_m^{(n)}(t)$, $t > 0$, converges in probability to a normalized sum of independent identically distributed random variables, so that the limiting process is the standardized Brownian motion $t^{-1/2}u(t)$.

PROOF OF THEOREM 2.1. If the convergence in (2.4) holds for $t > 0$, then $F \in \Delta_{\text{int}}(\Phi\bullet h, m(n)t)$, $t > 0$, by Definition 2.1. Now assume $F \in \Delta_{\text{int}}(\Phi\bullet h, m(n)t)$, $t > 0$, for some intermediate sequence $m(n)$ and define $F^{-1}(y) = \inf\{u: F(u) \geq y\}$. Then, using the notation of Lemma 2.1,

$$\begin{aligned}Z_{[mt]}^{(n)} &=_d F^{-1}(U_{[mt]}^{(n)}) = F^{-1}(1 - mtn^{-1}[1 - (mt)^{-1/2}u_m^{(n)}(t)])\\ &= \alpha_{mt}^{(n)}[h^{-1}(u_m^{(n)}(t)) + o_p(1)] + \beta_{mt}^{(n)} \quad \text{as} \quad n \to \infty\end{aligned} \tag{2.6}$$

by (2.2). The convergence in (2.4) now follows by Lemma 2.1 ($h^{-1}(y)$ is continuous for all $y \in R$). This completes the proof of the theorem.

Under the conditions of Theorem 2.1, the scale functions $\alpha_{mt}^{(n)}$ of (2.4) are not always of the same order as $n \to \infty$ for different values of $t > 0$. For example, if $F(x) = 1 - (\log x)^{-1}$, then $F \in \Delta_{\text{int}}(\Phi, m(n))$ only when $m(n)$ is an intermediate sequence such that $(m(n))^{-1} = o(n^{-2/3})$. In this case, the normalizing functions are of the following form as $n \to \infty$:

$$\alpha_{mt}^{(n)} = n(mt)^{-3/2}\exp(n/mt)[1 + o(1)];$$

$$\beta_{mt}^{(n)} = \exp(n/mt)[1 + o(\alpha_{mt}^{(n)})].$$

Consequently, locationally adjusted order statistics of the form $Z_{[mt]} - \beta_{mt}^{(n)}$, $t > 0$, are of different stochastic orders as $n \to \infty$ for different values of $t > 0$. But when $F \in \Delta_{\text{ext}}(\Lambda_c)$, one can always construct an intermediate sequence $m(n)$ such that $F \in \Delta_{\text{int}}(\Phi, m(n))$ and for any such intermediate sequence, the statistics $Z_{[mt]} - \beta_{mt}^{(n)}$, $t > 0$, are of the same stochastic order for all $t > 0$.

THEOREM 2.2. *If $F \in \Delta_{\text{ext}}(\Lambda_c)$, there exists an intermediate sequence $k(n)$ and normalizing functions $\alpha_y^{(n)}$ and $\beta_y^{(n)}$, $0 < y \leq n$, such that if $m = m(n)$ is any* *intermediate sequence for which* $m(n) = o(k(n))$ *as* $n \to \infty$, *then for* $t > 0$

$$(2.7) \qquad (Z_{[mt]} - \beta_{mt}^{(n)})/\alpha_m^{(n)} \to_D w(t) \quad \text{as} \quad n \to \infty$$

where $w(t)$, $t > 0$, *is the Gaussian process characterized by*:

$$(2.8) \qquad \begin{aligned} E(w(t)) &= 0, \quad t > 0; \\ \operatorname{Cov}(w(t_1), w(t_2)) &= t_1^{-c}\, t_2^{-c-1} \quad \textit{whenever} \quad 0 < t_1 \le t_2. \end{aligned}$$

Also in this case:

$$(2.9) \qquad \beta_{mt}^{(n)} = \beta_m^{(n)} - \alpha_m^{(n)}\, m^{1/2}\left[\int_1^t u^{-c-1}\, du\right][1 + o(1)] \quad \textit{as} \quad n \to \infty$$

for $-\infty < c < \infty$.

The proof will require the following lemma.

LEMMA 2.2. *Assume* $F \in \Delta_{\text{ext}}(\Lambda_c)$ *and let* $a_y > 0$ *and* b_y, $y > 0$, *be appropriate normalizing functions for* $Z_1^{([y])}$ *as* $y \to \infty$. *If as* $n \to \infty$, $n^* = n(t^{-1} + o(1))$, $t > 0$, *then as* $n \to \infty$:

$$\frac{a_{n^*}}{a_n} \to t^{-c}, \quad b_{n^*} = b_n - a_n\left[\left(\int_1^t u^{-c-1}\, du\right) + o(1)\right]$$

for $-\infty < c < \infty$.

PROOF. For all $x \in R$, as $n \to \infty$

$$F^n(a_{n^*}x + b_{n^*}) = F^{n^*(t+o(1))}(a_{n^*}x + b_{n^*}) \to \Lambda_c^t(x) = \Lambda_c(Ax + B)$$

where

$$A = t^{-c}, \quad B = -\int_1^t u^{-c-1}\, du, \quad -\infty < c < \infty.$$

Lemma 2.2 now follows since for all $x \in R$, $F^n(a_n x + b_n) \to \Lambda_c(x)$ as $n \to \infty$.

PROOF OF THEOREM 2.2. By assumption there exist normalizing functions $a_y > 0$ and b_y, $y > 0$ such that as $n \to \infty$

$$(2.10) \qquad F^n(a_n x + b_n) \to \exp(-g_c(x))$$

uniformly for $x \in R$. For an arbitrary intermediate sequence $m = m(n)$, define

$$\xi_m^{(n)}(x) \equiv nm^{-1}[1 - F(a_{n/m}x + b_{n/m})] - g_c(x).$$

Then

$$(2.11) \qquad \begin{aligned} 1 - F(a_{n/m}m^{-1/2}x + b_{n/m}) &= mn^{-1}[g_c(m^{-1/2}x) + \xi_m^{(n)}(m^{-1/2}x)] \\ &= mn^{-1}[1 - m^{-1/2}x + O(m^{-1} + \xi_m^{(n)}(m^{-1/2}x))] \end{aligned}$$

as $n \to \infty$. Therefore by (2.2), $F \in \Delta_{\text{int}}(\Phi, m(n))$ whenever $\xi_m^{(n)}(m^{-1/2}x) = o(m^{-1/2})$ as $n \to \infty$, for all $x \in R$. Let x_0 be any point such that $\Lambda_c^{-1}(0) < x_0 < 0$.

As $n \to \infty$, $|\xi_m^{(n)}(x)| \to 0$ uniformly for $x_0 \le x < \infty$ by (2.10), since $\xi_m^{(n)}(x)$ is the difference between two nondecreasing functions and $g_c(x)$ is bounded for $x_0 \le x < \infty$. Consequently,

$$\Xi_{m,x_0}^{(n)} \equiv \sup_{1 \le k \le m} \sup_{x_0 \le x < \infty} |\xi_k^{(n)}(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

It suffices to let

$$k_n \equiv \min[(\Xi_{N(n),x_0}^{(n)})^{-2}, N(n)]$$

where $N(n)$ is any intermediate sequence as $n \to \infty$, so that when $m = o(k_n)$ as $n \to \infty$, we have for all $x \in R$ (and for sufficiently large n):

$$(2.12) \qquad |\xi_m^{(n)}(m^{-1/2}x)| \le \Xi_{m,x_0}^{(n)} \le \Xi_{N(n),x_0}^{(n)} \le k_n^{-1/2} = o(m^{-1/2}) \quad \text{as} \quad n \to \infty.$$

(This particular choice of k_n depends on x_0, $N(n)$ and the normalizing functions $a_y > 0$ and b_y, $y > 0$.) Now define

$$(2.13) \qquad \alpha_{mt}^{(n)} \equiv (mt)^{-1/2} a_{n/mt}, \quad \beta_{mt}^{(n)} \equiv b_{n/mt} \quad \text{for} \quad t > 0.$$

We now have that $F \in \Delta_{\text{int}}(\Phi, m(n)t)$ for $t > 0$ whenever $m(n)$, an intermediate sequence, is such that $m(n) = o(k_n)$ as $n \to \infty$ by (2.11), (2.12) and condition (2.2), where the normalizing functions can be defined as in (2.13). The convergence in (2.7) now follows by Theorem 2.1 and Lemma 2.2; (2.9) follows by (2.13) and Lemma 2.2. This completes the proof of the theorem.

We now introduce a large class of distributions for which the convergence in (2.7) holds for all intermediate sequences. This class includes all continuous distributions that are typically used in statistical applications.

DEFINITION 2.2. We will say that F is in the first differentiable domain of attraction of the extremal distribution Λ_c, $F \in \Delta_{\text{dif}}(\Lambda_c)$, when:

(i) F is differentiable throughout some left neighborhood of

$$z_0 \equiv \sup\{u: F(u) < 1\};$$

(ii) There exist sequences $a_n > 0$ and b_n such that as $n \to \infty$, $(d/dx)F^n(a_n x + b_n) \to \Lambda_c'(x)$ uniformly in x for all finite subintervals in the support of Λ_c.

Clearly if $F \in \Delta_{\text{dif}}(\Lambda_c)$, then $F \in \Delta_{\text{ext}}(\Lambda_c)$, for the same normalizing sequences $a_n > 0$ and b_n, by the uniform convergence assumed in (ii). Furthermore, if $F \in \Delta_{\text{dif}}(\Delta_c)$ and if there exist sequences $A_n > 0$ and B_n such that $F^n(A_n x + B_n) \to \Lambda_c(x)$ as $n \to \infty$, for all x such that $0 < \Lambda_c(x) < 1$, then the convergence assumed in condition (ii) still applies for sequences $A_n > 0$ and B_n. This is an immediate consequence of Lemma 2.2 and the uniform convergence which is assumed to exist in condition (ii) for appropriate sequences. Condition (ii) can also be written in the form:

$$(2.14) \qquad n a_n f(a_n x + b_n) \to -g_c'(x) \quad \text{as} \quad n \to \infty$$

where $f(x) = F'(x)$ and this convergence is uniform in x for all finite subintervals in the support of Λ_c.

THEOREM 2.3. *If $F \in \Delta_{\rm dif}(\Lambda_c)$, then there exist normalizing functions $\alpha_y^{(n)}$ and $\beta_y^{(n)}$, $y > 0$, such that the convergence in* (2.7) *holds for all intermediate sequences $m(n)$.*

PROOF. The location normalizing function for the extremal domain of attraction, b_y, $y > 0$, must be such that as $y \to \infty$

$$y(1 - F(b_y)) = -\log \Lambda_c(0)(1 + o(1)) = (1 + o(1)) \tag{2.15}$$

since $F \in \Delta_{\rm ext}(\Lambda_c)$. It follows from (2.15) and Lemma 2.2 that b_y can be chosen as $b_y = \inf\{u\colon F(u) \ge 1 - y^{-1}\}$ so that when y is sufficiently large, $F(b_y) = 1 - y^{-1}$ since F must be continuous in a left neighborhood of $z_0 \equiv \sup\{u\colon F(u) < 1\}$, by assumption. For any $x \in R$, and for any intermediate sequence $m = m(n)$, there exists a sequence $x_0(n)$ (by the mean value theorem) such that $|x - x_0(n)| \le |x|$ and such that for sufficiently large n (it suffices that n be large enough so that $0 < g_c(m^{-1/2}x) < \infty$):

$$\begin{aligned} F(a_{n/m}m^{-1/2}x + b_{n/m}) - F(b_{n/m}) &= a_{n/m}f(a_{n/m}m^{-1/2}x_0(n) + b_{n/m})m^{-1/2}x \\ &= mn^{-1}m^{-1/2}x(1 + o(1)) \quad \text{as} \quad n \to \infty \end{aligned} \tag{2.16}$$

by (2.14). Therefore $F \in \Delta_{\rm int}(\Phi, m(n))$ for any intermediate sequence $m(n)$ by (2.16), condition (2.2), and our choice of b_y, $y > 0$. The convergence in (2.7) now follows for all intermediate sequences $m(n)$, by Theorem 2.1 and Lemma 2.2. This completes the proof of the theorem.

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