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TWO EXAMPLES CONCERNING A THEOREM OF BURGESS AND MAULDIN

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We show that for the transition kernels (μ_y) of a certain random walk in $\mathbb{R}^2$ and the Radon Transform in $\mathbb{R}^3$ there is no subset K of positive Lebesgue-measure such that $(\mu_y)_{y\in K}$ is completely orthogonal.

Let X and Y be complete metric spaces, and $\mathscr{B}(X)$, $\mathscr{B}(Y)$ the corresponding classes of Borel sets. Burgess and Mauldin have proved the following theorem for a transition kernel $(\mu_y)_{y\in Y}$ of probability-measures on $(X, \mathscr{B}(X))$.

THEOREM ([1]). *Assume that for distinct points $y_1, y_2 \in Y$ the measures μ_{y_1} and μ_{y_2} are always mutually singular. Then there is a set $K \subset Y$ homeomorphic to the Cantor set such that the kernel $(\mu_y)_{y\in Y}$ is completely orthogonal, i.e., there is a Borel map $\phi: X \to K$ such that*

$$\mu_y(X - \phi^{-1}(y)) = 0 \quad \text{for all } y \in Y.$$

Burgess and Mauldin asked if for a given atomless probability measure υ on Y one can choose K in such a way that $\upsilon(K) > 0$. In [2] Gardner gives a counterexample which even shows more, but is somewhat artificial. In this note we want to point out that there are also "classical kernels" like the transition kernel of certain random walks and the kernel of the Radon Transform for which the question posed by Burgess and Mauldin has a negative answer.

1. A random walk. Let $X = Y = \mathbb{R}^2$. υ is Lebesgue's measure on $\mathbb{R}^2$ and μ the rotation invariant probability measure supported by the unit circle S^1. Then

$$\mu_y(A) = \mu(A - y), \qquad y \in \mathbb{R}^2$$

defines the transition kernel of a random walk on $\mathbb{R}^2$ with the following properties:

a) μ_{y_1} and μ_{y_2} are mutually singular for $y_1 \neq y_2$.
b) If $(\mu_y)_{y\in K}$ is a completely orthogonal kernel for a Borel set $K \subset \mathbb{R}^2$ then $\upsilon(K) = 0$.

PROOF.
a) Clear since two distinct circles intersect in at most two points.

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b) Assume to the contrary that there is a Borel set $K_0 \subset \mathbb{R}^2$, $v(K_0) > 0$, and a Borel map $\phi: \mathbb{R}^2 \to K_0$ such that

$$\mu_y(\phi^{-1}(y)) = 1 \quad \text{for all} \quad y \in K_0.$$

Choose a bounded Borel set $K \subset K_0$ with $v(K) > 0$. Since the support of μ_y, $y \in K$, is contained in a large enough ball $B_r = \{x \in \mathbb{R}^2: \| x \| \le r\}$, we can modify ϕ in such a way that K and $\phi^{-1}(K)$ are contained in B_r and still

$$\mu_y(\phi^{-1}(y)) = 1 \quad \text{for all} \quad y \in K.$$

Choose Borel functions g_n on B_r with $| g_n | = 1$ on K and $g_n \to 0$ in the weak topology of the Hilbert space $L_2(B_r, v)$ and put $f_n := g_n \circ \phi$. Consider the convolution operator

$$T: L_2(B_r, v) \to L_2(B_r, v), \quad Tf = \chi B_r(f * \mu).$$

For $y \in K$, we obtain

$$Tf_n(y) = \int_{R^2} f(x)\, d\mu_y(x) = g_n(y)\mu_y(\phi^{-1}(y)) = g_n(y).$$

In particular, we have $\| \chi K T f_n \|_{L_2} \ge v(K)^{1/2}$ and $\chi K T f_n \to 0$ weakly for $n \to \infty$. But, this leads to a contradiction because we shall now show that T is a compact operator. First, we observe that the Fourier-Stieltjes transform $\hat{\mu}(x)$ goes to zero as $\| x \| \to \infty$. Indeed by the rotation invariance of μ and the substitution $u = \cos \alpha$, we have

$$\hat{\mu}(x) = \int e^{-i(x,y)}\, d\mu(y) = \int_{-\pi}^{\pi} e^{-i\|x\|\cos\alpha}\, d\alpha$$

$$= 2 \int_0^1 e^{-i\|x\|u} \frac{1}{\sqrt{1-u^2}}\, du.$$

The last term goes to zero for $\| x \| \to \infty$ by the Riemann-Lebesgue-lemma applied to the one-dimensional function $(1/\sqrt{1-u^2})\chi(0,1)(u)$. Given $\varepsilon > 0$ we choose now a large enough ball B_s such that $|\hat{\mu}(x)| \le \varepsilon$ for $x \notin B_s$. If $\mathscr{F}$ denotes the Fourier transform in $L_2(\mathbb{R}^2, v)$ and $M_{\hat{\mu}}$ stands for multiplication with the function $\hat{\mu}$ we can write

$$T = \chi B_r \mathscr{F} M_{\hat{\mu}} \mathscr{F}^{-1} = \chi B_r \mathscr{F} \chi B_s M_{\hat{\mu}} \mathscr{F}^{-1} + \chi B_r \mathscr{F} \chi B_s^c M_{\hat{\mu}} \mathscr{F}^{-1}.$$

Since $\chi B_r \mathscr{F} \chi B_s$ is a compact integral operator and $\| \chi B_s^c M_{\hat{\mu}} \| \le \varepsilon$ for arbitrary $\varepsilon > 0$ it follows that $T: L_2(B_r, v) \to L_2(B_r, v)$ is compact.

REMARK. As an example of a Cantor set $K \subset \mathbb{R}^2$ such that $(\mu_y)_{y\in K}$ is completely orthogonal (as in the theorem), we may take the classical Cantor subset K of $\{0\} \times [0, 1]$.

In order to find a separating function ϕ consider the continuous maps

$$p: K \times S^1 \to K, \quad p(x, y) = x \quad \text{and} \quad q: K \times S^1 \to \mathbb{R}^2, \quad q(x, y) = x + y.$$

If $B \subset K \times S^1$ is a Borel cross-section for the map $q: K \times S^1 \to Im(q)$ (see [5] I, Theorem 4.2), we define ϕ on Imq by $\phi = (p|_B) \circ (q|_B)^{-1}$ and extend it to a Borel map $\phi: \mathbb{R}^2 \to K$. For $x \in K$ we have

$$\phi^{-1}(x) \supset (x + S^1) \setminus \cup_{y \in K, y \neq x} y + S^1, \quad \mu_x(\cup_{y \in K, y \neq x} y + S^1) = 0$$

and therefore $\mu_x(\phi^{-1}(x)) = 1$.

2. The Radon transform. Let $X = \mathbb{R}^3$ and $Y = S^2 \times \mathbb{R}^+$. μ is the three-dimensional Lebesgue-measure on X and v the product of the rotation invariant measure on S^2 and the one-dimensional Lebesgue measure on $\mathbb{R}^+$. If $\mu_{(\omega,p)}$ for $(\omega, p) \in S^2 \times \mathbb{R}^+$ denotes the two-dimensional Lebesgue-measure on the hyperplane $\{x: (x, \omega) = p\}$, then the *Radon Transform* $R: L_1(X, \mu) \to L_1(Y, v)$ is given by

$$Rf(\omega, p) = \int_{(x,\omega)=p} f(x)\, d\mu_{(\omega,p)}(x).$$

Of course, in order to have a transition kernel $(\mu_{(\omega,p)})_{(\omega,p)\in Y}$ of probability measures, it would be necessary to introduce densities for the measures $\mu_{(\omega,p)}$ but we prefer to deal with the "classical" Radon transform. We claim:

a) μ_{y_1} and μ_{y_2} are mutually singular for $y_1, y_2 \in Y$, $y_1 \neq y_2$.
b) If $(\mu_y)_{y\in K}$ is a completely orthogonal kernel for a Borel set $K \subset Y$ then $v(K) = 0$.

PROOF.
a) Clear.
b) The Radon Transform also acts as an operator $R: B(D) \to B(E)$ where $B(D)$ and $B(E)$ are the bounded Borel functions on $D = \{x \in \mathbb{R}^3: |x| \leq 1\}$ and $E = S^2 \times [0, 1]$ resp. Assume that there is a $K \subset E$ with $v(K) > 0$ and a Borel function $\phi: D \to K$ such that

$$\mu_y(D - \phi^{-1}(y)) = 0 \quad \text{for all} \quad y \in K.$$

Then

$$(*) \qquad \chi_K R(\{f \in B(D): 0 \leq f \leq 1\}) = \{ag: g \in B(E), 0 \leq g \leq 1\}$$

where $a(\omega, p) = \mu_{(\omega,p)}(D)$ for $(\omega, p) \in K$ and zero otherwise. Indeed, for $g \in B(E)$, $0 \leq g \leq 1$, choose $f = g \circ \phi$ and observe that for $(\omega, p) \in K$

$$Rf(\omega, p) = \int_D g \circ \phi(x)\, d\mu_{(\omega,p)}(x) = \mu_{(\omega,p)}(D) \cdot f(\omega, p).$$

But $(*)$ leads to a contradiction because by [3], pages 28–29, $R(B(D))$ contains only functions for which the partial derivative $\partial/\partial p$ exists v– almost everywhere. For a characterization of complete orthogonality in terms of surjectivity conditions like $(*)$, see [6]. One could also prove b) with an argument similar to the one in 1b) since it is well known that the Radon-Transform $R: L_2(D) \to L_2(E)$ is a compact operator. □

REMARK. For a fixed $\omega \in S^2$, the kernel $(\mu_y)_{y \in R \times \{\omega\}}$ is completely orthogonal with an obvious separating function ϕ.

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