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BOCHNER'S THEOREM IN MEASURABLE DUAL OF TYPE 2 BANACH SPACE

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Let μ be a Radon probability measure on a type 2 Banach space E. The following Bochner's theorem is proved. For every continuous positive definite function ϕ ($\phi(0) = 1$) on E, there exists a Radon probability measure σ_ϕ on the measurable dual $H_0(\mu)$ of (E, μ) with the characteristic functional ϕ (in some restricted sense).

1. Introduction. Let E be a quasi-complete locally convex Hausdorff space and μ be a given Radon probability measure on E. The μ-measurable dual $H_0(\mu)$ of (E, μ) is the closure of E' (the topological dual of E) in $L^0(E, \mu)$ with the vector topology induced by $L^0(E, \mu)$. Let $R: E' \to H_0(\mu)$ be the natural mapping, $R(x') = \langle \cdot, x' \rangle$. Then R is Mackey continuous. This follows directly from the fact that the measure μ is supported on a countable union of compact convex sets. Thus the transpose R' maps $H_0(\mu)'$ into E.

The Bochner problem, formulated by Xia [5] (Chapter III, Section 3.3; see also Okazaki [3] and Sato [4]) as follows, is investigated.

Bochner problem. For every continuous positive definite function ϕ on E with $\phi(0) = 1$, is there a Radon probability measure σ_ϕ on $H_0(\mu)$ with the characteristic functional

$$\sigma_\phi\hat{\ }(\xi) = \int_{H_0(\mu)} \exp(i\langle \xi, f \rangle)\, d\sigma_\phi(f) = \phi \circ R'(\xi) \quad \text{for every} \quad \xi \in H_0(\mu)'?$$

In the case where μ is a Gaussian Radon measure, the Bochner problem was completely solved on a general locally convex Hausdorff space as follows

Lemma (Okazaki [3] and Sato[4]). *Let E be a locally convex Hausdorff space and ρ be a Gaussian Radon measure on E. Then for every continuous positive definite function ϕ on E with $\phi(0) = 1$, there is a Radon probability measure σ_ϕ on $H_0(\rho)$ with the characteristic functional $\sigma_\phi\hat{\ }(\xi) = \phi \circ R'(\xi)$ for every $\xi \in H_0(\rho)'$.*

In this paper, we shall restrict the space E to a type 2 Banach space, but the measure μ is general. We prove that the Bochner problem is valid in this case as stated in the abstract.

2. Bochner's theorem on type 2 Banach spaces. A Banach space

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E is of type 2 if and only if each Radon probability measure λ on E with $\int_E \|x\|^2 \, d\lambda(x) < +\infty$ is pre-Gaussian, that is, there exists a Gaussian Radon measure ρ on E with the characteristic functional $\rho\hat{}(x') = \int_E \exp(i\langle x, x'\rangle) \, d\rho(x) = \exp(-\frac{1}{2} \int |\langle x, x'\rangle|^2 \, d\lambda(x))$. For example, the Banach spaces L^p $(2 \leq p < +\infty)$ are of type 2 (for details, see Hoffmann-Jørgensen and Pisier [2]).

THEOREM. *Let E be a type 2 Banach space and μ be a Radon probability measure on E. Then for every continuous positive definite function ϕ on E with* $\phi(0) = 1$, *there is a Radon probability measure σ_ϕ on $H_0(\mu)$ with the characteristic functional $\sigma_\phi\hat{}(\xi) = \phi \circ R'(\xi)$ for every $\xi \in H_0(\mu)'$. Furthermore, we can find a Hilbertian subspace H $(RE' \subset H \subset H_0(\mu))$, depending only on μ, such that σ_ϕ is supported by H.*

PROOF. Take a Radon probability measure λ on E with $\int_E \|x\|^2 \, d\lambda(x) < +\infty$ such that μ and λ are mutually absolutely continuous. For example, consider the measure $\exp(-\|x\|^2)d\mu(x)$ and normalize it. Since μ and λ are mutually absolutely continuous, $H_0(\mu)$ and $H_0(\lambda)$ are topologically isomorphic; see Dudley [1] (Theorem 3). Let ρ be a Gaussian Radon measure on E with the characteristic functional $\rho\hat{}(x') = \exp(-\frac{1}{2} \int |\langle x, x'\rangle|^2 \, d\lambda(x))$ for $x' \in E'$. Then $H_0(\rho)$ coincides with the $L^2(\lambda)$-closure of E' since ρ is Gaussian. $H_0(\rho)$ is a Hilbert space. We have the continuous injection i: $H_0(\rho) \to H_0(\lambda) \simeq H_0(\mu)$. Let ϕ be a continuous positive definite function on E with $\phi(0) = 1$. Then by the Lemma, there exists a Radon probability measure ν_ϕ on $H = H_0(\rho)$ such that $\nu_\phi\hat{}(\xi) = \phi \circ R'(\xi)$ for every $\xi \in H'$. Considering the image $\sigma_\phi = i(\nu_\phi)$ on $H_0(\mu)$, we have the assertion.

This completes the proof.

REMARK. Let E be a locally convex Hausdorff space and μ be a Radon probability measure on E. Suppose that μ is absolutely continuous with respect to a pre-Gaussian measure. Then the Bochner problem is valid by a similar argument.

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