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ON THE RANGE OF BROWNIAN MOTION AND ITS INVERSE PROCESS

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Jump behavior of the first passage time processes for Brownian motion, its range and BES(3) are compared via their Poisson measures. Explicit results concerning Brownian motion up to a first passage time of its range are given.

1. Introduction. $X = \{X(t), t \geq 0\}$ will be either standard Brownian motion BM^x or the "three-dimensional" Bessel process BES^x, $x = X(0)$. In their canonical description on the space of continuous functions, $\mathscr{P}^x(\mathscr{P} = \mathscr{P}^0$ for BM) and $\mathscr{R}^x(\mathscr{R} = \mathscr{R}^0$ for BES) are their laws. Let $M(t) = \sup\{X(s), 0 \leq s \leq t\}$, $m(t) = \inf\{X(s), 0 \leq s \leq t\}$ and $R(t) = M(t) - m(t)$. The $\mathscr{P}$-law of this range was given by Feller [4]. We obtain the density of the Poisson measure for the pure jump independent, nonstationary increments process of first passage times of R under $\mathscr{P}$. Comparisons are made with the apparently not familiar corresponding density for the first passage time process of X under $\mathscr{R}$ and the well-known one under $\mathscr{P}$. This leads to relations between expected numbers of jumps $>s$ in given intervals. Some further results concerning X and R are given in the BM case, and it is noticed that the first passage time processes of R in the BM and BES^x cases are identical in law over $[0, x]$.

It is convenient to distinguish notationally between $\tau(y) = \inf\{s, X(s) = y\}$ and $\theta(y) = \inf\{s, R(s) = y\}$, to be used when X is BM, and the corresponding times $\tau_+(y)$, $\theta_+(y)$ to be used only when X is BES, respectively BES^x, $x > 0$. In this way $E\theta(y)$ and $E\tau_+(y)$ automatically mean $\mathscr{P}$- and $\mathscr{R}$-expectations, respectively. We use basic notation from [8] which gives shorter formulas than the one in [7]. For $t > 0$ and all $x \in \mathbb{R}$,

$$p_t(x) = (2\pi t)^{-1/2}\exp\{-x^2/2t\}, \quad g_t(x) = -(\partial/\partial x)p_t(x).$$

For $t > 0$ and reals x, y, z with $yz \neq 0$ the following sums are defined,

$$\begin{aligned} P_t(x, y) &= \sum p_t(x + 2ny), \\ G_t(x, y) &= \sum g_t(x + 2ny) = -(\partial/\partial x)P_t(x, y), \qquad (1.1) \\ Q_t(x, y, z) &= P_t(x - y, z) - P_t(x + y, z). \end{aligned}$$

Sums with no limits indicated are always over $n \in \mathbb{Z}$.

For $x > 0$, $g_t(x)dt = \mathscr{P}(\tau(x) \in dt)$ and for $0 < x < y$ ([3], Proposition 8) $G_t(x, y)dt = \mathscr{P}(\tau(x) \in dt, \tau(x - y) > t)$. It is well-known ([2], (11.10)) that

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$Q_t(x, y, z)dy = \mathscr{P}^x(X(t) \in dy, \tau(0) \wedge \tau(z) > t)$ when $0 < x \wedge y \leq x \vee y < z$. The probabilistic interpretation, for the first equation below, and Example 2 of [7] for the second equation give, when $0 < x \leq y < z$,

$$(1.2) \qquad \int_0^\infty G_t(x, y)\, dt = \frac{y - x}{y}, \quad \int_0^\infty Q_t(x, y, z)\, dt = \frac{2x(z - y)}{z}.$$

A further useful function, and integral, are for $t, y > 0$,

$$(1.3) \qquad E_t(y) = \int_0^y G_s(x, y)G_{t-s}(y - x, y)\, dx, \quad \int_0^\infty E_t(y)\, dt = \frac{1}{2y}.$$

The definition of $E_t(y)$ does not depend on s, $0 < s < t$. The integral results, e.g., from ([7], Theorem 4)

$$(1.4) \qquad \mathscr{R}(\tau_+(y) \in dt) = 2yE_t(y)\, dt.$$

One has ([7], Lemma 1)

$$(1.5) \qquad E_t(y) = (\partial/\partial t)P_t(y, y) = -(2t)^{-1}(\partial/\partial y)\{yP_t(y, y)\}.$$

2. Comparison of hitting time processes. Fix $y > 0$ and abbreviate when convenient $\theta(y) = \theta$. As $\mathscr{P}(\tau(x) \in dt, R(\tau(x)) < y) = G_t(x, y)dt$, one has for $0 < x < y$

$$(2.1) \qquad \mathscr{P}(\theta \in dt, X(\theta) \in dx) = (\partial/\partial y)G_t(x, y)dtdx.$$

Using (1.2) and symmetry, there follows

$$(2.2) \qquad \mathscr{P}(X(\theta) \in dx) = |x| y^{-2}dx, \quad |x| < y.$$

Integration of (2.1) in x can be taken under $\partial/\partial y$ so by (1.1), doubling to account for $X(\theta) < 0$,

$$(2.3) \qquad \mathscr{P}(\theta(y) \in dt) = 2(\partial/\partial y)Q_t(y/2, y/2, y)dt, \quad t > 0.$$

The argument of L.T.'s (Laplace transforms) will be denoted λ, and we abbreviate $(2\lambda)^{1/2} = \lambda_*$. Direct computation or [7], Example 3, give for (2.3) the L.T. $\mathrm{ch}^{-2}\,\tfrac{1}{2}y\lambda_*$. This was found in a different context in [6]. For BM, the process $\theta = \{\theta(y), y \geq 0\}$ has independent increments. If $0 \leq y < z$ the L.T. for the density of $\theta(z) - \theta(y)$ is therefore $\mathrm{ch}^2\,\tfrac{1}{2}y\lambda_* \cdot \mathrm{ch}^{-2}\,\tfrac{1}{2}z\lambda_*$. The density is given in the next section.

A well-known p.d.f. (probability density function) is

$$H(y) = \sum (-1)^n \exp(-\tfrac{1}{2}n^2y^2)$$

over $0 < y < \infty$ ([2], (11.39)). One has

$$(2.4) \qquad Q_s(y/2, y/2, y) = (2\pi s)^{-1/2}H(s^{-1/2}y).$$

LEMMA 1. *Fix $y > 0$. The three functions of s $(0 < s < \infty)$*

$$(2.5) \qquad 2P_s(0, y) - (1/y), \quad (1/y) - 2P_s(y, y), \quad Q_s(y/2, y/2, y)$$

are positive, decreasing with limit 0 as $s \to \infty$, and for $s \downarrow 0$, $P_s(0, y) \sim Q_s(y/2, y/2, y) \sim (2\pi s)^{-1/2}$.

PROOF. A standard theta function transformation formula ([1], (19.2)) gives

$$2P_s(0, y) = (1/y) \sum \exp\{-n^2\pi^2 s/2y^2\},$$

$$2P_s(y, y) = (1/y) \sum (-1)^n \exp\{-n^2\pi^2 s/2y^2\}.$$

Positivity, decrease and the limit 0 at infinity for the first two functions (2.5) follow by inspection, and therefore hold also for their mean $Q_s(y/2, y/2, y)$. Writing $(2\pi s)^{1/2}P_s(0, y) = 1 + 2\sum_{n>0}\exp\{-2n^2y^2/s\}$, and (2.4), give the behavior near 0.

For BM, the Poisson measure describing the hitting time process τ has density $f(y, s) = (2\pi s^3)^{-1/2}y, s > 0$ ([9], p. 27). Call respectively $f_+(y, s)$ and $f^R(y, s)$ the densities of the Poisson measures relative to the pure jump processes τ_+ and θ of hitting times of BES, and of the range of BM. Also, let $N(y, s)$, $N_+(y, s)$ and $N^R(y, s)$ be the numbers of jumps $>s$ taken by τ, τ_+ and θ up to y. Thus, e.g.,

$$(2.6)\quad E\theta(y) = \int_0^y \int_0^\infty sf^R(z, s)\, ds\, dz, \quad EN^R(y, s) = \int_0^y \int_s^\infty f^R(z, t)\, dt\, dz.$$

THEOREM 1. *For* $y, s > 0$,

$$(2.7)\quad f^R(y, s) = -2(\partial/\partial s)Q_s(y/2, y/2, y) > f_+(y, s) = -2(\partial/\partial s)P_s(0, y).$$

PROOF. The density $f^R(y, s)$ is determined ([9], p. 146) by

$$\mathrm{ch}^2\, \frac{1}{2}\, y\lambda_* = \exp\left\{\int_0^y \int_0^\infty (1 - e^{-\lambda s}) f^R(z, s)\, ds\, dz\right\}.$$

This is satisfied if

$$\int_0^\infty (1 - e^{-\lambda s}) f^R(y, s)\, ds = \lambda_* \mathrm{th}\, \frac{1}{2}\, y\lambda_*$$

or, setting $f^R(y, s) = -(\partial/\partial s)F(y, s)$, if $sF(y, s) \to 0$ for $s \downarrow 0$ and $F(y, s)$ is a decreasing function of s vanishing at infinity having L.T. $(\mathrm{th}\, \tfrac{1}{2}y\lambda_*)/\tfrac{1}{2}\lambda_*$. Thus, $F(y, s) = 2Q_s(y/2, y/2, y)$ ([7], Example 3) is, refering to Lemma 1, the solution. Proceeding similarly for $f_+(y, s)$, $\tau_+(y)$ has the L.T. $y\lambda_*/\mathrm{sh}\, y\lambda_*$ and one now seeks $F(y, s)$ with L.T. $(2\,\mathrm{ch}\, y\lambda_*)(\lambda_*\, \mathrm{sh}\, y\lambda_*)^{-1} - (\lambda y)^{-1}$. This gives $F(y, s) = 2P_s(0, y) - y^{-1}$ which, by Lemma 1, has the required behavior. Finally, writing

$$2Q_s(y/2, y/2, y) = [2P_s(0, y) - (1/y)] + [(1/y) - 2P_s(y, y)],$$

the desired inequality follows from Lemma 1.

REMARK 1. For an intuitive justification of $f^R > f_+$, one may interpret $f^R(y, \cdot)$ as a density for the duration of a Brownian excursion from $(-\infty, 0] \cup [y, \infty)$ while $f_+(y, \cdot)$. is one for the duration of an excursion of BES from $[y, \infty)$, or equivalently of a Brownian excursion from $(-\infty, 0] \cup [y, \infty)$ starting at y and conditioned to end at y, indicating that $f^R > f_+$. Notice also that $E\theta(y) = y^2/2 > E\tau_+(y) = y^2/3$.

LEMMA 2. *For $0 < y < \infty$, respectively $0 < s < \infty$, the functions*

$$\bar{H}(y) = \frac{1}{y}\int_0^y H(z)\,dz, \quad H_1(s) = 2P_s(1, 1), \tag{2.8}$$

$$H_2(s) = 2P_s(0, 1) - 2(2\pi s)^{-1/2}$$

are p.d.f.'s. The density of H_2 is the convolution of the density $2E_s(1)$ of H_1 with the density $g_s(1)$.

PROOF. The fact is obvious for $\bar{H}$. For H_1, refer to (1.4) and (1.5). For H_2, definition (1.1) gives the limit 0 for $s \downarrow 0$, and Lemma 1 gives the limit 1 for $s \uparrow \infty$. The L.T. $2\exp\{-\lambda_*\}/(\lambda_* \operatorname{sh}\lambda_*)$ of H_2 therefore gives for $(d/ds)H_2(s)$ the L.T. $(\lambda_*/\operatorname{sh}\lambda_*)\cdot\exp\{-\lambda_*\}$, product of those of the stated densities.

REMARK 2. The above convolution and reference to Theorem 3.5 of [10] show that $H_2(s) = \mathscr{P}(L \le s)$, where L is the duration of the Brownian excursion across $\tau(1)$. It will be shown elsewhere that the time L^- between the start and the maximum of said excustion has p.d.f. $\mathscr{P}(L^- \le s) = \bar{H}(\pi s^{1/2})$.

We have from (2.6), (2.7),

$$EN^R(y, s) = 2\int_0^y Q_s\left(\frac{z}{2}, \frac{z}{2}, z\right)dz,$$
$$EN_+(y, s) = \int_0^y \left[2P_s(0, z) - \frac{1}{z}\right] dz. \tag{2.9}$$

For $EN(y, s)$, the corresponding integrand is $2(2\pi s)^{-1/2}$. By (2.4) and the inequality in (2.7), one has therefore

$$EN(y, s) > EN^R(y, s) > EN_+(y, s).$$

Specifically, $EN^R(y, s) = \bar{H}(ys^{-1/2})EN(y, s)$ and, at $y = 1$ for brevity, $f(1, s) - f^R(1, s) = (\partial/\partial s)(H_2(s) - H_1(s))$ which, as $H_2 < H_1$, shows $f(1, s) < f^R(1, s)$ for $0 < s <$ some s_0, $f(1, s) > f^R(1, s)$ for $s > s_0$. This is clear also when considering the pathwise passage from BM to its range.

Further comparisons are better made about $N(y, z, s) = N(z, s) - N(y, s)$, $0 < y < z$, $0 < s$, and the corresponding $N^R(y, z, s)$, $N_+(y, z, s)$. Let

$$\Delta_1 = EN^R(y, z, s) - EN_+(y, z, s), \quad \Delta_2 = EN(y, z, s) - EN_+(y, z, s).$$

COROLLARY. *For $i = 1, 2$, with Φ being the unit-normal c.d.f.,*

$$\Delta_i = \ln(z/y) - 4\sum_{n\ge 0}(2n + i)^{-1}[\Phi((2n + i)zs^{-1/2}) - \Phi((2n + i)ys^{-1/2})],$$
$$1 - H_i(sy^{-2}) < \Delta_i/\ln z/y < 1 - H_i(sz^{-2}).$$

PROOF. Considering in (2.9), and correspondingly for $EN(y, z, s)$, integrals over (y, z) shows that

$$\Delta_i = \int_y^z \left[\frac{1}{v} - 4\sum_{n\ge 0} p_s((2n + i)v)\right] dv = \int_y^z \frac{1}{v}[1 - H_i(sv^{-2})]\,dv.$$

Term-by-term integration on the one hand and use of the bounds on $H_i(sv^{-2})$ at $v = y, z$ on the other gives the results.

3. Brownian motion prior to $\theta(z)$. Let $0 < y < z$. The density for $\theta(z) - \theta(y)$ can be obtained by conditioning for instance on $X(\theta(y)) > 0$, reviewing possibilities and carrying out some calculations. This gives

$$\mathscr{P}(\theta(z) - \theta(y) \in dt) = (\partial/\partial z)\{Q_t((z-y)/2, (z-y)/2, z) + Q_t((z-y)/2, (z+y)/2, z)\}dt.$$

The L.T. is easily checked from [7], Example 2.

Theorem 2 below parallels results of Williams ([10], Theorems 3.1 and 3.5). We give the proof because the same pattern can serve to establish in elementary fashion several known path decompositions. As pointed out in [10], direct comparison of $\mathscr{P}^x$ and $\mathscr{R}^x$ transition densities gives when $x, z > 0$, for all A in the σ-field $\mathscr{F}_t = \sigma\{X(s), 0 \le s < t\}$,

$$\mathscr{R}^x(A, X(t) \in dz) = (z/x)\mathscr{P}^x(A, X(t) \in dz, \tau(0) > t). \tag{3.1}$$

Consider the formal event $A = \{X(u_j) \in dx_j, j = 1, \cdots, k\}$, $0 = u_0 < u_1 < \cdots < u_k < r$, $0 < x, x_1, \cdots, x_k < z$. Writing $t_j = u_j - u_{j-1}$ and extending (3.1) to optional t ([5], page 100) one has with the abbreviation

$$Q = \textstyle\prod_{j=2}^{k} Q_{t_j}(x_{j-1}, x_j, z)dx_j,$$

$$\mathscr{R}^x(A, \tau_+(z) \in dr) = (z/x)Q_{t_1}(x, x_1, z)dx_1 QG_{r-u_k}(z - x_k, z)dr. \tag{3.2}$$

Now as $[(\partial/\partial x)Q_t(x, x_1, z)]_{x=0} = 2G_t(x_1, z)$ and $Q_t(0, x_1, z) = 0$, one has for $x \downarrow 0$, $\lim Q_t(x, x_1, z)/x = 2G_t(x_1, z)$. Thus (3.2) leads to

$$\mathscr{R}(A, \tau_+(z) \in dr) = 2zG_{t_1}(x_1, z)dx_1 QG_{r-u_k}(z - x_k, z)dr. \tag{3.3}$$

Those finite dimensional densities determine the law of BES considered up to $\tau_+(z)$.

Going back to BM, let $\theta = \theta(z)$, $\sigma = \inf\{s > 0, X(s) = M(\theta)\}$ if $X(\theta) < 0$, $\sigma = \inf\{s > 0, X(s) = m(\theta)\}$ if $X(\theta) > 0$ and let $\rho = \theta - \sigma$.

THEOREM 2. *For Brownian motion X, the process $\{|X(\sigma + u) - X(\sigma)|, 0 \le u \le \rho\}$ is BES considered up to $\tau_+(z)$.*

PROOF. Suppose, e.g., $X(\sigma) > 0$. For $s, t > 0$ let σ^* be the time when $M(s + t)$ is (first) achieved. One has for $0 < w, y < z$ ([7], Theorem 2).

$$\mathscr{P}(M(s+t) \in dy, \sigma^* \in ds, X(s+t) \in y - dw, \tau(y - z) > s + t) = 2G_s(y, z)G_t(w, z)dydsdw. \tag{3.4}$$

Therefore, in the notation of (3.2),

$$\mathscr{P}(\sigma \in ds, X(\sigma) \in dy, X(\sigma + u_j) \in y - dx_j, j = 1, \cdots, k, \rho \in dr) = 2G_s(y, z)G_{t_1}(x_1, z)dydsdx_1 \cdot QG_{r-u_k}(z - x_k, z)dr.$$

As

$$2 \int_0^z \int_0^\infty G_s(y, z)\, ds\, dy = z,$$

comparison with (3.3) gives

$$\mathscr{P}(X(\sigma) > 0,\ X(\sigma) - X(\sigma + u_j) \in dx_j,\ j = 1, \cdots, k,\ \rho \in dr) \\ = \tfrac{1}{2}\mathscr{R}(A,\ \tau_+(z) \in dr).$$

Adding the corresponding result for $X(\sigma) < 0$ gives the conclusion.

Maximality of the density (2.2) at the ends of its support suggests that the last zero before σ generally occurs "close" to σ. Let $\gamma = \sup\{s < \sigma, X(s) = 0\}$ and $\eta = \sigma - \gamma$. We can by scaling consider $z = 1$ only, so now $\theta = \theta(1)$. For $t, u, r > 0$ and $|x| < 1$, one has

$$\mathscr{P}(\gamma \in dt,\ \eta \in du,\ \rho \in dr,\ X(\theta) \in dx) \\ = 2Q_t(|x|, |x|, 1)E_u(1 - |x|)E_r(|x|)dtdudrdx.$$

This is obtained by choosing times $s' \in (t, t + u)$, $s'' \in (t + u, t + u + r)$, using (5.2) of [7] for the pre-s' part of the path, (3.4) above for the s'–s'' part and (1.3) when integrating over possible values of $X(s')$ and $X(s'')$. The integrals in (1.2) and (1.3) therefore give

$$\mathscr{P}(\eta \in du,\ X(\theta) \in dx) = 2|x|(1 - |x|)E_u(1 - |x|)dudx.$$

This corresponds to the fact that conditionally on $X(\theta) = x$, the γ-to-σ part of $|X|$ is BES over $[0, \tau_+(1 - |x|)]$, and σ is a splitting time, so (1.4) and (2.2) can be applied. Using (1.5) one obtains after some calculations

$$\mathscr{P}(\eta \le u) = \pi(u\pi/2)^{1/2} - 8u \textstyle\sum_{n>0} 1/(2n - 1)^3[\Phi((2n - 1)u^{-1/2}) - \tfrac{1}{2}].$$

The following values confirm the smallness of η:

u:	1/100	1/16	1/4	1/2.25
$\mathscr{P}(\eta \le u)$:	.35	.72	.96	.99

Further information can be obtained along similar lines. For instance,

$$\mathscr{P}(M(\gamma)/M(\theta) \in dx \mid X(\theta < 0) = 2(1 - x)^{-3} \cdot (1 - x^2 + 2x \ln x)dx, \quad 0 \le x \le 1.$$

We end with some remarks concerning the range of BES^x, $x > 0$. Suppose the event A in (3.1) is symmetric with respect to level x: $A = 2x - A$, and such that $A \subset \{\tau(0) \wedge \tau(2x) > t\}$. For $0 < y < x$, (3.1) gives

$$R^x(A,\ |X(t) - x| \in dy) \\ = \mathscr{R}^x(A,\ X(t) \in x + dy) + \mathscr{R}^x(A,\ X(t) \in x - dy) \\ = (1 + (y/x))\mathscr{P}^x(A,\ X(t) \in x + dy) + (1 - (y/x))\mathscr{P}^x(A,\ X(t) \in x - dy)$$

The latter two probabilities are equal by hypothesis, hence

$$\mathscr{R}^x(A,\ |X(t) - x| \in dy) = \mathscr{P}^x(A,\ |X(t) - x| \in dy).$$

This implies in particular that $\{R(t), 0 \leq t \leq \theta(x)\}$ and $\{\theta(y), 0 \leq y \leq x\}$ have the same law under $\mathscr{R}^x$ (when we should write θ_+) as under $\mathscr{P}$. For $y > x$ on the other hand, the increments of θ_+ are no more independent. From (2.1) and (3.1), one has for $0 < z < x \wedge y$,

$$\mathscr{R}^x(\theta_+(y) \in dt, X(\theta_+(y)) \in x - dz) = (1 - (z/x))(\partial/\partial y)G_t(z, y)dtdz,$$

hence

$$\mathscr{R}^x(X(\theta_+(y)) \in x - dz) = (1 - zx^{-1})zy^{-2}dz, \quad 0 < z < x \wedge y.$$

If "$y, \downarrow$" stands for "$\theta_+(y)$ occurs while X is decreasing", this gives

$$\mathscr{R}^x(y, \downarrow) = \begin{cases} 1/2 - y/3x & \text{for} \quad 0 < y \leq x, \\ x^2/6y^2 & \text{for} \quad x < y. \end{cases}$$

For $0 < y \leq x$, one obtains more specifically the simple result

$$\mathscr{R}^x(\theta_+(y) \in dt, \downarrow) = 1/2\mathscr{P}(\theta(y) \in dt) - 2(t/x)E_t(y)dt.$$

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